MAT 307 LECTURE 2.

HW 1 DUE MONDAY IN CLASS

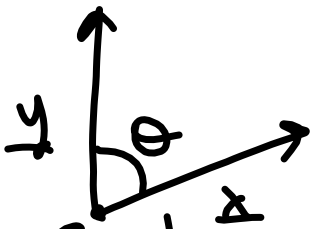
AVAILABLE FROM:

math.stonybrook.edu/~rdhough/mat307_fall20/math307.html

<u>RECALL</u>: GIVEN $\underline{x}, \underline{y} \in \mathbb{R}^n$,

$$\underline{x} \cdot \underline{y} = x_1 y_1 + x_2 y_2 + \dots + x_n y_n.$$

IN 2-D: THE LAW OF COSINES SAYS



$$\frac{\underline{x} \cdot \underline{y}}{|\underline{x}| \cdot |\underline{y}|} = \cos\theta.$$

IN PARTICULAR $|\underline{x} \cdot \underline{y}| \leq |\underline{x}| \cdot |\underline{y}|$.

DEFINITION: IF $\underline{x} \cdot \underline{y} = 0$ WE SAY THAT $\underline{x}$ AND $\underline{y}$ ARE **ORTHOGONAL** OR **PERPENDICULAR**.

PYTHAGOREAN THEOREM: IF

$\underline{x} \cdot \underline{y} = 0$, THEN

$|\underline{x} - \underline{y}|^2 = |\underline{x}|^2 + |\underline{y}|^2$.

[THIS IS VALID IN ANY DIMENSION:

$|\underline{x}-\underline{y}|^2 = (\underline{x}-\underline{y}) \cdot (\underline{x}-\underline{y}) = \underline{x} \cdot \underline{x} + \underline{y} \cdot \underline{y} - \cancel{2\,\underline{x} \cdot \underline{y}}$ $= 0$

$= |\underline{x}|^2 + |\underline{y}|^2$

THE CAUCHY-SCHWARZ INEQUALITY:

IF $\underline{x}, \underline{y}$ ARE VECTORS IN $\mathbb{R}^n$

THEN $|\underline{x} \cdot \underline{y}| \leq |\underline{x}| \cdot |\underline{y}|$.

EQUALITY HOLDS IF AND ONLY IF $\underline{x}$ AND $\underline{y}$ ARE LINEARLY DEPENDENT.

PROOF: WE MAY ASSUME $\underline{y} \neq \underline{0}$ OR ELSE $\underline{x}, \underline{y}$ LIN. DEP. AND $\underline{x} \cdot \underline{y} = 0 = |\underline{x}| \cdot |\underline{y}|$.

CONSIDER, FOR A REAL NUMBER $t \in \mathbb{R}$,

$$0 \leq |\underline{x} - t\underline{y}|^2 = (\underline{x} - t\underline{y}) \cdot (\underline{x} - t\underline{y})$$
$$= |\underline{x}|^2 - 2t\, \underline{x} \cdot \underline{y} + t^2 \cdot |\underline{y}|^2.$$

DIVIDE BY $|\underline{y}|^2$

$$0 \leq \left(\frac{|\underline{x}|}{|\underline{y}|}\right)^2 - \frac{2t\, \underline{x} \cdot \underline{y}}{|\underline{y}|^2} + t^2$$

