$\boxed{\text{MAT 307 LECTURE 4}}$

- REDUCED ROW ECHELON FORM
- LINEAR INDEPENDENCE

RECALL FROM LAST CLASS:

A LINEAR SYSTEM HAS THE FORM: m EQUATIONS, $x_1, \dots, x_n$ UNKNOWNS.

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$
$$\vdots \qquad\qquad\qquad \vdots$$
$$a_{m1}x_1 + \dots + \quad a_{mn}x_n = b_m$$

EXAMPLE:

$$2x + 3y = 4$$
$$x - y = 1.$$

BASIC STRATEGY: FORM COMBINATIONS OF EQUATIONS TO ELIMINATE VARIABLES:

EQ'N 1 $- 2 \cdot$ EQ'N 2 $= 5y = 2$

$y = 2/5$

$x = 7/5$

ROW REDUCTION PROCESS:

IN EACH ROW OF A MATRIX

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$$

THE FIRST NON-ZERO ENTRY IS CALLED A LEADING ENTRY

(1) PICK A COLUMN IN WHICH SOME ROW HAS A LEADING ENTRY

(2) DIVIDE THE ROW BY THE LEADING ENTRY SO THAT THE LEADING ENTRY IS NOW 1.

(3) SUBTRACT MULTIPLES OF THE ROW FROM OTHER ROWS TO CLEAR THE REST OF THE COLUMN.

REPEAT WHILE THE ABOVE CHANGES THE MATRIX.

EACH TIME THAT THIS COLUMN PROCESS OCCURS, THE NUMBER OF COLUMNS WITH 1 ENTRY 1, REMAINING ENTRIES 0 INCREASES BY 1, SO TERMINATES IN FINITELY MANY STEPS.

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 2 & 3 & 4 \\ 1 & 0 & 1 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 0 & 2 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 & 4 \\ 1 & 0 & 1 & 1 & 1 \end{pmatrix}$$

<u>CLAIM</u>: AFTER COLUMN 1 PROCESS, THE 3rd ROW ALWAYS HAS LEADING ENTRY IN COLUMN 1, THIS ROW IS NEVER PICKED AGAIN, AND COLUMN 1 REMAINS UNCHANGED IN THE REST OF THE PROCESS.

THIS IS BECAUSE, TO CHANGE COLUMN 1 NOW, WE WOULD HAVE TO SUBTRACT ROW 3 FROM ANOTHER ROW, BUT THIS NEVER OCCURS SINCE THE LEADING 1 IS ALWAYS IN THE FIRST COLUMN AND THE REST OF THE COLUMN IS 0.

$$\begin{pmatrix} 0 & 2 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 & 4 \\ 1 & 0 & 1 & 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 0 & -2 & -3 & -4 \\ 0 & 1 & 2 & 3 & 4 \\ 1 & 0 & 1 & 1 & 1 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 0 & 0 & 1 & \frac{3}{2} & 2 \\ 0 & 1 & 2 & 3 & 4 \\ 1 & 0 & 1 & 1 & 1 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 0 & 0 & 1 & \frac{3}{2} & 2 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -\frac{1}{2} & -1 \end{pmatrix}$$

THIS MATRIX IS REDUCED!

- EACH LEADING ENTRY IN A ROW IS 1, IT BELONGS TO A COLUMN WITH REMAINING ENTRIES 0.

THUS THERE ARE NO FURTHER MOVES TO MAKE.

IT IS COMMON TO INCLUDE A FURTHER OPERATION, SWAP TWO ROWS.

★ ONCE NO FURTHER COLUMN OPERATIONS CAN BE PERFORMED, ORDER THE ROWS SO THAT THEIR LEADING ENTRY MOVES LEFT TO RIGHT.

REDUCED ROW ECHELON FORM

(R.R.E.F.):

$$\begin{pmatrix} 0 & \cdots & 0 & 1 & * & * & 0 & \cdots & & * \\ 0 & & & 0 & \cdots & 0 & 1 & * & * & \cdots \\ \vdots & & & & & & & & & \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & 1 & * & * & \cdots * \\ 0 & \cdots & & & & & & & & 0 \\ 0 & \cdots & & & & & & & & 0 \end{pmatrix}$$

☆ 0 OR MORE ROWS OF ALL ZEROS AT BOTTOM.

