LEIBNIZ RULE:

LET $g(x,y)$ A FUNCTION ON $a \le x \le b$, $c \le y \le d$ WITH CONTINUOUS DERIVATIVE.

THEN: $\frac{d}{dy} \int_a^b g(x,y)\, dx$

$= \int_a^b \frac{\partial}{\partial y} g(x,y)\, dx.$

EXAMPLE:

$$G(y) = \int_0^1 \sin(ye^x)\,dx$$

$$G'(y) = \int_0^1 \cos(ye^x)\cdot e^x\,dx$$

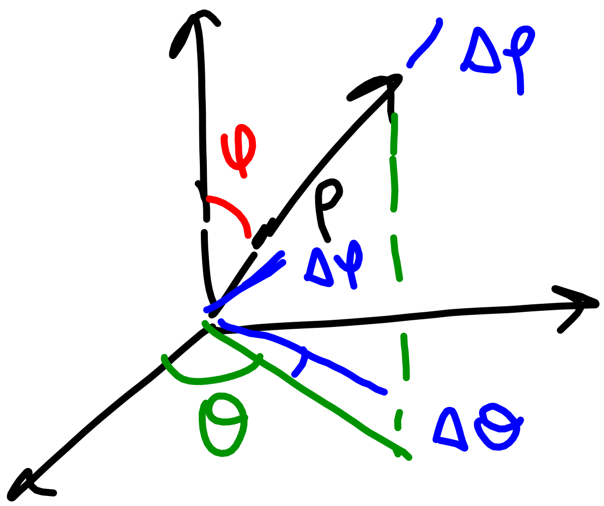
SUBSTITUTE $u = e^x$, $du = e^x\,dx$

$$= \int_1^e \cos(yu)\,du$$

$$= \frac{1}{y}\left[\sin yu\right]_1^e$$

$$= \frac{1}{y}\left[\sin ye - \sin y\right].$$

INTEGRATION IN SPHERICAL COORDINATES:



AREA ELEMENT

$r^2 \Delta r \Delta\theta \Delta\varphi$
$\sin\varphi$

JACOBI'S THEOREM: LET

$U \xrightarrow{T} \mathbb{R}^n$ BE CONTINUOUSLY

OPEN SET

DIFFERENTIABLE, LET $R \subset U$ BE A REGION WITH FINITELY MANY SMOOTH COMPONENTS.

SUPPOSE:

(i) T IS 1-1 ON INT(R)

(ii) $\det T' \neq 0$ ON INT(R).

IF f IS BOUNDED AND CONTINUOUS ON $T(R)$

THEN

$$\int_{T(R)} f(\underline{x})\, dV = \int_R f(T(\underline{y}))\, |\det T'(\underline{y})|\, dV_{\underline{y}}.$$

IN TWO DIMENSIONS:

$$T(u,v) = (F(u,v), G(u,v)).$$

$$T'(u,v) = \begin{pmatrix} \frac{\partial F}{\partial u} & \frac{\partial F}{\partial v} \\ \frac{\partial G}{\partial u} & \frac{\partial G}{\partial v} \end{pmatrix}$$

$$\det T'(u,v) = \frac{\partial F}{\partial u}\frac{\partial G}{\partial v} - \frac{\partial F}{\partial v}\frac{\partial G}{\partial u}$$

JACOBI'S FORMULA:

$$\int_{T(R)} f(x,y)\,dx\,dy = \int_R f(F(u,v), G(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv.$$

CYLINDRICAL COORDINATES:

$$T: \begin{pmatrix} r\cos\theta \\ r\sin\theta \\ z \end{pmatrix}, \quad T' = \begin{pmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

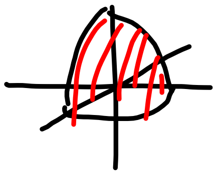
$\det T' = r.$

EXAMPLE INTEGRAL:

GIVEN A REGION R DEFINED IN SPHERICAL COORDINATES BY

$$\left\{0\le\rho\le 1,\quad 0\le\varphi\le\frac{\pi}{2},\quad 0\le\theta\le\frac{\pi}{2}\right\}$$

LET THIS CORRESPOND TO A REGION B IN CARTESIAN COORDINATES



$$\int_B x^2+y^2\,dx\,dy\,dz$$

$$= \int_R \rho^2\sin^2\varphi\,(\rho^2\sin\varphi)\,d\rho\,d\theta\,d\varphi.$$

$x=\rho\sin\varphi\cos\theta$ $\qquad x^2+y^2$

$y=\rho\sin\varphi\sin\theta$ $\qquad =\rho^2\sin^2\varphi.$

$$\underbrace{\int_0^1\rho^4\,d\rho}_{=\frac{1}{5}}\int_0^{\pi/2}\sin^3\varphi\,d\varphi\underbrace{\int_0^{\pi/2}1\,d\theta}_{=\pi/2}$$

$$\sin\varphi=\frac{e^{i\varphi}-e^{-i\varphi}}{2i}$$

EULER'S FORMULA

$\approx \sin 3\varphi$

$$(\sin\varphi)^3=\frac{e^{i3\varphi}-3e^{i\varphi}+3e^{-i\varphi}-e^{-i3\varphi}}{8i^3}.$$

$\approx \sin\varphi$

$$\int_0^{\pi/2}e^{i\varphi}\,d\varphi=\frac{1}{i}e^{i\varphi}\Big]_0^{\pi/2}.$$

ETC.

