MAT 307 LECTURE 21

PATH INDEPENPENT INTEGRALS, SURFACE INTEGRALS, FLUX

FLUX VERSION OF GREEN'S

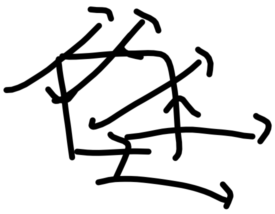
THEOREM:

$$\iint_R \operatorname{div} F \, dA = \int_{\partial R} F \cdot N \, ds$$

N: OUTWARD POINTING UNIT NORMAL

ds: ARC LENGTH

THIS GIVES A WAY OF INTERPRETING THE DIVERGENCE AS INFINITESMAL CHANGE OF DENSITY AT A POINT



F = FLUID FLOW

WE'RE INTERESTED IN CHANGE OF PRESSURE

$$\int_{\partial R} F \cdot N \, ds$$ = CHANGE IN FLUID CONTENT IN THE REGION.

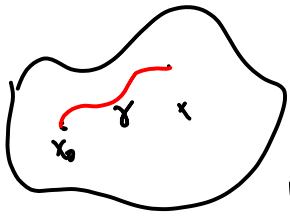
THIS IS EQUAL TO THE INTEGRAL OF THE DIVERGENCE OVER THE SQUARE.

THEOREM: LET $\mathbb{F}$ BE A CONTINUOUS VECTOR FIELD IN A POLYGONALLY DEFINED REGION R, SO CUT OUT BY STRAIGHT LINE SEGMENTS.

IF THE PATH INTEGRAL $\int_\gamma \mathbb{F} \cdot dx$ IS INDEPENDENT OF THE PATH BETWEEN THE ENDPOINTS, THEN $\mathbb{F} = \nabla f(x)$ WHERE $f(x) = \int_{\gamma : [x_0, x]} \mathbb{F}\, dx$

WHERE x_0 IS AN ARBITRARY BASE POINT AND γ IS ANY PATH FROM x_0 TO x.

PROOF OF THEOREM:



DEFINE $f(x) = \int_{x_0}^{x} \mathbb{F}\, dx$

OVER ANY PATH.

$$f(\underline{x} + t \cdot \underline{e}_j) = \int_{\underline{x}_0}^{x} \mathbb{F}\, dx + \int_{\underline{x}}^{\underline{x}+t\underline{e}_j} \mathbb{F}\, d\underline{x}.$$

WHERE THE LAST INTEGRAL IS OVER A STRAIGHT LINE PATH FROM $\underline{x}$ TO $\underline{x} + t\underline{e}_j$.

THUS $f(\underline{x} + t\underline{e}_j) - f(\underline{x})$

$$= \int_{\underline{x}}^{\underline{x}+t\underline{e}_j} \mathbb{F} \cdot d\underline{x}.$$

$\mathbb{F} = \begin{pmatrix} \mathbb{F}_1 \\ \mathbb{F}_2 \\ \vdots \\ \mathbb{F}_n \end{pmatrix}$, ON THE LINE INTEGRAL THIS

IS $\int_0^t \mathbb{F}_j(\underline{x} + u\underline{e}_j)\, du$

WE HAVE $\lim_{t \downarrow 0} \frac{1}{t} \int_0^t \mathbb{F}_j(\underline{x} + u\underline{e}_j)\, du$

IS $\mathbb{F}_j(\underline{x}) + \lim_{t \downarrow 0} \frac{1}{t} \int_0^t \mathbb{F}_j(\underline{x} + u\underline{e}_j) - \mathbb{F}_j(\underline{x})\, du.$

BY CONTINUITY, THE INTEGRAND TENDS TO 0 AS $t \downarrow 0$, SO THE INTEGRAL CAN BE MADE ARBITRARILY SMALL COMPARED TO t.

THUS $\lim_{t \downarrow 0} \frac{1}{t} \int_0^t \mathbb{F}_j(\underline{x} + u\underline{e}_j)\, du = \mathbb{F}_j(\underline{x}).$

THUS $\frac{\partial}{\partial x_j} f(\underline{x}) = \mathbb{F}_j(\underline{x})$ AS DESIRED. $\square$

GIVEN A CONSERVATIVE FORCE FIELD $\mathbb{F}$ IN $D \subset \mathbb{R}^3$, THE WORK DONE IN MOVING BETWEEN TWO POINTS IS $W(\underline{x}_1, \underline{x}_2) = \int_{\underline{x}_1}^{\underline{x}_2} \mathbb{F} \cdot d\underline{x}$ WHICH IS PATH INDEPENDENT.

WE CALL THIS THE CHANGE IN POTENTIAL ENERGY.

THEOREM: INVERSE SQUARE FIELDS, SUCH AS GRAVITY, OR THE ELECTRIC FIELD OF A CHARGED PARTICLE, ARE CONSERVATIVE.

IN GENERAL, IF $\mathbb{F}$ IS CONSERVATIVE

$$U(x) = -\int_{x_0}^{x} \mathbb{F} \cdot d\mathbf{x}$$

IS THE <u>POTENTIAL ENERGY</u>.

