MAT 307 LECTURE 20

GREEN'S THEOREM.

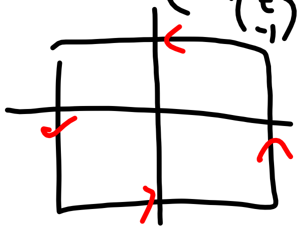
GREEN'S THEOREM:

LET D BE A PLANE REGION BOUNDED BY A SMOOTH CURVE γ, WHICH HAS A PARAMETERIZATION $g:[a,b] \to \mathbb{R}^2$ SUCH THAT, AS t INCREASES FROM a TO b, g TRACES γ ONCE IN THE COUNTERCLOCKWISE DIRECTION. THEN

$$\int_{\lambda} F\,dx + G\,dy = \int_{R} \left(\frac{\partial G}{\partial x} - \frac{\partial F}{\partial y} \right) dx\,dy.$$

PARAMETRIZE THE BOUNDARY IN 4 PARTS

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{cases} \begin{pmatrix} 1 \\ t \end{pmatrix} & -1 \le t \le 1 \\ \begin{pmatrix} -t \\ 1 \end{pmatrix} & -1 \le t \le 1 \\ \begin{pmatrix} -1 \\ -t \end{pmatrix} & -1 \le t \le 1 \\ \begin{pmatrix} t \\ -1 \end{pmatrix} & -1 \le t \le 1. \end{cases}$$



WE WISH TO INTEGRATE $F\,dx + G\,dy$.
THE dx INTEGRAL VANISHES EXCEPT ON TOP, BOTTOM

$$\int_{-1}^{1} F\begin{pmatrix} -t \\ 1 \end{pmatrix} \cdot (-1)\, dt = \int_{-1}^{1} e^{-t}(-1)\, dt$$
$$= -\left(e - \frac{1}{e}\right).$$

THE BOTTOM INTEGRAL IS

$$\int_{-1}^{1} F\begin{pmatrix} t \\ -1 \end{pmatrix} \cdot 1\, dt = \int_{-1}^{1} -e^{t}\, dt$$
$$= -\left(e - \frac{1}{e}\right).$$

THE LEFT AND RIGHT INTEGRALS AGAINST $G\,dy$ CONTRIBUTE THE SAME QUANTITY. THUS

$-4\left(e - \frac{1}{e}\right)$. THIS

IS CONSISTENT WITH GREEN'S THEOREM.

<u>PROOF:</u> LET $g : [0,1] \to \mathbb{R}^n$ BE A PARAMETRIZATION OF γ.

THEN $\tilde{g}(t) = g(1-t)$ PARAMETRIZES γ^{-1}.

[RUN TIME IN REVERSE.]

IT FOLLOWS THAT

$$\int_{\gamma^-} E \cdot dx = \int_0^1 F(\tilde{g}(t)) \cdot \tilde{g}'(t)\, dt$$

$$= \int_0^1 F(g(1-t)) \cdot (-1)\, g'(1-t)\, dt$$

$$= -\int_0^1 F(g(t)) \cdot g'(t)\, dt$$

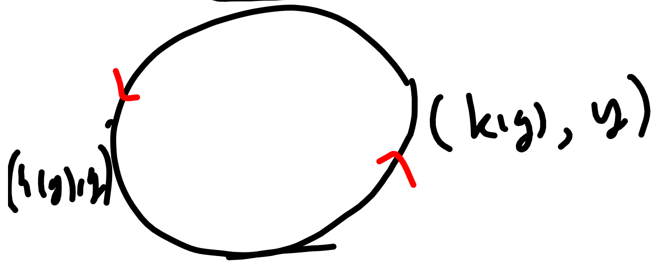
$$= -\int_\gamma E \cdot dx. \qquad \square$$

GREEN'S THEOREM: LET D BE A BOUNDED PLANE REGION WHICH IS THE FINITE UNION OF SIMPLE REGIONS.

