MAT 307 LECTURE 3

- THE CROSS PRODUCT
- LINEAR SYSTEMS.

THE CROSS PRODUCT OF TWO VECTORS $\underline{u}, \underline{v} \in \mathbb{R}^3$ IS

$$\underline{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}, \quad \underline{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}.$$

$$\underline{u} \times \underline{v} = \begin{pmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{pmatrix}$$

$$= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \underline{i} \cdot \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - \underline{j} \cdot \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + \underline{k} \cdot \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}$$

$$= \underline{i}\,(u_2 v_3 - u_3 v_2) - \underline{j} \cdot (u_1 v_3 - u_3 v_1) + \underline{k}\,(u_1 v_2 - u_2 v_1).$$

EXAMPLE:

$$\underline{u} = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}, \quad \underline{v} = \begin{pmatrix} 2 \\ 4 \\ -5 \end{pmatrix}$$

$$\underline{u} \times \underline{v} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & -3 & 2 \\ 2 & 4 & -5 \end{vmatrix} = \underline{i} \cdot \begin{vmatrix} -3 & 2 \\ 4 & -5 \end{vmatrix} - \underline{j} \cdot \begin{vmatrix} 1 & 2 \\ 2 & -5 \end{vmatrix} + \underline{k} \cdot \begin{vmatrix} 1 & -3 \\ 2 & 4 \end{vmatrix}$$

$$= \underline{i} \cdot (15 - 8) - \underline{j} \cdot (-5 - 4) + \underline{k}(4 + 6)$$

$$= 7\underline{i} + 9\underline{j} + 10\underline{k}.$$

THEOREM: $\underline{u} \times \underline{v}$ IS ORTHOGONAL TO BOTH $\underline{u}$ AND $\underline{v}$.

PROOF: WE CAN CALCULATE

$$\underline{u} \cdot (\underline{u} \times \underline{v}) = (u_1 \underline{i} + u_2 \underline{j} + u_3 \underline{k}) \cdot \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}.$$

$$= (u_1 \underline{i} + u_2 \underline{j} + u_3 \underline{k}) \cdot \left(\underline{i} \cdot \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - \underline{j} \cdot \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + \underline{k} \cdot \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \right).$$

MATCH $\underline{i}, \underline{j}, \underline{k}$ COORDS.

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

$$= u_1 \cdot \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - u_2 \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + u_3 \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}$$

$$= \begin{vmatrix} u_1 & u_2 & u_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = 0$$

(TWO ROWS ARE EQUAL).

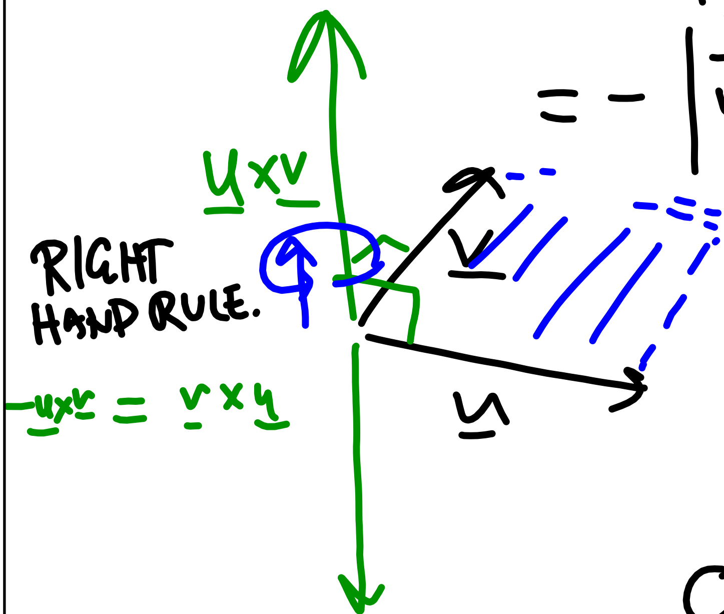
SIMILARLY $\underline{v} \cdot (\underline{u} \times \underline{v})$

$$= \begin{vmatrix} v_1 & v_2 & v_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = 0. \qquad \square$$

THEOREM: $\underline{u} \times \underline{v} = -\underline{v} \times \underline{u}$

(× PRODUCT IS ANTI-COMMUTATIVE).