$\boxed{\text{COMPLETE THE SQUARE.}}$

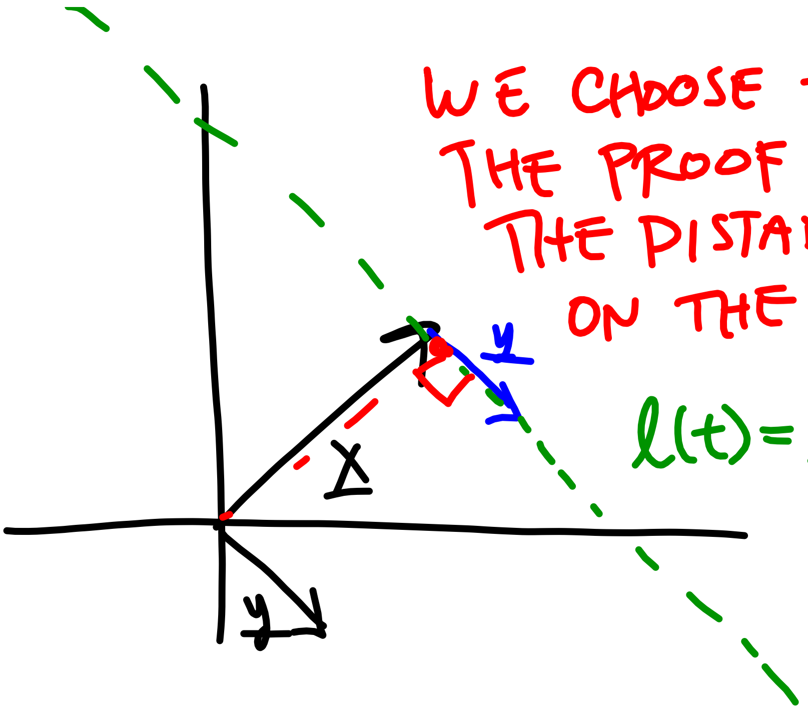
$$0 \leq \left(t - \frac{\underline{x} \cdot \underline{y}}{|\underline{y}|^2}\right)^2 + \left(\frac{|\underline{x}|}{|\underline{y}|}\right)^2 - \left(\frac{\underline{x} \cdot \underline{y}}{|\underline{y}|^2}\right)^2.$$

CHOOSE $t = \frac{\underline{x} \cdot \underline{y}}{|\underline{y}|^2} \Rightarrow$

$$\left(\frac{|\underline{x}|}{|\underline{y}|}\right)^2 \geq \left(\frac{\underline{x} \cdot \underline{y}}{|\underline{y}|^2}\right)^2$$

OR $|\underline{x}| \cdot |\underline{y}| \geq |\underline{x} \cdot \underline{y}|$.

NOTE: EQUALITY OCCURS IFF THERE EXISTS t SO THAT $|\underline{x} - t\underline{y}| = 0$.



WE CHOOSE t IN THE PROOF TO MIN. THE DISTANCE TO O ON THE LINE.

$\ell(t) = \underline{x} + t \cdot \underline{y}.$

DEFINITION: GIVEN TWO NON-ZERO VECTORS IN $\mathbb{R}^n$, $\underline{x}, \underline{y}$, DEFINE THE ANGLE θ BETWEEN $\underline{x}$ AND $\underline{y}$ BY

$$\cos\theta = \frac{\underline{x} \cdot \underline{y}}{|\underline{x}| \cdot |\underline{y}|}.$$

[THIS IS THE SAME AS CALCULATING THE ANGLE BETWEEN TWO VECTORS IN THE TWO DIMENSIONAL PLANE WHICH CONTAINS THEM.]

UNIT VECTORS: GIVEN A VECTOR $\underline{x}$, $\underline{x}$ IS A UNIT VECTOR IF $|\underline{x}| = 1$.