LET F AND G BE CTSLY DIFFERENTIABLE. THEN

$$\int_D \frac{\partial G}{\partial y} - \frac{\partial F}{\partial x} \, dx\,dy = \oint_\gamma F\,dx + G\,dy.$$

TO PERFORM THE G PART OF THE INTEGRAL



THE BOUNDARY IS PARAMETRIZED BY $\{(k(y), y) : c \le y \le d\}$

AND IN REVERSE BY

$$\{(h(y), y) : c \le y \le d\},$$

THUS THE G INTEGRAL IS

$$\int_c^d G(k(y), y) - G(h(y), y) \, dy.$$

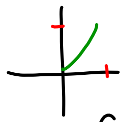
THE CORRESPONDING AREA INTEGRAL IS

$$\int_c^d \int_{h(y)}^{k(y)} \frac{\partial G}{\partial x} \, dx \, dy.$$

WHICH IS EQUAL, AGAIN, BY INTEGRATION BY PARTS.

EXAMPLE: GIVEN THE CURVE γ, PARAMETERIZED

$$g(t) = \begin{pmatrix} t \\ t^2 \end{pmatrix}, \quad 0 \le t \le 1.$$



LET $\mathbb{F}(x,y) = \begin{pmatrix} F(x,y) \\ G(x,y) \end{pmatrix}$

IF $G_y = F_x$ THEN YOU CAN CHANGE THE PATH OF INTEGRATION SO LONG AS IT HAS THE SAME ENDPOINTS

E.G. IF $F(x,y) = xy^2$

$F_y = 2xy.$

$G(x,y) = x^2 y$ HAS THE SAME x DERIVATIVE.

IN THIS CASE

$$\int_\gamma F(x,y)\,dx + G(x,y)\,dy = \int_{\gamma'} F(x,y)\,dx + G(x,y)\,dy.$$

WHERE γ' HAS THE SAME ENDPOINTS. $\gamma : \left\{ \begin{pmatrix} t \\ t^2 \end{pmatrix}.\ 0 \le t \le 1 \right\}.$

$\gamma' : \left\{ \begin{pmatrix} t \\ t \end{pmatrix}.\ 0 \le t \le 1 \right\}.$

$$\int_\gamma F\,dx + G\,dy = \int_0^1 (t\,t^4)\cdot 1 + (t^2\cdot t^2)2t\,dt$$

$F: xy^2$
$G: x^2y$

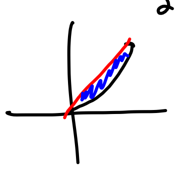
$$= \int_0^1 t^5 + 2t^5\,dt$$

$$= \frac{3t^6}{6}\Big]_0^1 = \frac{1}{2}.$$

$$\int_{\gamma'} F\,dx + G\,dy$$

$$= \int_0^1 t^2\cdot t + t\,t^2\,dt$$

$$= 2\cdot\int_0^1 t^3\,dt = 2\cdot\frac{t^4}{4}\Big]_0^1 = \frac{1}{2}.$$



$G_x - F_y$ VANISHES IN THE REGION BETWEEN THE TWO CURVES,

SO THE LINE INTEGRALS ARE EQUAL BY GREEN'S THEOREM.

IF, INSTEAD, WE WRITE

$$G(x,y) = \begin{pmatrix} G \\ -F \end{pmatrix},$$ AS A

VECTOR FIELD, THEN

$$\int_D \operatorname{div} G \, dA = \int_D G_x - F_y \, dA$$

$$= \int_{\partial D} G\,dx - F\,dy$$

$$= \underbrace{\int_{\partial D} G \cdot \underline{N}(s) \, ds}_{\text{FLUX. OF } G.}$$

= INTEGRAL OF DIVERGENCE.

THIS IS A 2-DIM'L VERSION OF GAUSS'S THEOREM.

EXAMPLE:

$\mathbb{F}(x,y) = -y\,dx + x\,dy.$

THEN CURL $F = 2$

SO $\iint_R \text{CURL } F \, dA = 2 \text{ AREA}(R).$

$= \int_\gamma -y\,dx + x\,dy.$

SO ITS POSSIBLE TO COMPUTE THE AREA OF A REGION BY INTEGRATING A LINE INTEGRAL ABOUT ITS BOUNDARY.