PROOF:
$$\underline{v} \times \underline{u} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ v_1 & v_2 & v_3 \\ u_1 & u_2 & u_3 \end{vmatrix} = - \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = -\underline{u} \times \underline{v}. \qquad \square$$



START FROM $\underline{u}$, ROTATE CIRCLOCKWISE TO $\underline{v}$, THUMB POINTS UP AT $\underline{u} \times \underline{v}$.

EXAMPLE: GIVEN TWO PLANES

$2x - y + z = 10, \quad -3x + 2y - z = -7.$

THEIR LINE OF INTERSECTION CAN BE FOUND AS FOLLOWS.

$$\underline{n}_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \quad \underline{n}_2 = \begin{pmatrix} -3 \\ 2 \\ -1 \end{pmatrix}$$

FIRST PLANE $\perp \underline{n}_1$

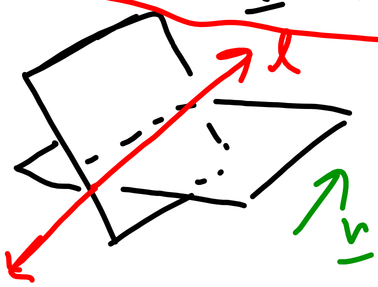
SECOND PLANE $\perp \underline{n}_2$.

THE VECTOR $\underline{v} = \underline{n}_1 \times \underline{n}_2$ IS PERPENDICULAR TO BOTH $\underline{n}_1$ AND $\underline{n}_2$, AND HENCE IS PARALLEL TO BOTH PLANES.

$$\underline{v} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & -1 & 1 \\ -3 & 2 & -1 \end{vmatrix} = \underline{i} \cdot \begin{vmatrix} -1 & 1 \\ 2 & -1 \end{vmatrix} - \underline{j} \cdot \begin{vmatrix} 2 & 1 \\ -3 & -1 \end{vmatrix} + \underline{k} \cdot \begin{vmatrix} 2 & -1 \\ -3 & 2 \end{vmatrix}$$

$$= -\underline{i} - \underline{j}(-2+3) + \underline{k}(4-3)$$

$$= -\underline{i} - \underline{j} + \underline{k} = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}.$$



$\ell = \underline{x} + t\underline{v}$

WHERE $\underline{x}$ IS ANY POINT ON THE LINE.

WE MUST FIND $\underline{x} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}$ SUCH THAT

$$2x_0 - y_0 + z_0 = 10$$
$$-3x_0 + 2y_0 - z_0 = -7.$$

LET $x_0 = 0$: $z_0 - y_0 = 10$
$2y_0 - z_0 = -7$

$y_0 = 3$, $z_0 = 13$ PARTICULAR SOLUTION.

$$\boxed{\ell(t) = \begin{pmatrix} 0 \\ 3 \\ 13 \end{pmatrix} + t \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}}$$

THE CROSS PRODUCT SATISFIES:

(1) ANTI-COMMUTATIVITY: $\underline{u} \times \underline{v} = -\underline{v} \times \underline{u}$

(2) ADDITIVITY: $(\underline{u} + \underline{w}) \times \underline{v} = \underline{u} \times \underline{v} + \underline{w} \times \underline{v}$

(3) HOMOGENEITY: FOR $r \in \mathbb{R}$,
$(r \cdot \underline{u}) \times \underline{v} = r \cdot (\underline{u} \times \underline{v})$.

NOTE: (2),(3) TOGETHER SAY THAT $\underline{u} \times \underline{v}$ IS LINEAR AS A FUNCTION OF $\underline{u}$. BY ANTI-COMMUTATIVITY $\underline{u} \times \underline{v}$ IS ALSO LINEAR IN $\underline{v}$.

PROOF OF (2) AND (3):

$$(a\underline{u}+b\underline{w})\times\underline{v} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ (au_1+bw_1) & (au_2+bw_2) & (au_3+bw_3) \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$\boxed{a\underline{u}+b\underline{w} = \begin{pmatrix} au_1+bw_1 \\ au_2+bw_2 \\ au_3+bw_3 \end{pmatrix}}$$

BY PROPERTIES OF DET

$$= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ au_1 & au_2 & au_3 \\ v_1 & v_2 & v_3 \end{vmatrix} + \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ bw_1 & bw_2 & bw_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

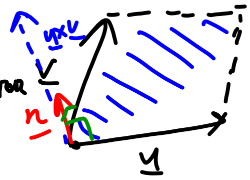
$$= a\cdot\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} + b\cdot\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ w_1 & w_2 & w_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$= a\cdot \underline{u}\times\underline{v} + b\cdot\underline{w}\times\underline{v}.$$

NOTE: WE USED THE DET. OF A 3x3 MATRIX IS LINEAR AS A FN OF EACH ROW SEPARATELY, SWAPPING ROWS FLIPS THE SIGN.