☆ EACH NON-ZERO ROW HAS LEADING ENTRY 1 (PIVOT) 0's BEFORE IT AND IN REST OF THE COLUMN.

☆ THE PIVOTS MOVE LEFT-TO-RIGHT AS YOU MOVE DOWN THE MATRIX.

$\boxed{\text{THEOREM: ANY MATRIX CAN BE PUT IN R.R.E.F. BY ROW OPS.}}$

<u>RECALL:</u> A LINEAR SYSTEM

$$\begin{matrix} a_{11}x_1 + \dots + a_{1n}x_n = b_1 \\ \vdots \qquad\qquad \vdots \\ a_{m1}x_1 + \dots + a_{m1}x_n = b_m \end{matrix}$$

CAN BE WRITTEN AS A MATRIX EQUATION $A \cdot \underline{x} = \underline{b}$

$$A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix}, \quad \underline{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad \underline{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}.$$

RECALL IF A IS $m \times n$, B IS $n \times p$ THEN A HAS ROWS $\alpha_i \in \mathbb{R}^n$

$$\begin{pmatrix} - & \alpha_1 & - \\ & \vdots & \\ - & \alpha_m & - \end{pmatrix},$$

B HAS COLUMNS $\beta_j \in \mathbb{R}^n$

$$\begin{pmatrix} | & & | \\ \beta_1 & \dots & \beta_p \\ | & & | \end{pmatrix}.$$

THEN $C = A \cdot B$ HAS i,j ENTRY $\alpha_i \cdot \beta_j$. (DOT PRODUCT OF VECTORS)

THEOREM: MATRIX MULTIPLICATION IS LINEAR, THAT IS,

$$(aM+bN) \cdot A = aMA + bNA.$$
$$B \cdot (aM+bN) = aBM + bBN.$$

PROOF: THE LINEAR COMBINATION $aM+bN$ FORMS A LINEAR COMBINATION OF THE ROWS OF M AND N. THE DOT PRODUCT IS LINEAR.

IF M HAS ROWS $\begin{pmatrix} - & r_1 & - \\ & \vdots & \\ - & r_m & - \end{pmatrix}$

N HAS ROWS $\begin{pmatrix} - & s_1 & - \\ & \vdots & \\ - & s_m & - \end{pmatrix}$ THEN

$aM+bN$ HAS ROWS

$$\begin{pmatrix} - & a\underline{r}_1 + b\underline{s}_1 & - \\ - & a\underline{r}_2 + b\underline{s}_2 & - \\ & \vdots & \\ - & a\underline{r}_m + b\underline{s}_m & - \end{pmatrix}.$$

IF $A = \begin{pmatrix} | & & | \\ \alpha_1 & \dots & \alpha_p \\ | & & | \end{pmatrix}$

$(aM+bN) \cdot A$ HAS i,j ENTRY

$$(a\underline{r}_i + b\underline{s}_i) \cdot \underline{\alpha}_j = a(\underline{r}_i \cdot \underline{\alpha}_j) + b(\underline{s}_i \cdot \underline{\alpha}_j).$$

$\square$

<u>THEOREM</u>: EVERY SOLUTION OF $A\underline{x} = \underline{b}$ AS $\underline{x} = \underline{x}_p + \underline{x}_h$ WHERE $\underline{x}_p$ IS A SINGLE <u>PARTICULAR</u> SOLUTION AND $\underline{x}_h$ SATISFIES $A\underline{x}_h = \underline{0}$ IS A SOLUTION OF THE HOMOGENEOUS SYSTEM.

PROOF: SUPPOSE $\underline{x}_1, \underline{x}_2$ ARE ANY SOLUTIONS OF $A\underline{x}_1 = \underline{b}$, $A\underline{x}_2 = \underline{b}$.

THEN $A(\underline{x}_1 - \underline{x}_2) = \underline{b} - \underline{b}$
$= \underline{0}$

SO IF ANY $\underline{x}_p$ IS CHOSEN A SOLUTION $A\underline{x} = \underline{b}$ CAN BE WRITTEN $\underline{x} = \underline{x}_h + \underline{x}_p$ WHERE $\underline{x}_h = \underline{x} - \underline{x}_p$ SATISFIES $A\underline{x}_h = 0$. $\square$

WE HAVE SEPARATE STRATEGIES
FOR SOLVING THE
HOMOGENEOUS SYSTEM
AND THE PARTICULAR
PART.