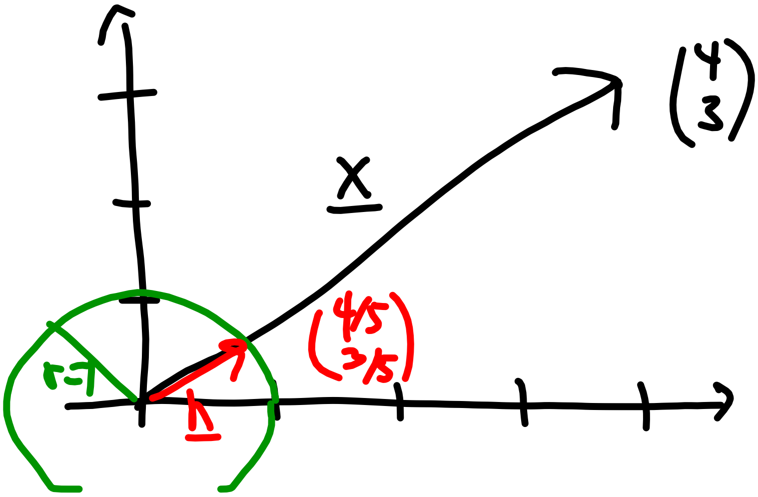
IF $\underline{x} \neq \underline{0}$, THE UNIT VECTOR IN THE DIRECTION OF $\underline{x}$ IS

$$\underline{n} = \frac{\underline{x}}{|\underline{x}|}.$$

$$|\underline{x}| = \sqrt{4^2+3^2} = \sqrt{25} = 5.$$

EXAMPLE:

$$\underline{n} = \frac{1}{|\underline{x}|} \cdot \underline{x} = \begin{pmatrix} 4/5 \\ 3/5 \end{pmatrix}.$$



<u>THEOREM:</u> GIVEN $\underline{x} \in \mathbb{R}^n$ AND $\underline{u} \in \mathbb{R}^n$ A UNIT VECTOR, $\underline{x}$ HAS A UNIQUE REPRESENTATION $\underline{x} = \underline{p} + \underline{q}$ WHERE

(1) $\underline{p} = t \cdot \underline{u} = (\underline{x} \cdot \underline{u}) \cdot \underline{u}$

(2) $\underline{q} \cdot \underline{u} = 0$.

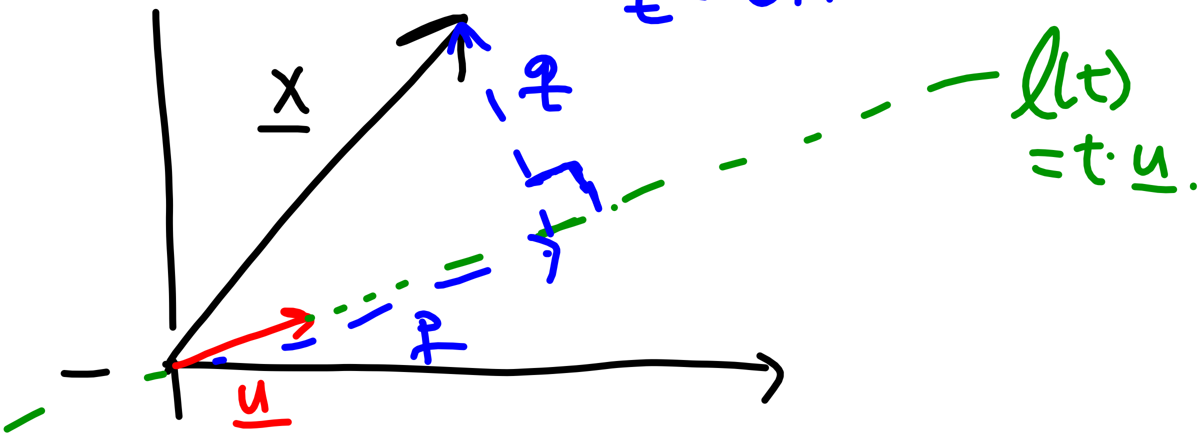
$\underline{p}$ IS CALLED THE ORTHOGONAL PROJECTION OF $\underline{x}$ ONTO $\underline{u}$,

$\underline{q}$ IS CALLED THE COMPONENT OF $\underline{x}$ ORTHOGONAL TO $\underline{u}$.

ILLUSTRATION:

$\underline{p}$ = UNIQUE PT ON $\ell(t)$ CLOSEST TO $\underline{x}$.

$\underline{q}$ = ORTHOGONAL TO $\underline{u}$.



