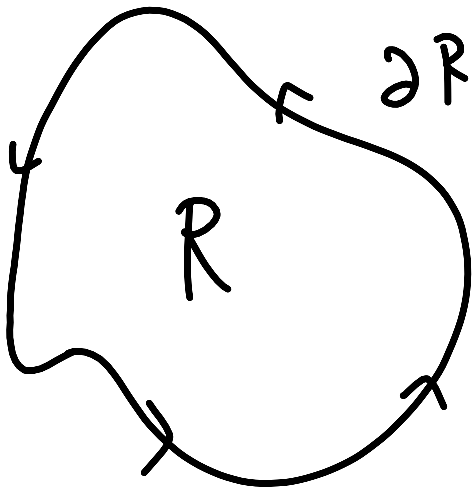


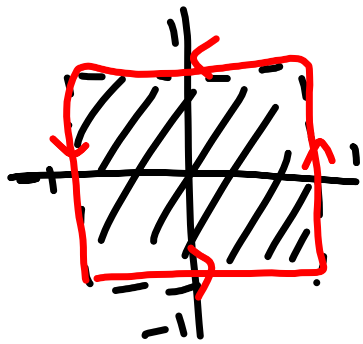
GREEN'S THEOREM RELATES INTEGRALS AROUND THE BOUNDARY ∂R TO INTEGRALS OVER THE PLANE REGION R.

IN THIS WAY IT GENERALIZES THE FUNDAMENTAL THEOREM OF CALCULUS

INTEGRAL OVER INTERVAL $\leftrightarrow$ EVALUATION AT ENDPOINTS.

EXAMPLE:

$$R = [-1,1]^2 =$$



LET $F(x,y) = ye^x$

$G(x,y) = -xe^y.$

THEN $\frac{\partial F}{\partial y} = e^x, \quad \frac{\partial G}{\partial x} = -e^y.$

THEN $\frac{\partial G}{\partial x} - \frac{\partial F}{\partial y} = -(e^x + e^y).$

$$\iint_R \frac{\partial G}{\partial x} - \frac{\partial F}{\partial y}\, dx\,dy = \int_{-1}^{1}\int_{-1}^{1} -(e^x + e^y)\, dx\,dy$$

$$= -2 \times \left\{ 2 \times \int_{-1}^{1} e^x\, dx \right\} = -4(e - e^{-1}).$$

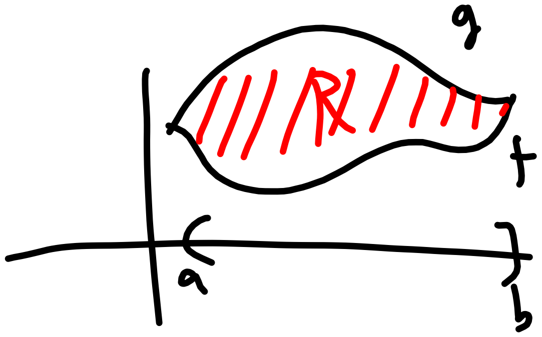
THEOREM: LET γ BE A SMOOTH CURVE AND F A CTS VECTOR FIELD DEFINED ON γ. DENOTE BY γ^{-1} THE CURVE TRACED IN THE OP. DIRECTION. THEN

$$\int_\gamma F \cdot dx = - \int_{\gamma^{-1}} F \cdot dx$$

[DOING A LINE INTEGRAL IN THE OPPOSITE DIRECTION OBTAINS THE NEGATIVE VALUE.]

DEFINITION: WE SAY A REGION $R \subset \mathbb{R}^2$ IS A (x) SIMPLE REGION IF THERE ARE CONTINUOUS FUNCTIONS f AND g, $f \le g$ WITH $f(a) = g(a)$, $f(b) = g(b)$

$$R = \left\{ (x,y) : \begin{array}{l} a \le x \le b \\ f(x) \le y \le g(x) \end{array} \right\}$$



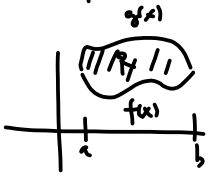
AND IF THERE ARE FUNCTIONS $h, k : [c,d] \to \mathbb{R}$

SO THAT

$$R = \{ (x,y) : c \le y \le d,\ h(y) \le x \le k(y) \}$$

PROOF: ASSUME THAT D IS IN FACT A SIMPLE REGION, GIVEN BY

$$\{(x,y) : a \le x \le b,\ f(x) \le y \le g(x)\}.$$



A PARAMETERIZATION OF THE LOWER BOUNDARY IS $\{(x, f(x)) : a \le x \le b\}$.

