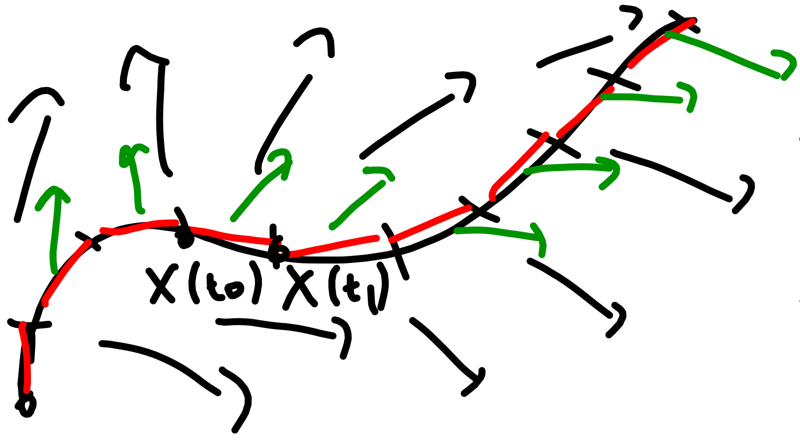
# LINE INTEGRALS:

GIVEN A VECTOR FIELD $F : \mathbb{R}^n \to \mathbb{R}^n$, AND A CURVE $\gamma : [a,b] \to \mathbb{R}^n$ WE WISH TO INTEGRATE THE VECTOR FIELD ALONG THE CURVE.



$X(t_1) - X(t_0) \simeq (t_1 - t_0)\, X'(t_0).$

ADD UP $\rightarrow$ . $\rightarrow$ TAKE THE LIMIT AS THE SEGMENT LENGTHS $\rightarrow 0$.

TAKE SUM: $\approx \sum \vec{F}(x(t_i))(X(t_{i+1}) - X(t_i))$

$$\approx \sum \vec{F}(x(t_i)) \cdot X'(t_i)(t_{i+1} - t_i)$$

$\Delta t$

In limit $\rightarrow$ $\boxed{\displaystyle\int_{t=t_0}^{t_1} \vec{F}(x(t)) \cdot x'(t)\, dt.}$

EXAMPLE:

$F: \mathbb{R}^3 \to \mathbb{R}^3$,

$$F(x,y,z) = \begin{pmatrix} x^2 \\ y^2 \\ z^2 \end{pmatrix}.$$

$$g(t) = \begin{pmatrix} t \\ t^2 \\ t^3 \end{pmatrix}$$

THEN $F(g(t)) = \begin{pmatrix} t^2 \\ t^4 \\ t^6 \end{pmatrix}$

$$g'(t) = \begin{pmatrix} 1 \\ 2t \\ 3t^2 \end{pmatrix}$$

$$F(g(t)) \cdot g'(t) = = t^2 + 2t^5 + 3t^8$$

IF $\gamma$ IS PARAMETERIZED BY $g(t)$, $0 \leq t \leq 1$,

$$\int_\gamma F \cdot dx = \int_0^1 F(g(t)) \cdot g'(t) \, dt$$

$$= \int_0^1 t^2 + 2t^5 + 3t^8 \, dt$$

$$= \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = \underline{1}.$$

EXAMPLE:

$$F(x,y,z) = \begin{pmatrix} x-y \\ y-z \\ z-x \end{pmatrix}$$

$$g(t) = \begin{pmatrix} t \\ -t \\ t^2 \end{pmatrix}$$

$$F(g(t)) = \begin{pmatrix} 2t \\ -t-t^2 \\ t^2-t \end{pmatrix}$$

$$g'(t) = \begin{pmatrix} 1 \\ -1 \\ 2t \end{pmatrix}$$

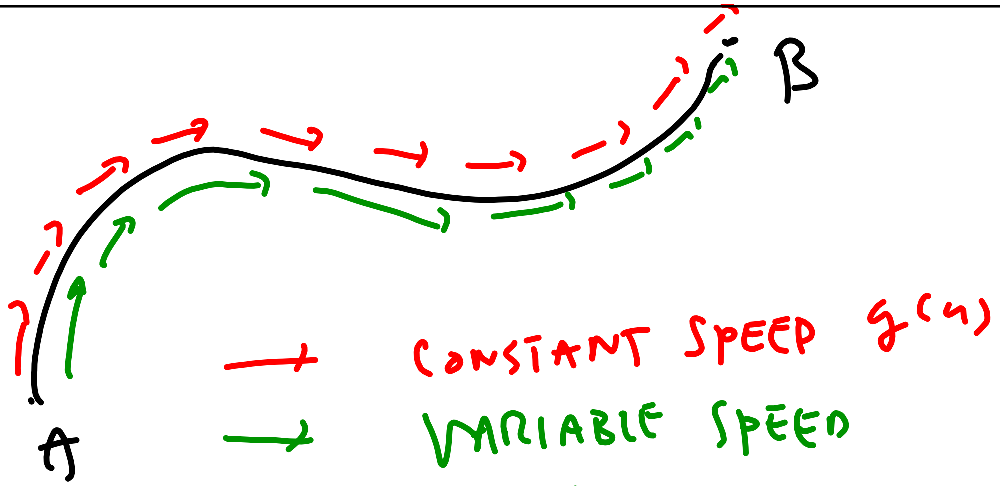
so $F(g(t)) \cdot g'(t) = 2t + t + t^2 + 2t^3 - 2t^2$

$= 3t - t^2 + 2t^3.$

$$\int_0^1 F(g(t)) \cdot g'(t)\,dt$$

$$= \int_0^1 3t - t^2 + 2t^3\,dt$$

$$= \frac{3}{2} - \frac{1}{3} + \frac{1}{2} = \frac{5}{3}.$$




$f:[a,b]\to\mathbb{R}^n$, $g:[c,d]\to\mathbb{R}^n$

AS THE FIGURE IS DRAWN THE LOCATION $f$ AT TIME $t$ IS THE SAME AS THE LOCATION OF $g$ AT TIME $u=u(t)$

$$f(t)=g(u(t)).$$

SAY THE TWO PARAMETERIZATIONS ARE EQUIVALENT IF THERE IS A SMOOTH CHANGE OF VARIABLE $u=u(t)$, CONTINUOUSLY DIFFERENTIABLE, WITH

$$u(a)=c$$
$$u(b)=d$$

AND $f(t)=g(u(t))$.

DEFINITION: $F : \mathbb{R}^n \to \mathbb{R}^n$ IS CONSERVATIVE IF THERE EXISTS A POTENTIAL FUNCTION $f : \mathbb{R}^n \to \mathbb{R}$ SUCH THAT $F = \nabla f$.

PROOF: LET $g(t)$, $0 \le t \le 1$
BE A PARAM. OF $\gamma$,
$g(0)=a$, $g(1)=b$.

CONSIDER

$$\frac{d}{dt} f(g(t)) = \nabla f(g(t)) \cdot g'(t)$$
$$= F(g(t)) \cdot g'(t).$$

THUS

$$\int_\gamma \vec{F} \cdot dx = \int_0^1 \frac{d}{dt} f(g(t))\, dt$$
$$= f(g(1)) - f(g(0))$$
$$= f(b) - f(a).$$

$\square$

EXAMPLE: $f(x,y) = x \cdot y$.

$$\nabla f = \begin{pmatrix} y \\ x \end{pmatrix}.$$

$$Df \cdot dx = y \cdot dx + x \cdot dy.$$

$$\int_\gamma Df \cdot dx = \int_\gamma y\,dx + x\,dy.$$

$$= x_2 y_2 - x_1 y_1. \quad \square$$

MORE GENERALLY IF IT IS KNOWN THAT $\vec{F}$ IS CONSERVATIVE, A POTENTIAL FUNCTION CAN BE CONSTRUCTED BY DEFINING

$$f(x) = \int_{x_0}^{x} \vec{F} \cdot dx$$

WHERE THE INTEGRAL IS OVER ANY PATH FROM $X_0$ TO $X$.