PROOF: FIRST EXISTENCE:

DEFINE $\underline{p} = (\underline{x} \cdot \underline{u}) \cdot \underline{u}$.

$\underline{q} = \underline{x} - \underline{p}$.

THEN $\underline{q} \cdot \underline{u} = \underline{x} \cdot \underline{u} - \underline{p} \cdot \underline{u}$

$= \underline{x} \cdot \underline{u} - (\underline{x} \cdot \underline{u}) \cdot \underbrace{\underline{u} \cdot \underline{u}}_{1}$

$= 0.$

THIS GIVES THE REQUIRED DECOMP

PROOF OF UNIQUENESS:

IF $\underline{x} = \underline{p}_1 + \underline{q}_1 = \underline{p}_2 + \underline{q}_2$

WHERE $\underline{p}_1 = t_1 \underline{u}$, $\underline{p}_2 = t_2 \underline{u}$,

$\underline{q}_1 \cdot \underline{u} = \underline{q}_2 \cdot \underline{u} = 0$

THEN $\underline{0} = \underline{p}_1 + \underline{q}_1 - (\underline{p}_2 + \underline{q}_2)$

SO $\underline{p}_1 - \underline{p}_2 = \underline{q}_2 - \underline{q}_1 \Rightarrow$

$$|\underline{p}_1 - \underline{p}_2|^2 = (\underline{p}_1 - \underline{p}_2) \cdot (\underline{p}_1 - \underline{p}_2)$$
$$= (\underline{p}_1 - \underline{p}_2) \cdot (\underline{q}_2 - \underline{q}_1)$$
$$= (t_1 - t_2) \underline{u} \cdot (\underline{q}_2 - \underline{q}_1) = 0.$$

THUS $\underline{p}_1 = \underline{p}_2$, $\underline{q}_1 = \underline{q}_2$. $\square$

THEOREM: LET $\underline{x} \in \mathbb{R}^n$, $\underline{u} \in \mathbb{R}^n$, $\underline{u}$ A UNIT VECTOR. THE DISTANCE $|\underline{x} - t\underline{u}|$ IS MINIMIZED BY $t = \underline{x} \cdot \underline{u}$. $(t\underline{u} = \underline{p})$

AND $|\underline{x}|^2 = |\underline{p}|^2 + |\underline{q}|^2$.

PROOF: $|\underline{x} - t\underline{u}|^2 = |\underline{x}|^2 + t^2 \underbrace{\cancel{|\underline{u}|^2}}_{=1} - 2t\, \underline{x} \cdot \underline{u}$.

COMPLETE THE SQUARE!

$= \underbrace{\boxed{|\underline{x}|^2 - (\underline{x} \cdot \underline{u})^2}}_{\text{CONSTANT}} + (t - \underline{x} \cdot \underline{u})^2$.

THIS IS MINIMIZED BY $t = \underline{x} \cdot \underline{u}$.

ALSO, $\underline{x} = \underline{p} + \underline{q}$ AND $\underline{p} \cdot \underline{q} = 0$

$\Rightarrow |\underline{x}|^2 = |\underline{p}|^2 + |\underline{q}|^2$. (PYTHAG. THM)

<u>EXAMPLE:</u> $\underline{u} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \quad \underline{x} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}.$

$$\underline{x} \cdot \underline{u} = \frac{1}{\sqrt{2}} + \frac{4}{\sqrt{2}} = \frac{5}{\sqrt{2}}.$$

$$(\underline{x} \cdot \underline{u}) \cdot \underline{u} = \frac{5}{\sqrt{2}} \cdot \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} 5/2 \\ 5/2 \end{pmatrix} = \underline{p}.$$

$$\underline{q} = \underline{x} - \underline{p} = \begin{pmatrix} -3/2 \\ 3/2 \end{pmatrix}. \qquad \underline{p} \cdot \underline{q} = 0.$$