THEOREM: $|\underline{u} \times \underline{v}|$ = AREA OF THE PARALLELOGRAM DESCRIBED BY $\underline{u}, \underline{v}$.

PROOF: $\underline{n}$ = UNIT VECTOR $\perp$



THE LENGTH OF $\underline{u} \times \underline{v}$ IS THE AREA ////.

LET $\underline{n}$ BE A UNIT VECTOR $\perp$ TO THE PLANE DETERMINED BY $\underline{u}, \underline{v}$

$$\text{THUS } |\underline{u} \times \underline{v}| = |\underline{n} \cdot (\underline{u} \times \underline{v})|$$

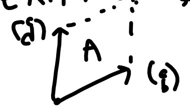
SINCE $\underline{u} \times \underline{v}$ IS IN THE DIRECTION OF $\underline{n}$.

$$\underline{n} \cdot (\underline{u} \times \underline{v}) = (n_1 \underline{i} + n_2 \underline{j} + n_3 \underline{k}) (\underline{u} \times \underline{v})$$

$$= \begin{vmatrix} n_1 & n_2 & n_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}.$$

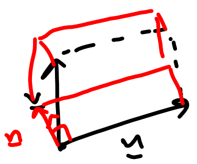
FACT TO BE PROVED LATER IN THE CLASS: $\left| \det \begin{pmatrix} - \underline{v}_1 - \\ - \underline{v}_2 - \\ \vdots \\ - \underline{v}_n - \end{pmatrix} \right|$ n VECTORS IN $\mathbb{R}^n$

IS THE VOLUME OF THE PARALLELEPIPED DETERMINED BY $\underline{v}_1, \dots, \underline{v}_n$.



$A = |ad - bc|$.

ASSUMING THIS, $\left| \begin{vmatrix} n_1 & n_2 & n_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \right|$ = VOLUME OF PARALLELEPIPED DESCRIBED BY $\underline{n}, \underline{u}, \underline{v}$.



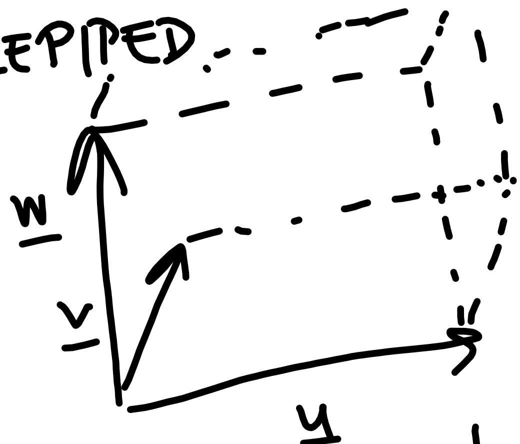
AREA BASE = AREA OF PARALLELOGRAM DESCRIBED BY $\underline{u}, \underline{v}$. SINCE $\underline{n}$ IS $\perp$, OF HEIGHT 1, THIS IS ALSO THE VOLUME. $\square$

DEFINE: THE SCALAR TRIPLE PRODUCT OF $\underline{u}, \underline{v}$ AND $\underline{w}$ IS

$$\underline{u} \cdot (\underline{v} \times \underline{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

WE SAY THE SIGN IS POSITIVE OR NEGATIVE ACCORDING AS $\underline{u}, \underline{v}, \underline{w}$ ARE ORIENTED POSITIVELY BY THE RIGHT HAND RULE.

THE MAGNITUDE IS THE VOLUME OF THE SPANNED PARALLELEPIPED.



|SCALAR TRIPLE PRODUCT| = VOLUME.

LINEAR SYSTEMS:

$$a_{11}x + a_{12}y + a_{13}z = b_1$$
$$a_{21}x + a_{22}y + a_{23}z = b_2$$
$$a_{31}x + a_{32}y + a_{33}z = b_3.$$

3x3 SYSTEM

WE SAY TWO SYSTEMS ARE EQUIVALENT IF THEY HAVE THE SAME SOLUTION SET.

