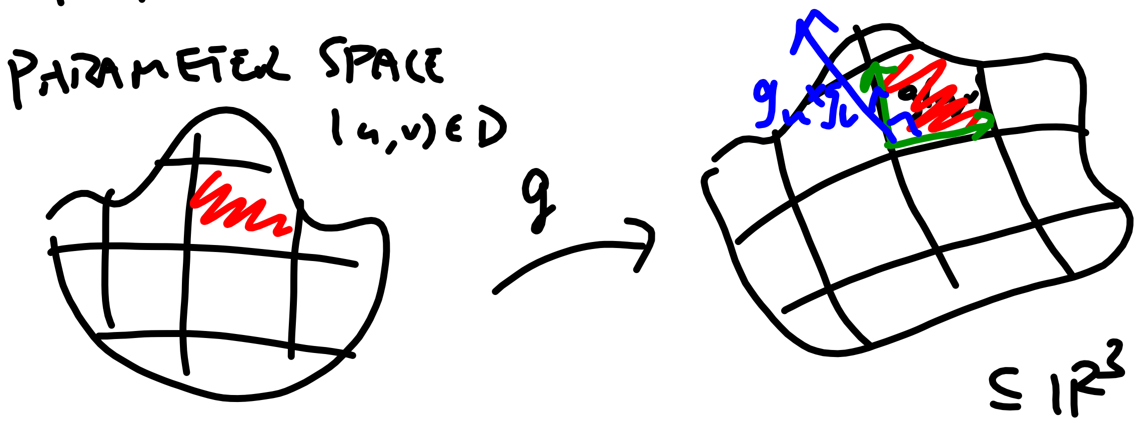
SURFACE INTEGRALS:

WE ASSUME GIVEN A

PARAMETRIC SMOOTH SURFACE.



DEFECT IN AREA, AREA ELEMENT

$|g_u \times g_v|$ APPROXIMATE

RATIO OF AREA ON THE SURFACE TO THE AREA ON THE PARAMETER GRID.

TO PERFORM THE SURFACE INTEGRAL OF A FUNCTION $\mu(x)$ DEFINED ON THE SURFACE

WE INTEGRATE

$$\int_S \mu(x)\, dS = \int_D \mu(g(u,v))\, |g_u \times g_v|\, du\, dv.$$

EXAMPLE: GIVEN A PARABOLIC DISH

$$\left\{ \begin{pmatrix} x \\ y \\ x^2+y^2 \end{pmatrix} : 0 \le x^2+y^2 \le 1 \right\}.$$

THE SURFACE AREA IS

$$\iint_{x^2+y^2\le 1} \sqrt{1+(2x)^2+(2y)^2}\, dx\, dy.$$

$$= \int_0^{2\pi}\int_0^1 \sqrt{1+4r^2}\; r\, dr\, d\theta.$$

$u = 1+4r^2$

$du = 8r\, dr$

$$= \frac{2\pi}{8}\int_1^5 \sqrt{u}\, du$$

$$= \frac{2\pi}{8} \cdot \left.\frac{u^{3/2}}{3/2}\right]_1^5$$

$$= \frac{\pi}{6}\left(5^{3/2}-1\right). \quad \square$$

INTEGRATING VECTOR FIELDS OVER A SURFACE IN $\mathbb{R}^3$:

$$\mathbb{R}^3 \xrightarrow{F} \mathbb{R}^3$$

THE UNIT NORMAL TO THE PARAMETRIC SURFACE $g(u,v)$ IS

$$N = \frac{g_u \times g_v}{|g_u \times g_v|}.$$

IN PARAMETRIC FORM,
THE VECTOR FIELD AT THE POINT IS

$$\mathbb{F}(g(u,v)).$$

THE NORMAL VECTOR IS

$$\frac{g_u \times g_v}{|g_u \times g_v|}$$

THE AREA ELEMENT IS $d\sigma = |g_u \times g_v|\, du\, dv.$

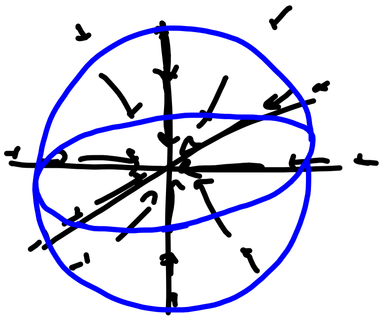
<u>FLUX:</u> $\displaystyle\iint_{(u,v)\in D} F(g(u,v)) \cdot (g_u \times g_v)\, du\, dv.$

INVERSE SQUARE FIELDS:

FOR GRAVITY:

$$F(\underline{x}) = -\frac{GMm}{|\underline{x}|^3} \cdot \underline{x}.$$

THIS IS THE FIELD FROM A POINT MASS AT O.



SPHERE OF RADIUS a. CALLED S_a.

THE FLUX ACROSS S_a:
THE FIELD POINTS IN THE DIRECTION OPPOSITE N,

SO $\mathbb{F} \cdot N = -\frac{GMm}{a^2}$.