<u>REMARK:</u> FOR THOSE USED TO MATRIX MULTIPLICATION

$$\underline{x} \cdot \underline{y} = \underline{x}^t \cdot \underline{y}.$$

$$[x_1 \ \dots \ x_n] \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = x_1 y_1 + \dots + x_n y_n.$$

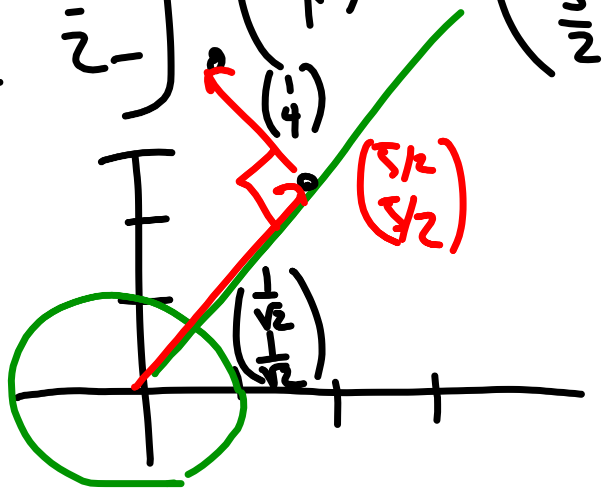
$$\underline{u} \cdot \underline{x} = \underline{u}^t \cdot \underline{x}.$$

SO $P = \underline{u} \cdot (\underline{u}^t \cdot \underline{x}) = (\underline{u}\,\underline{u}^t) \cdot \underline{x}.$

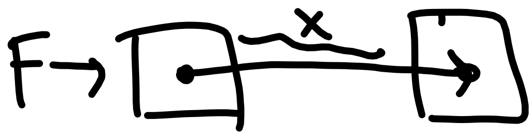
$\underline{u} = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$; $\underline{u} \cdot \underline{u}^t = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$

$$= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \cdot \begin{pmatrix} 1 \\ 4 \end{pmatrix} = \begin{pmatrix} \frac{5}{2} \\ \frac{5}{2} \end{pmatrix}.$$



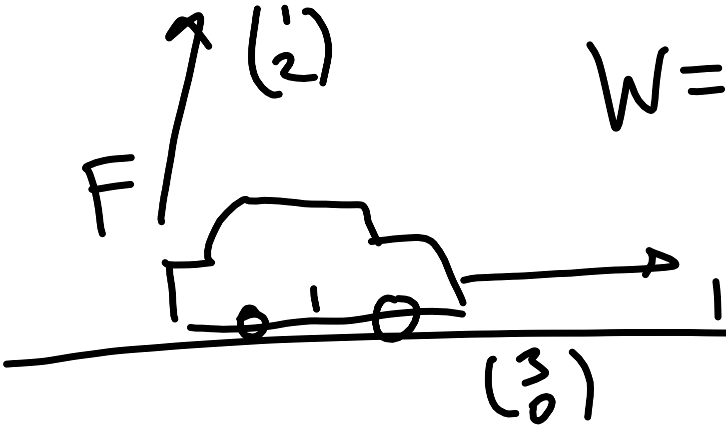
<u>EXAMPLE:</u> THE (1-DIM'L) <u>WORK</u> DONE BY A CONSTANT FORCE APPLIED TO AN OBJECT MOVING IN A STRAIGHT LINE DURING DISPLACEMENT X IS $F \cdot X$.



