MAT 307 LECTURE 23:

STOKES' THEOREM

STOKES' THEOREM: LET S BE A PIECE OF A SMOOTH SURFACE IN $\mathbb{R}^3$, PARAMETRIZED BY A TWICE CONTINUOUSLY DIFFERENTIABLE MAP $g : D \to \mathbb{R}^3$ WHERE D IS THE FINITE DISJOINT UNION OF SIMPLE REGIONS.

LET D BE BOUNDED BY A PIECEWISE SMOOTH CURVE.

IF $\mathbb{F}$ IS CONTINUOUSLY DIFF. VECTOR FIELD IN $\mathbb{R}^3$ THEN

$$\int_S \underbrace{\operatorname{curl} F \cdot dS}_{\operatorname{curl} F \cdot N \, d\sigma} = \int_{\partial S} \underbrace{\mathbb{F} \cdot dx}_{\mathbb{F} \cdot T \cdot ds}.$$

UNIT NORMAL CORRESPONDING TO THE ORIENTATION

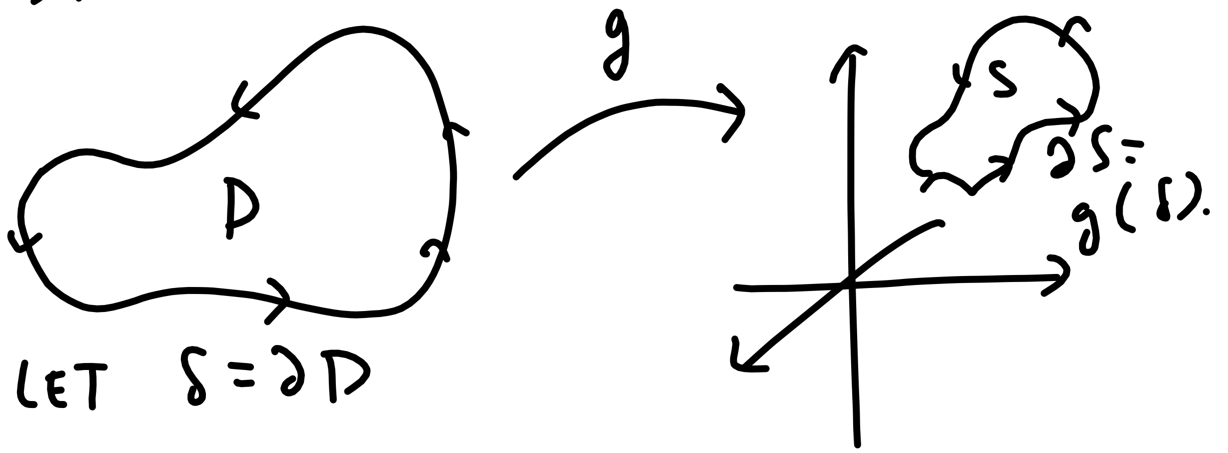
SURFACE AREA ELEMENT

UNIT TANGENT VECTOR

ARC LENGTH

DETERMINED BY THE RIGHT-HAND RULE.

SINGLE SIMPLE REGION CASE



LET $\delta = \partial D$

LET $h(t) = (u(t), v(t))$ BE

A SMOOTH, COUNTERCLOCKWISE ORIENTED PARAMETERIZATION OF δ,

AND, THUS, $g(h(t))$ IS

A POSITIVELY ORIENTED PARAMETERIZATION OF ∂S.

BY GREEN'S THEOREM IN THE PLANE, WE CAN REPLACE A TWO DIMENSIONAL LINE INTEGRAL WITH THE AREA INTEGRAL OF THE SCALAR FLUX.