SUPPOSE $A\underline{x} = \underline{b}$ HAS BEEN <u>REDUCED</u> TO

$$\tilde{A}\underline{x} = \tilde{b}$$

WHERE

$$\tilde{A} = \begin{pmatrix} 0 & \cdots & 0 & 1 & * & \cdots & * & * \\ 0 & \cdots & & 0 & 0 & \cdots 1 & * & \cdots * \\ \vdots & & & & & & & \\ 0 & \cdots & & & & 0 & 1 & * \; * \\ 0 & 0 & \cdots & & & & & 0 \end{pmatrix} \Bigg|_{k} \quad \text{R.R.E.F.}$$

$$\tilde{b} = \begin{pmatrix} \tilde{b}_1 \\ \vdots \\ \tilde{b}_k \\ \hline \vdots \\ \tilde{b}_m \end{pmatrix}.$$

BELOW k ROW WE HAVE ALL ZEROS

SO <u>NO SOLUTION</u> IF $\tilde{b}_j \neq 0$ FOR $j > k$.

TO FIND A PARTICULAR SOLUTION IF $\tilde{b}_j = 0$ FOR $j > k$:
FOR EACH $j \leq k$, SUPPOSE THE PIVOT IN ROW j IS AT POSITION $p(j)$. THEN SET $x_{p(j)} = \tilde{b}_j$.
THE x's NOT CORRESPONDING TO PIVOTS CAN BE SET TO 0.

THIS GIVES A METHOD OF FINDING A PARTICULAR SOL'N.

EXAMPLE: GIVEN THE SYSTEM:

$$\begin{pmatrix} 1 & -2 & 1 \\ 2 & 1 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -2 & 1 \\ 0 & 5 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2/5 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 9/5 \\ 2/5 \end{pmatrix}$$

x, y PIVOT VARIABLES; SET $z=0$.

$$x = 9/5, \quad y = 2/5.$$

THIS IS A PARTICULAR SOLUTION. $\square$

HOMOGENEOUS SYSTEM:

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

$$\left.\begin{aligned} x - z &= 0 \\ y - z &= 0 \end{aligned}\right\} \quad \begin{aligned} x &= z \\ y &= z \end{aligned}$$

THIS HAS SOLUTION $\left\{ \begin{pmatrix} z \\ z \\ z \end{pmatrix} : z \in \mathbb{R} \right\}$.

EACH VARIABLE CORRESPONDING TO A PIVOT CAN BE EXPRESSED AS A LINEAR COMBINATION OF THE NON-PIVOT VARIABLES, WHICH ARE FREE PARAMETERS.

<u>DEFINITION</u>: VECTORS $\underline{u}_1, \ldots, \underline{u}_n$ IN $\mathbb{R}^m$ ARE <u>LINEARLY INDEPENDENT</u> IF THE ONLY SOLUTION OF

$$x_1 \underline{u}_1 + \ldots + x_n \underline{u}_n = \underline{0}$$

IS $x_1 = x_2 = \ldots = x_n = 0$.

EQUIVALENTLY, NO $\underline{u}_i$ MAY BE EXPRESSED AS A LINEAR COMBINATION OF THE REMAINING VECTORS.

THEOREM: THE TWO DEFINITIONS ARE EQUIVALENT.

PROOF: SUPPOSE W.L.O.G.

$\underline{u}_n = c_1 \underline{u}_1 + c_2 \underline{u}_2 + \dots + c_{n-1} \underline{u}_{n-1}$.

THEN $c_1 \underline{u}_1 + \dots + c_{n-1} \underline{u}_{n-1} - \underline{u}_n = \underline{0}$

IS A LINEAR RELATION.