<u>THEOREM:</u> LET $\mathbb{F}$ BE A CONTINUOUS VECTOR FIELD DEFINED IN A POLYGONALLY DETERMINED REGION. THE FOLLOWING ARE EQUIVALENT.

(a) PATH INDEPENDENCE

(b) THE INTEGRAL AROUND A SMOOTH LOOP IS 0.

(c) THERE IS A CONTINUOUSLY DIFF. POTENTIAL FUNCTION f SO THAT $\mathbb{F} = \nabla f$.

WE ALREADY PROVED (a) $\Leftrightarrow$ (b) AND (a) $\Leftarrow$ (c).

PROOF: $\mathbb{F} = \nabla f = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{pmatrix}$

$$\frac{\partial}{\partial x_i} \mathbb{F}_j = \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} f$$

$$= \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_i} f = \frac{\partial}{\partial x_j} \mathbb{F}_i .$$

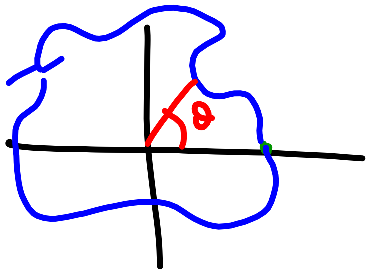
SINCE MIXED PARTIALS COMMUTE. $\square$

THIS GIVES AN EASY CRITERIA TO CHECK IF $\mathbb{F} : \mathbb{R}^3 \to \mathbb{R}^3$ OR $\mathbb{R}^2 \to \mathbb{R}^2$, NAMELY IT IS NECESSARY $\nabla \times \mathbb{F} = \underline{0}$.

PROOF: TO OBTAIN SUCH A VECTOR FIELD TAKE

$\theta = \arg(x,y)$ FOR $x,y \neq 0$.

$= \arctan \frac{y}{x} + n\pi$.



$\theta_0 = 0$

$\theta_1 = 2\pi$

θ VARIES CONTINUOUSLY ALONG CLOSED PATHS, DIFFERENTIABLE

SO BY MIXED PARTIALS COMMUTE

θ IS NOT A SINGLE VALUED FUNCTION

YOU CAN "DEFINE"

$\nabla\theta = \nabla \arctan\left(\frac{y}{x}\right)$

$\text{curl } \nabla\theta = 0$.

$$\mathbb{F}(x,y) = \begin{pmatrix} \frac{-y}{x^2+y^2} \\ \frac{x}{x^2+y^2} \end{pmatrix}$$

$$\text{curl } \mathbb{F} = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$$ SCALAR CURL

$$= \frac{1}{x^2+y^2} - \frac{2x^2}{(x^2+y^2)^2} - \left[\frac{-1}{x^2+y^2} + \frac{2y^2}{(x^2+y^2)^2}\right]$$

$= 0$.

$$\int_{x^2+y^2=1} \mathbb{F}\cdot dx = \int_{x^2+y^2=1} -y\,dx + x\,dy = 2 \text{ TIMES AREA} = 2\pi.$$

PROOF: FIX TWO COORDINATES i, j, $1 \le i < j \le n$.

CONSIDER A SMALL RECTANGLE R, OF SIDE LENGTH δ, WITH CORNERS $\underline{x}, \underline{x} + \delta \underline{e}_i, \underline{x} + \delta \underline{e}_i + \delta \underline{e}_j, \underline{x} + \delta \underline{e}_j$ TRACED IN THIS ORDER.

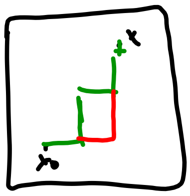
HOLDING THE OTHER COORDINATES FIXED, WE CAN TREAT $\mathbb{F}$ AS A FN. OF x_i, x_j ONLY.

BY GREEN'S THEOREM

$$\int_{\partial R} \mathbb{F} \cdot dx = \iint_R \frac{\partial \mathbb{F}_j}{\partial x_i} - \frac{\partial \mathbb{F}_i}{\partial x_j} \, dA$$

$$= 0.$$

THIS MEANS THAT IF WE DEFINE $f(x)$ BY $f(x) = \int_{x_0}^{x} \mathbb{F}\, dx$ FOLLOWING ANY PATH CONSISTING OF ONLY EDGES IN THE DIRECTION OF ONE OF THE AXES, THEN THE INTEGRAL IS INDEPENDENT OF THE PARTICULAR PATH.



THIS TYPE OF PATH IS ALL THAT IS NEEDED TO CALCULATE THE GRADIENT. $\Box$

SURFACE INTEGRALS:

WE'RE INTERESTED IN PARAMETRIC SURFACES

$$g(u,v) = \begin{pmatrix} g_1(u,v) \\ g_2(u,v) \\ g_3(u,v) \end{pmatrix}$$

TWO FREE PARAMETERS

$\rightarrow$ SURFACE IS TWO DIM'L.

THE VECTOR $g_u \times g_v$

IS CALLED THE

"STANDARD NORMAL VECTOR"

THE STANDARD UNIT NORMAL

IS $\dfrac{g_u \times g_v}{\|g_u \times g_v\|}$.