IT FOLLOWS THAT THE INTEGRAND IS

$$\frac{\partial}{\partial u}\left(F_1(g)\frac{\partial g_1}{\partial v}\right) - \frac{\partial}{\partial v}\left(F_1(g)\cdot\frac{\partial g_1}{\partial u}\right)$$

$$= \frac{\partial F_1}{\partial y}\left(\frac{\partial g_2}{\partial u}\frac{\partial g_1}{\partial v} - \frac{\partial g_2}{\partial v}\frac{\partial g_1}{\partial u}\right) + \frac{\partial F_1}{\partial z}\left(\frac{\partial g_3}{\partial u}\frac{\partial g_1}{\partial v} - \frac{\partial g_3}{\partial v}\frac{\partial g_1}{\partial u}\right)$$

$$= -\frac{\partial F_1}{\partial y}\cdot\frac{\partial(g_1,g_2)}{\partial(u,v)} + \frac{\partial F_1}{\partial z}\frac{\partial(g_3,g_1)}{\partial(u,v)}$$

IT FOLLOWS THAT

$$\oint_{\partial S} F_1\,dx = \int_D -\frac{\partial F_1}{\partial y}\frac{\partial(g_1,g_2)}{\partial(u,v)} + \frac{\partial F_1}{\partial z}\frac{\partial(g_3,g_1)}{\partial(u,v)}\,du\,dv$$

$$= \int_S -\frac{\partial F_1}{\partial y}\,dx\,dy + \frac{\partial F_1}{\partial z}\,dz\,dx.$$

$\square$

THIS PROVES STOKES' THEOREM.

INTERPRETATION:

(i) THE DIRECTION OF curl F IS THE AXIS ABOUT WHICH F ROTATES THE MOST.

(ii) |curl F| DESCRIBES THIS AMOUNT OF ROTATION.

AS WE ALREADY REMARKED, curl F IS ALWAYS A DIVERGENCE FREE FIELD.

ACTUALLY, ANY FIELD G WITH div $G = 0$ CAN BE WRITTEN AS $G = \text{curl } F$ FOR SOME FIELD F.

WE WON'T PROVE THIS FOR NOW.

CLASSICAL NOTATION AND THE

∇ OPERATOR.

IN CLASSICAL NOTATION,

STOKES' FORMULA READS

$$\int_S (\nabla \times \vec{F}) \cdot \underline{N} \, d\sigma = \int_{\partial S} \vec{F} \cdot \vec{T} \, ds.$$

GAUSS'S FORMULA READS:

$$\int_R \nabla \cdot F \, dV = \int_{\partial R} F \cdot \underline{N} \, d\sigma.$$

PROOF:

(2) $$\nabla(fg) = \begin{pmatrix} \frac{\partial}{\partial x_1}(fg) \\ \vdots \\ \frac{\partial}{\partial x_n}(fg) \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial x_1} g + f \frac{\partial g}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_n} g + f \frac{\partial g}{\partial x_n} \end{pmatrix}$$

$$= f \cdot \nabla g + \nabla f \cdot g.$$

(4) $$\nabla \times (f \cdot \mathbb{F}) = \begin{pmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ fF_1 & fF_2 & fF_3 \end{pmatrix}$$

$$= \underline{i} \left(\frac{\partial(fF_3)}{\partial y} - \frac{\partial(fF_2)}{\partial z} \right)$$

$$- \underline{j} \left(\frac{\partial(fF_3)}{\partial x} - \frac{\partial(fF_1)}{\partial z} \right)$$

$$+ \underline{k} \left(\frac{\partial(fF_2)}{\partial x} - \frac{\partial(fF_1)}{\partial y} \right)$$

THE DERIVATIVES ON THE FIELD ARE AS IF f WAS CONSTANT,

$$= f \cdot \nabla \times F.$$

THE TERMS COMING FROM DERIVATIVES ON f ARE THE SAME AS

$$\det \begin{pmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\ F_1 & F_2 & F_3 \end{pmatrix} = \nabla f \times \mathbb{F}.$$

THE PROOF OF (6) IS SIMILAR.

(7) IS AN EXERCISE.

GREEN'S IDENTITIES:

IF $F = \nabla f$ IS A CONSERVATIVE FIELD, GAUSS'S THEOREM GIVES

$$\int_R \nabla \cdot \nabla f = \int_R \Delta f = \int_{\partial R} \nabla f \cdot N \, d\sigma.$$

$$= \int_{\partial R} \frac{\partial f}{\partial N} \, d\sigma.$$

DIRECTIONAL DERIV.