IN GENERAL, FORCE IS A VECTOR $\vec{F}$ AND ONLY THE COMPONENT IN THE DIRECTION OF THE DISPLACEMENT $\underline{X}$ CONTRIBUTES TO THE WORK

$$\boxed{\text{WORK} = \vec{F} \cdot \vec{X}.}$$

<u>EXAMPLE:</u> $\vec{F} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad \vec{x} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$

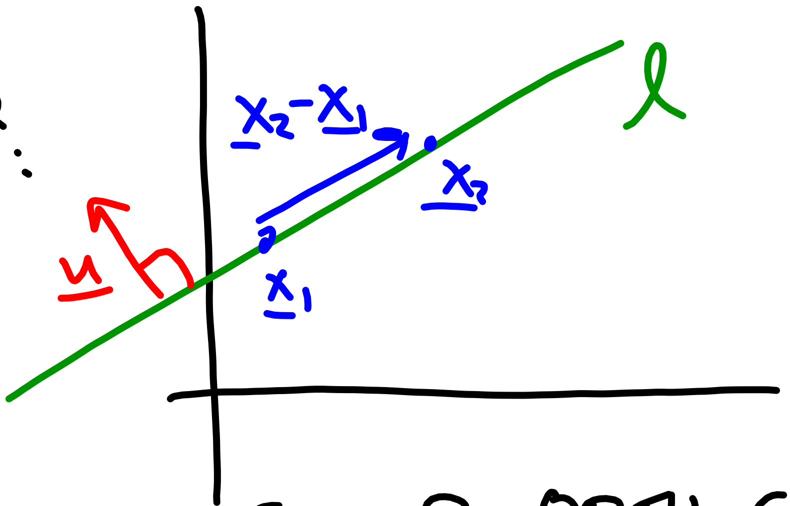


$W = \vec{F} \cdot \vec{x} = 3.$

$\vec{F} = \begin{pmatrix} F_x \\ F_y \end{pmatrix}, \quad F_x = \underline{1}.$

DUAL FORM OF LINE OR PLANE:

IN $\mathbb{R}^2$:



$\underline{u}$ IS A UNIT VECTOR ORTHOGONAL TO THE LINE ℓ.

IF $\underline{x}_1, \underline{x}_2$ ARE TWO POINTS OF ℓ, THEN

$$\underline{u} \cdot (\underline{x}_1 - \underline{x}_2) = 0.$$

IN DUAL FORM, A LINE IS DETERMINED BY A POINT $\underline{x}_0$ ON THE LINE, AND A (UNIT) VECTOR $\underline{u}$ ORTHOGONAL TO THE LINE

$$\ell = \{\underline{x} \in \mathbb{R}^2 : \underline{u} \cdot (\underline{x} - \underline{x}_0) = 0\}.$$
$$= \{\underline{x} \in \mathbb{R}^2 : \underline{u} \cdot \underline{x} = \underline{u} \cdot \underline{x}_0\}.$$

<u>EXAMPLE:</u> LET

$$\underline{P} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad x_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

THE PLANE THROUGH $\underline{x_0}$, $\perp$ TO $\underline{P}$

HAS

$$\underline{P} \cdot (\underline{x} - \underline{x_0}) = 0$$

$$\iff \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} x-1 \\ y \\ z \end{pmatrix} = 0 \iff x - 1 + y + z = 0$$

$$\iff x + y + z = \underline{1}$$

IN 3-DIMENSIONS, A PLANE IS DETERMINED BY A POINT ON THE PLANE AND A VECTOR PERP TO THE PLANE.

THE LINE OR PLANE IS
IN NORMAL FORM IF
THE ORTHOGONAL VECTOR
IS A UNIT VECTOR.

THIS THE SAME AS DEFINING THE LINE OR PLANE WITH A SINGLE LINEAR EQ'N.