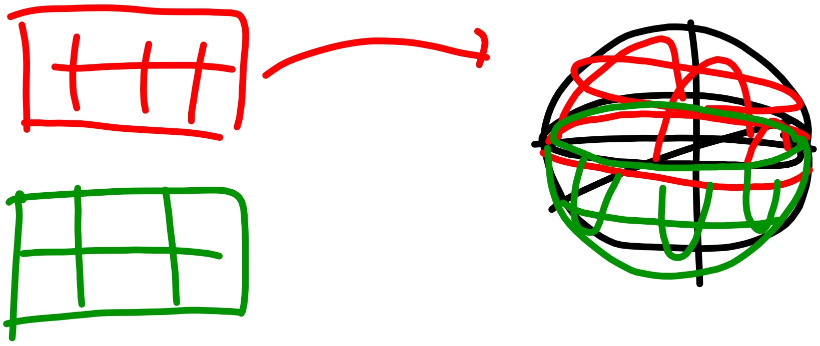
$$\text{FLUX} = -\frac{GMm}{a^2} \int_{S_a} d\sigma$$

$$\int_{S_a} d\sigma = a^2 \cdot \underbrace{4\pi}_{\text{SURFACE AREA OF A UNIT SPHERE.}}$$

FLUX = $-4\pi GMm$.

THE FLUX THROUGH ANY SPHERICAL SHELL IS THE SAME.

GIVEN SOME OBJECT, LIKE THE EARTH, WE COVER IT WITH A COLLECTION OF OVERLAPPING PARAMETERIZING CHARTS, CALLED AN ATLAS.



ASSUME MY OBJECT IS COVERED BY A COLLECTION OF PARAMETRIC SURFACES, AND THAT WHEN THEY OVERLAP THEY HAVE THE SAME ORIENTATION. IN THIS CASE, THE SURFACE IS ORIENTABLE.

(THE OBJECT IS CALLED A MANIFOLD).

<u>RECALL</u>: A <u>SIMPLE</u> REGION IN $\mathbb{R}^3$ IS A REGION BOUND BETWEEN THE GRAPHS OF TWO FUNCTIONS FROM THE xy-PLANE, xz-PLANE, yz-PLANE.

RECALL $\mathbb{F} = \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix}$

$$\operatorname{div} \mathbb{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}.$$

WE'LL SHOW THE PROOF FOR

$\mathbb{F} = \begin{pmatrix} F_1 \\ 0 \\ 0 \end{pmatrix}$, THE REMAINING ARGUMENT HOLDING

BY LINEARITY.

THE DIVERGENCE INTEGRAL IS

$$\int_{(y,z)\in D} \int_{g(y,z)}^{h(y,z)} \frac{\partial F_1}{\partial x}(x,y,z)\, dx\, dy\, dz.$$

BY THE FUNDAMENTAL THEOREM OF CALCULUS, THIS INTEGRAL IS

$$\iint_{(y,z)\in D} \boxed{F_1(h(y,z),y,z) - F_1(g(y,z),y,z)}\, dy\, dz.$$

SINCE THE VECTOR FIELD POINTS ONLY IN THE x DIRECTION,

AND SINCE THE SURFACES ARE GRAPHS, $\begin{pmatrix} h(y,z) \\ y \\ z \end{pmatrix}, \begin{pmatrix} g(y,z) \\ y \\ z \end{pmatrix}$

THEIR STANDARD NORMAL VECTORS ARE $\begin{pmatrix} 1 \\ -h_y \\ -h_z \end{pmatrix}, \begin{pmatrix} 1 \\ -g_y \\ -g_z \end{pmatrix}$.

SO THE INTEGRAL GIVES THE FLUX ACROSS THE SURFACE;

[TO PRESERVE OUTWARD POINTING NORMAL, TAKE THE POSITIVE STANDARD NORMAL VECTOR FOR THE FIRST SURFACE, AND NEGATIVE FOR SECOND.]

SURFACE INDEPENDENCE PRINCIPLE

IF A VECTOR FIELD F SATISFIES $\operatorname{div} F = 0$ (INCOMPRESSIBLE) AND S_1, S_2 ARE TWO SURFACES WHICH MATCH ON THE BOUNDARY THEN THE FLUX ACROSS S_1 IS EQUAL TO THE FLUX ACROSS S_2 IF TAKEN IN THE SAME DIRECTION.

PROOF: $S_1 \cup -S_2$ BOUNDS A REGION OVER WHICH THE DIVERGENCE IS 0. APPLY GAUSS'S THEOREM.

OPP. ORIENTATION

THE LAPLACIAN

$$\Delta f = \operatorname{div} \nabla f = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) f.$$

IF $\Delta f = 0$ THEN WE SAY f IS "HARMONIC."

IN THIS CASE, THE GRADIENT FIELD ∇f IS DIVERGENCE FREE, SO ITS FLUX IS SURFACE INDEPENDENT.