RECALL FROM LAST CLASS, A SURFACE OF REVOLUTION IS OBTAINED BY ROTATING A PLANE CURVE ABOUT A LINE IN THE PLANE.

WHEN THE CURVE IS PARAM. BY A GENERAL PARAMETER,

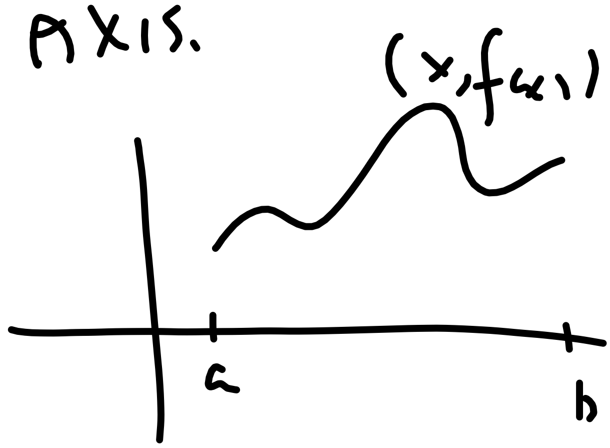
$$g(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

THEN

$$\text{SURFACE AREA} = 2\pi \int_a^b r(g(t)) \sqrt{x'(t)^2 + y'(t)^2}\, dt$$

EXAMPLE: CONSIDER

$y = f(x)$, ROTATED ABOUT X AXIS.



$$ds = \sqrt{1 + f'(x)^2}\,dx.$$

$$r(x) = |f(x)|.$$

SURFACE AREA:

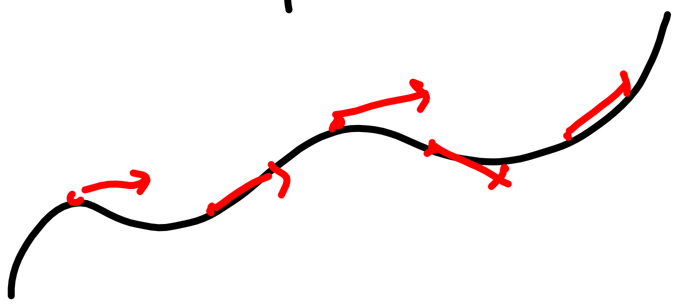
$$2\pi \int_a^b |f(x)| \sqrt{1 + f'(x)^2}\,dx.$$

RECALL: SPEED $= \frac{ds}{dt} = |x'(t)|$

VELOCITY $= v(t) = x'(t)$.

WHEN $\frac{ds}{dt} \neq 0$, DEFINE THE UNIT TANGENT VECTOR TO A CURVE BY

$$T(t) = \frac{x'(t)}{|x'(t)|}.$$



THUS $x'(t) = s'(t) \cdot T(t)$

SPEED DIRECTION

IN $\mathbb{R}^3$ AN ORTHONORMAL COORDINATE SYSTEM IS GIVEN BY

$T(t), N(t), B(t) = T(t) \times N(t)$.

THESE ARE UNIT NORMAL COORDINATE VECTORS RELATIVE TO THE CURVE

EXAMPLE:

HELIX: $X(t) = \begin{pmatrix} \cos t \\ \sin t \\ t \end{pmatrix}$

$$\dot{X}(t) = \begin{pmatrix} -\sin t \\ \cos t \\ 1 \end{pmatrix}$$

$$T(t) = \frac{1}{\sqrt{2}} \begin{pmatrix} -\sin t \\ \cos t \\ 1 \end{pmatrix}$$

$$T'(t) = \frac{1}{\sqrt{2}} \begin{pmatrix} -\cos t \\ -\sin t \\ 0 \end{pmatrix}$$

$$N(t) = \begin{pmatrix} -\cos t \\ -\sin t \\ 0 \end{pmatrix}$$

$$B(t) = T(t) \times N(t)$$

$$= \frac{1}{\sqrt{2}} \begin{vmatrix} i & j & k \\ -\sin t & \cos t & 1 \\ -\cos t & -\sin t & 0 \end{vmatrix}$$