RECALL:

$\underline{x}(t)$ POSITION

$\underline{x}'(t)$ VELOCITY

$|\underline{x}'(t)|$ SPEED.

SO $\displaystyle\int_a^b |x'(t)|\,dt$

REPRESENTS THE TOTAL

DISTANCE TRAVELED.

EXAMPLE:

$\left(a \cos \frac{s}{a}, a \sin \frac{s}{a}\right) = g(s)$

PARAMETERIZES A CIRCLE OF RADIUS $a$ BY ARC-LENGTH, SINCE $g'(s) = \left(-\sin \frac{s}{a}, \cos \frac{s}{a}\right)$

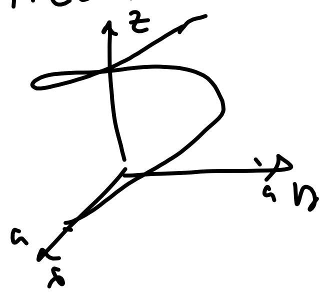
$\Rightarrow |g'(s)| = 1.$ $\square$

THE TOTAL MASS OF
A WIRE WITH MASS
DENSITY FUNCTION $\mu(x)$
IS

$$M = \int_{s_0}^{s_1} \mu(h(s))\, ds$$

WHERE $h : [s_0, s_1] \to \mathbb{R}^n$
IS THE ARC LENGTH
PARAMETERIZATION.

HELICAL COIL (SOLENOID)




$$g(t) = \begin{pmatrix} a\cos t \\ a\sin t \\ t \end{pmatrix}$$

LET THE MASS DENSITY AT POINT $x$ BE THE SQUARE OF THE DISTANCE FROM

$$q = \begin{pmatrix} 0 \\ 0 \\ \pi \end{pmatrix}.$$

$$|g(t) - q|^2 = \left| \begin{pmatrix} a\cos t \\ a\sin t \\ t-\pi \end{pmatrix} \right|^2$$

$$= a^2\cos^2 t + a^2 \sin^2 t + (t-\pi)^2.$$

$$= a^2 + (t-\pi)^2.$$

$$g'(t) = \begin{pmatrix} -a\sin t \\ a\cos t \\ 1 \end{pmatrix}$$

$$\Rightarrow |g'(t)| = \sqrt{a^2+1}.$$

THE MASS OF COIL BETWEEN $t=0$ AND $t=2\pi$ IS

$$\int_0^{2\pi} |g(t) - q|^2 \, |g'(t)| \, dt$$

$$= \int_0^{2\pi} \left( a^2 + (t-\pi)^2 \right) \sqrt{a^2+1} \, dt$$

$$= 2 \int_0^{\pi} \left( a^2 + (t-\pi)^2 \right) \sqrt{a^2+1} \, dt$$

$$= 2\sqrt{a^2+1} \int_0^{\pi} a^2 + t^2 \, dt$$

$$= 2\sqrt{a^2+1} \left[ a^2\pi + \frac{\pi^3}{3} \right].$$

DEFINITION: BY A SURFACE OF REVOLUTION WE MEAN A SURFACE $S \subset \mathbb{R}^3$ OBTAINED BY ROTATING A PLANE CURVE ABOUT A FIXED LINE IN THE PLANE OF THE CURVE.

SURFACE AREA:

$$\sigma(S) = \int_{s_1}^{s_2} 2\pi R(s)\, ds$$

WHERE $ds$ IS THE ARC LENGTH PARAMETERIZATION OF THE CURVE.

WHEN THE CURVE IS

$$g(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix},$$

$$ds = \sqrt{x'(t)^2 + y'(t)^2}.$$

$R(g(t))$ ABOUT $x$-AXIS IS $y(t)$.

$$\boxed{\sigma(S) = 2\pi \int_{t_0}^{t_1} y(t) \sqrt{x'(t)^2 + y'(t)^2}\, dt}$$