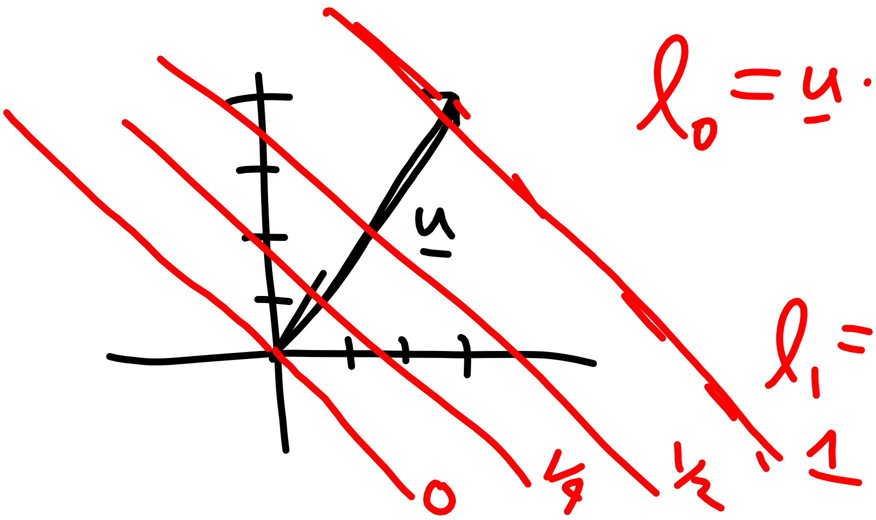
E.G. $3x+4y=1$.

$$\frac{3}{5}x+\frac{4}{5}y=\frac{1}{5}$$

NORMAL FORM, $\underline{u}=\begin{pmatrix} 3/5 \\ 4/5 \end{pmatrix}$.

EXAMPLE: $\underline{u} = \begin{pmatrix} 3/5 \\ 4/5 \end{pmatrix}$

$\ell_0 = \underline{u} \cdot \underline{x} = 0$



$\ell_1 = \underline{u} \cdot \underline{x} = 1.$

<u>THEOREM:</u> LET $\underline{n} \cdot (\underline{x} - \underline{x}_0) = 0$ BE A LINE OR PLANE IN NORMAL FORM. LET

$$\delta = \underline{n} \cdot (\underline{y} - \underline{x}_0)$$

FOR SOME POINT $\underline{y}$. THEN THE DISTANCE OF $\underline{y}$ TO THE LINE OR PLANE IS $|\delta|$.

PROOF: ANY POINT $\underline{w}$ IN THE LINE OR PLANE HAS $\underline{n} \cdot (\underline{w} - \underline{x}_0) = 0$. WRITE

$$\underline{y} = (\underline{y} \cdot \underline{n})\underline{n} + \underline{z} \quad \text{WHERE } \underline{n} \cdot \underline{z} = 0.$$

THEN

$$|\underline{y} - \underline{w}|^2 = |(\underline{y} \cdot \underline{n} - \underline{w} \cdot \underline{n})\underline{n} + \underline{v}|^2$$

WHERE $\underline{v}$ IS ORTHOGONAL TO $\underline{n}$, SINCE $\underline{n}, \underline{v}$ ORTHOG.

$$= (\underline{y} \cdot \underline{n} - \underline{w} \cdot \underline{n})^2 + |\underline{v}|^2.$$

$$= \delta^2 + |\underline{v}|^2.$$

THE CHOICE OF $\underline{v}$ IS ARBITRARY SUBJECT TO $\underline{n} \cdot \underline{v} = 0$ SO THE MINIMUM IS δ^2.

<u>EXAMPLE</u>: GIVEN A LINE $\begin{pmatrix}3\\4\end{pmatrix}\cdot \underline{x} = 1 \iff 3x+4y=1.$

IN NORMAL FORM $\frac{3}{5}x+\frac{4}{5}y=1.$

THE DISTANCE FROM $\begin{pmatrix}2\\3\end{pmatrix}$ TO THE LINE IS

$|\delta|$ WHERE $\delta=\begin{pmatrix}\frac{3}{5}\\\frac{4}{5}\end{pmatrix}\cdot\begin{pmatrix}2\\3\end{pmatrix}-\frac{1}{5}$

$=\boxed{17/5}$

EXAMPLE

$$\delta = \begin{pmatrix} \frac{3}{5} \\ \frac{4}{5} \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 3 \end{pmatrix} - \frac{1}{5}$$