THE REVERSE PARAMETERIZATION OF THE UPPER BOUNDARY IS

$$\{(x, g(x)) : a \le x \le b\}.$$

SO THE LINE INTEGRAL IS GIVEN BY

$$\int_a^b \begin{pmatrix} F(x,f(x)) \\ G(x,f(x)) \end{pmatrix} \cdot \begin{pmatrix} 1 \\ f'(x) \end{pmatrix} dx - \int_a^b \begin{pmatrix} F(x,g(x)) \\ G(x,g(x)) \end{pmatrix} \cdot \begin{pmatrix} 1 \\ g'(x) \end{pmatrix} dx$$

$$= -\int_a^b F(x,g(x)) - F(x,f(x))\, dx - \int_a^b G(x,g(x))\,g'(x) - G(x,f(x))f'(x)\,dx.$$

THE ITERATED INTEGRAL IS

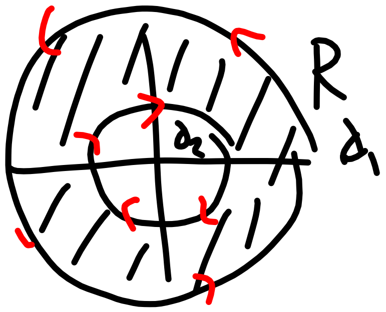
$$\int_a^b \int_{f(x)}^{g(x)} \frac{\partial G}{\partial x} - \frac{\partial F}{\partial y}\, dx\, dy.$$

$$-\int_a^b \int_{f(x)}^{g(x)} \frac{\partial F}{\partial y}\, dy\, dx = -\int_a^b F(x,g(x)) - F(x,f(x))\, dx$$

BY THE FUNDAMENTAL THEOREM OF CALCULUS.

THIS HANDLES THE F PART OF THE LINE INTEGRAL.

EXAMPLES: CONSIDER THE ANNULAR REGION



$$R = \{(x,y) : 1 \le x^2+y^2 \le 4\}.$$

DEFINE
$$F(x,y) = \frac{-y}{x^2+y^2}$$
$$G(x,y) = \frac{x}{x^2+y^2}.$$

$$-\frac{\partial F}{\partial y} = \frac{1}{x^2+y^2} - \frac{2y^2}{(x^2+y^2)^2}$$

$$\frac{\partial G}{\partial x} = \frac{1}{x^2+y^2} - \frac{2x^2}{(x^2+y^2)^2}.$$

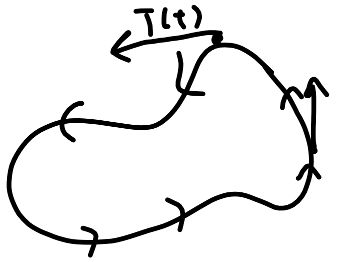
THEN
$$\frac{\partial G}{\partial x} - \frac{\partial F}{\partial y} = \frac{1}{x^2+y^2} - \frac{2y^2}{(x^2+y^2)^2} + \frac{1}{x^2+y^2} - \frac{2x^2}{(x^2+y^2)^2}$$
$$= 0.$$

THUS THE INTEGRAL OVER R IS ZERO. $\square$

A SECOND PERSPECTIVE:

LET $g(t)$ PARAMETERIZE THE BOUNDARY. THEN THE UNIT TANGENT VECTOR IS GIVEN BY

$$\left(\frac{g_1'(t)}{|g'(t)|}, \frac{g_2'(t)}{|g'(t)|} \right) = T(t).$$



THE OUTWARD POINTING UNIT NORMAL

IS $N(t) = \left(\frac{g_2'(t)}{|g'(t)|}, -\frac{g_1'(t)}{|g'(t)|} \right)$.

IF $\mathbb{F} = \begin{pmatrix} F \\ G \end{pmatrix}$ IS A CTSLY VARYING VECTOR FIELD,

$$ds = |g'(t)|\,dt$$

THEN $\int F\,dx = \int_\gamma \underline{F} \cdot T(s)\,ds.$

THE SCALAR CURL OF $\mathbb{F}$ IS

$G_x - F_y$.

THUS:

$$\int_D \text{curl}\,\mathbb{F}\,dA = \int_\gamma \mathbb{F} \cdot T(s)\,ds$$

EXAMPLE: THE FIELD

$$\mathbb{F}(x,y) = y \cdot \underline{i} + x \cdot \underline{j}$$

IS INCOMPRESSIBLE SINCE

$\operatorname{div} \mathbb{F} = 0$.

THUS $\int_{\gamma} \mathbb{F} \cdot \underline{N} \, ds = 0$

FOR ALL SMOOTH LOOPS.