CONVERSELY, IF

$x_1 \underline{u}_1 + \dots + x_n \underline{u}_n = \underline{0}$ SOME $x_i \neq 0$

THEN $-x_i \underline{u}_i = \sum_{j \neq i} x_j \underline{u}_j$

$$\Leftrightarrow \quad \underline{u}_i = \frac{-1}{x_i} \sum_{j \neq i} x_j \underline{u}_j$$

$$= \sum_{j \neq i} \left(\frac{-x_j}{x_i} \right) \underline{u}_j. \quad \square$$

<u>DEFINITION</u>: A k-PLANE IN $\mathbb{R}^n$ HAS THE FORM

$$c_1\underline{u}_1 + \dots + c_k\underline{u}_k + \underline{v}$$

WHERE $\underline{u}_1, \dots, \underline{u}_k \in \mathbb{R}^n$ ARE LINEARLY INDEPENDENT, $\underline{v} \in \mathbb{R}^n$ AND $c_1, \dots, c_k \in \mathbb{R}$.

<u>EX</u>: $l(t) = t\underline{u} + \underline{v}$ (LINE).

$$P(s,t) = s\underline{u}_1 + t\underline{u}_2 + \underline{v}$$

(PLANE).

PARAMETRIC EQUATIONS.

THEOREM: IF A MATRIX B IS OBTAINED FROM A MATRIX A BY ELEMENTARY ROW OPERATIONS, ANY LINEAR RELATION IN THE COLUMNS OF A IS PRESENT IN THE COLUMNS OF B.

PROOF: LET $A = \begin{pmatrix} | & | & & | \\ \underline{\alpha}_1 & \underline{\alpha}_2 & \cdots & \underline{\alpha}_n \\ | & | & & | \end{pmatrix}$ $\alpha_i \in \mathbb{R}^m$. SUPPOSE $c_1\underline{\alpha}_1 + \dots + c_n\underline{\alpha}_n = \underline{0}$, $c_1, \dots, c_n \in \mathbb{R}$.

THIS CAN BE WRITTEN AS A MATRIX PRODUCT

$$\begin{pmatrix} | & & | \\ \underline{\alpha}_1 & \cdots & \underline{\alpha}_n \\ | & & | \end{pmatrix} \cdot \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

ELEMENTARY $m \times m$ MATRICES

(1) $\begin{pmatrix} 1 & & & 0 \\ & \ddots & & \\ & & r & \\ 0 & & & \ddots & 1 \end{pmatrix}$ $M_{i,r}$ HAS 1's ON DIAGONAL EXCEPT iTH ENTRY r, 0's OFF DIAGONAL.

(2) $\begin{array}{c c} & j \\ i \rightarrow & \begin{pmatrix} 1 & & 0 \\ & 1 & r \\ 0 & & \ddots & 1 \end{pmatrix} \end{array}$ E^r_{ij} HAS 1's ON DIAGONAL, r AT i,j ENTRY 0's ELSEWHERE

(3) $\begin{array}{c c} & i \quad j \\ \begin{matrix} i \rightarrow \\ j \rightarrow \end{matrix} & \begin{pmatrix} 1 & & & 0 \\ & 0 & 1 & \\ 0 & 1 & 0 & \\ & & & 1 \end{pmatrix} \end{array}$ S_{ij} HAS 1's ON DIAGONAL EXCEPT i,j HAS 1 AT (i,j) AND (j,i).

$M_{i,r} \cdot A$ HAS rTH ROW OF A MULTIPLIED BY r TIMES, OTHERS THE SAME

$E_{i,j}^r \cdot A$ ADDS ROW j TO ROW i

S_{ij} SWAPS ROW i AND ROW j.

PERFORMING ELEMENTARY ROW OPERATIONS TO A MAY BE WRITTEN AS A SEQUENCE OF LEFT MULTIPLICATIONS BY ELEM MATRICES, SO IF

$$B = E_1 E_2 E_3 \dots E_k \cdot A$$

WHERE E_i IS AN ELEM MATRIX

THEN $B = \begin{pmatrix} | & & | \\ \underline{\beta}_1 & \cdots & \underline{\beta}_n \\ | & & | \end{pmatrix}$

THEN $B \cdot \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = c_1\underline{\beta}_1 + \dots + c_n\underline{\beta}_n$

$$= E_1 \cdots E_k \boxed{\left(A \cdot \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}\right)}^{\,\text{“}0\text{''}}$$

$$= 0. \qquad \square$$