DEFINITION: AN ELEMENTARY MULTIPLICATION MULTIPLIES BOTH SIDES OF AN EQUATION BY A SCALAR

$$2x+3y=5 \iff 4x+6y=10.$$

<u>DEFINITION</u>: AN ELEMENTARY MODIFICATION ADDS A MULTIPLE OF ONE EQUATION TO ANOTHER

$$\begin{array}{lll} E_1 & x+2y=5 & \\ E_2 & 3x+y=7 & \end{array} \iff \begin{array}{ll} x+2y=5 & E_1 \\ 5x+5y=17 & E_2+2E_1 \end{array}$$

THEOREM: BOTH ELEMENTARY OPERATIONS LEAVE THE SOL'N SET UNCHANGED.

PROOF: SUPPOSE $x_1, x_2, \ldots, x_n \in \mathbb{R}$ SATISFY A SYSTEM

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \qquad & \quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_n \end{aligned}$$

MULTIPLYING BOTH SIDES OF AN EQUALITY, OR ADDING EQUALITIES PRESERVES EQUALITY; AND YOU CAN PASS MULT / ADDITION TO COEFFICIENTS.

$$r(a_1x_1 + \cdots + a_nx_n) = (ra_1)x_1 + \cdots + (ra_n)x_n.$$
$$= rb$$

$$a_{11}x_1 + \cdots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + \cdots + a_{2n}x_n = b_2$$

$$(a_{11} + a_{21})x_1 + \cdots + (a_{1n} + a_{2n})x_n = b_1 + b_2. \checkmark$$

THIS SHOWS THAT SOLUTIONS ARE PRESERVED.

ANY ELEMENTARY OP CAN BE UNDONE BY A DIFFERENT ELEMENTARY OPERATION. THIS MEANS THAT NO NEW SOLUTIONS ARE CREATED. $\square$

WE SAY THAT A SYSTEM IS HOMOGENEOUS IF THE VECTOR

$$\begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} = \underline{0}, \quad \text{i.e.} \quad \begin{aligned} a_{11}x_1 + \dots + a_{1n}x_n &= 0 \\ a_{21}x_1 + \dots + a_{2n}x_n &= 0 \\ &\vdots \\ a_{m1}x_1 + \dots + a_{mn}x_n &= 0. \end{aligned}$$

YOU CAN WRITE A LINEAR SYSTEM AS A MATRIX EQUATION

$$A\underline{x} = \underline{b} \qquad \underline{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad \underline{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$$

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$$

ROWS OF MATRIX: ROW VECTORS

COLUMNS OF MATRIX: COLUMN VECTORS.

E.G.

$$\boxed{\begin{aligned} 2x + 3y - 4z &= 1 \\ x - y + 2z &= -1 \end{aligned}}$$

$$\Leftrightarrow \begin{pmatrix} 2 & 3 & -4 \\ 1 & -1 & 2 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

REDUCTION PROCESS:

THE FIRST NON-ZERO ENTRY IN A ROW IS CALLED A LEADING ENTRY

$$\begin{array}{cl} \begin{pmatrix} 1 & 3 & 4 \\ 0 & 2 & 2 \\ 1 & 0 & 7 \end{pmatrix} & \begin{array}{l} \leftarrow \text{Row } j. \\ \\ \\ \end{array} \end{array}$$

- PICK A COLUMN WITH A LEADING ENTRY SAY r IN ROW j.
- DIVIDE ROW j BY r SO LEADING ENTRY IS 1
- SUBTRACT MULTS OF ROW FROM OTHER ROWS TO CLEAR COLUMN

EXAMPLE:

$$\begin{pmatrix} 1 & 3 & 4 \\ 0 & 2 & 2 \\ 1 & 0 & 7 \end{pmatrix}$$

$$\xrightarrow{r_3 - r_1} \begin{pmatrix} 1 & 3 & 4 \\ 0 & 2 & 2 \\ 0 & -3 & 3 \end{pmatrix} \xrightarrow{\frac{r_2}{2}} \begin{pmatrix} 1 & 3 & 4 \\ 0 & 1 & 1 \\ 0 & -3 & 3 \end{pmatrix}$$

$$\xrightarrow[r_3+3r_2]{r_1-3r_2} \begin{pmatrix} 1 & 0 & 1 \\ & 1 & 1 \\ & 0 & 6 \end{pmatrix} \xrightarrow{r_3/6} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow[r_2-r_3]{r_1-r_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$