$$= \frac{1}{\sqrt{2}} \left\{ i \begin{vmatrix} \cos t & 1 \\ -\sin t & 0 \end{vmatrix} - j \begin{vmatrix} -\sin t & 1 \\ -\cos t & 0 \end{vmatrix} + k \begin{vmatrix} -\sin t & \cos t \\ -\cos t & -\sin t \end{vmatrix} \right\}$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} \sin t \\ -\cos t \\ 1 \end{pmatrix}$$

$$T(t) = \frac{1}{\sqrt{2}} \begin{pmatrix} -\sin t \\ \cos t \\ 1 \end{pmatrix}, \quad N(t) = \begin{pmatrix} -\cos t \\ -\sin t \\ 0 \end{pmatrix},$$

$$B(t) = \frac{1}{\sqrt{2}} \begin{pmatrix} \sin t \\ -\cos t \\ 1 \end{pmatrix}$$

THEOREM: FOR A TWICE DIFFERENTIABLE CURVE $X(t)$ THE ACCELERATION IS IN THE PLANE OF $T(t), N(t)$.

AND

$$X''(t) = s'' T(t) + s'|T'| \cdot N(t)$$

a_T = TANGENTIAL COMPONENT

$a_N = s'|T'|$

NORMAL COMPONENT

<u>DEFINITION</u>: LET S DENOTE THE ARC-LENGTH PARAMETER, T(s) THE UNIT TANGENT VECTOR TO THE CURVE AT DISTANCE S.

THE <u>SCALAR CURVATURE</u> IS

$$K = \left| \frac{dT(s)}{ds} \right|.$$

THEOREM: SUPPOSE $\underline{X}(t)$ IS TWICE DIFFERENTIABLE. THEN

$$\left| \frac{dT}{dt} \right| = \frac{ds}{dt} \cdot \kappa$$

AND HENCE

$$\underline{X}''(t) = s'' \cdot T(t) + (s')^2 \cdot \kappa \cdot N(t).$$

EXAMPLE:

$$\underline{x}(t) = \begin{pmatrix} a\cos t \\ a\sin t \\ bt \end{pmatrix}, \quad a > 0$$

$$x'(t) = \begin{pmatrix} -a\sin t \\ a\cos t \\ b \end{pmatrix}$$

$$T(t) = \frac{1}{\sqrt{a^2+b^2}} \begin{pmatrix} -a\sin t \\ a\cos t \\ b \end{pmatrix}$$

$$s(t) = \sqrt{a^2+b^2} \cdot t, \quad \text{SINCE}$$

$$|x'(t)| = \sqrt{a^2+b^2}.$$

$$T'(t) = \frac{1}{\sqrt{a^2+b^2}} \begin{pmatrix} -a\cos t \\ -a\sin t \\ 0 \end{pmatrix}$$

$$\kappa = \frac{|T'(t)|}{s'} = \frac{a}{a^2+b^2}.$$

EXAMPLE: $F: \mathbb{R}^2 \to \mathbb{R}^2$

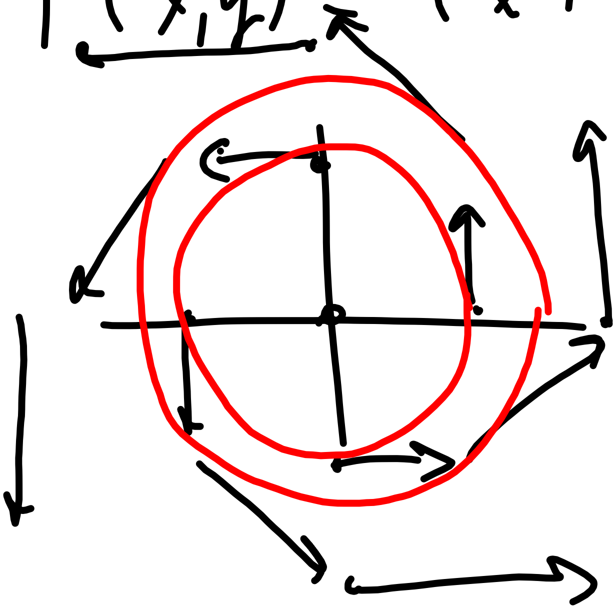
$F(x,y) = \begin{pmatrix} -y \\ x \end{pmatrix}$

$x(t) = \begin{pmatrix} a\cos t \\ a\sin t \end{pmatrix}$

$x'(t) = \begin{pmatrix} -a\sin t \\ a\cos t \end{pmatrix}$

$= \begin{pmatrix} -y \\ x \end{pmatrix}$

$= F(x(t))$.



FLOW LINES FOR THIS VECTOR FIELD ARE CIRCLES.

IF $F = \nabla f$ IS A GRADIENT OR CONSERVATIVE VECTOR FIELD, THEN THE LEVEL SETS $\{f(x) = k\}$ ARE PERPENDICULAR TO THE FLOW LINES.

THIS IS BECAUSE, IF $f(\gamma(t)) = k$ IS CONSTANT

THEN

$$\frac{d}{dt} f(\gamma(t)) = 0 = \nabla f(\gamma(t)) \cdot \gamma'(t)$$

SO $\gamma'(t) \perp F(\gamma(t))$.

OPERATIONS ON VECTOR FIELDS:

GIVEN $F: \mathbb{R}^n \to \mathbb{R}^n$ A VECTOR FIELD, THE DIVERGENCE DIV F IS

$$\operatorname{DIV} F = \sum_{i=1}^{n} \frac{\partial F_i}{\partial x_i}.$$

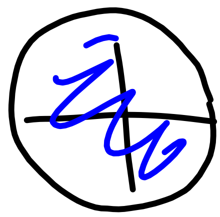
$$\nabla \cdot F$$

E.G. $\operatorname{DIV} F(x,y,z) = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}.$

$$\nabla = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix}, \quad F = \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix}.$$

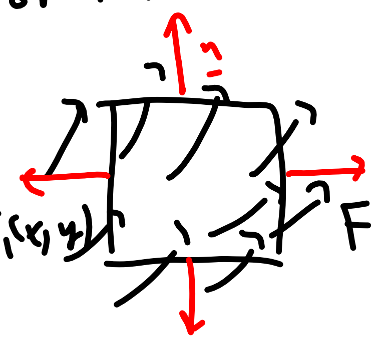
EXAMPLE: $F(x,y,z) = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$\operatorname{div} F = 3.$



QUANTITY INSIDE FLOWING ALONG A VECTOR FIELD F

THE DIVERGENCE GIVES A MEASURE OF THE NET GAIN OR LOSS COMING IN OR LEAVING A REGION.



$-F_1(x,y)$ $\quad F_1(x+\Delta x, y)$

$\text{FLUX} = \underline{n} \cdot \underline{F}$

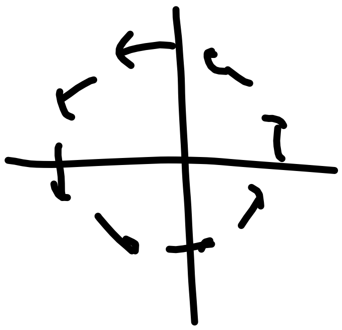
NET GAIN/LOSS IN HORIZONTAL DIRECTION: $\approx F_1(x+\Delta x, y) - F_1(x,y).$

$$\approx \Delta x \frac{\partial F_1}{\partial x}.$$

SUMMING IN ALL DIRECTIONS AND DIVIDING BY Δ GIVES THE DIVERGENCE.

EXAMPLE:

$$\underline{G}(x,y) = \begin{pmatrix} -\frac{1}{4}\dfrac{y}{\sqrt{x^2+y^2}} \\ \frac{1}{4}\dfrac{x}{\sqrt{x^2+y^2}} \end{pmatrix}$$



$$\operatorname{div} G = -\frac{1}{4}\frac{xy}{(x^2+y^2)^{3/2}} + \frac{1}{4}\frac{xy}{(x^2+y^2)^{3/2}} = 0.$$

THE SCALAR CURL OF A TWO DIMENSIONAL VECTOR FIELD $F(x,y) = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}$ IS $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$.