THE MOMENT OF A SYSTEM OF PARTICLES $(\underline{x}_1, m_1), \ldots, (\underline{x}_n, m_n)$ IN A PLANE P WITH UNIT NORMAL $\underline{n}$

$$M_P = \sum_{k=1}^{N} m_k \underline{n} \cdot (\underline{x}_k - \underline{x}_0).$$

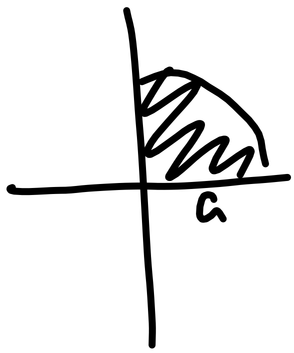
PROOF:

$$M_P = \sum_{k=1}^{N} m_k \underline{n} \cdot (\underline{x}_k - \bar{\underline{x}})$$

$$= \underline{n} \cdot \left[\sum_{k=1}^{N} m_k \underline{x}_k - \left(\sum_{k=1}^{N} m_k \right) \bar{\underline{x}} \right]$$

$$= \underline{n} \cdot \underline{0} .$$

EXAMPLES: $\mu(x,y) = x^2+y^2$.



QUARTER DISC, RADIUS a

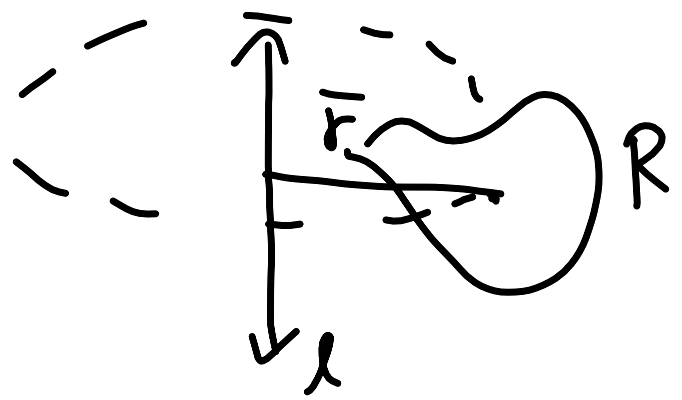
$$M = \int_Q \mu(x,y)\,dx\,dy$$

$$= \int_0^{\pi/2}\int_0^a \underbrace{(x^2+y^2)}_{r^2}\, r\,dr\,d\theta$$

$$= \pi/2 \int^a r^3\,dr$$

$$= \pi/2 \; \frac{r^4}{4}\Big]_0^a = \frac{\pi}{8}a^4.$$

PAPPUS'S THEOREM: THE VOLUME OF A SOLID OF REVOLUTION B ABOUT A LINE L IS EQUAL TO THE AREA OF A CROSS-SECTION TIMES THE AVERAGE CIRCUMFERENCE $2\pi\bar{r}$, THAT IS THE CIRCUMFERENCE OF THE CIRCLE TRACED BY THE CENTROID.



IMPROPER INTEGRALS:

THESE ARE INTEGRALS IN WHICH EITHER THE INTEGRAND IS UNBOUNDED, OR THE REGION OF INTEGRATION IS UNBOUNDED.

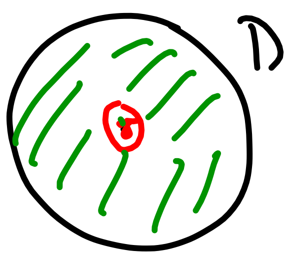
EXAMPLE: $D = \{x^2+y^2 \leq 1\}$

$$f(x,y) = \begin{cases} 0 & \text{IF } (x,y) = (0,0) \\ \frac{1}{\sqrt{x^2+y^2}} & \text{OTHERWISE.} \end{cases}$$

TO COMPUTE THESE INTEGRALS, LET D_δ BE A SEQUENCE OF SETS SO THAT $\bigcup_\delta D_\delta = D$

$D_{\delta_1} \subset D_{\delta_2}$ IF $\delta_2 < \delta_1$

"EXHAUSTION."



$/// = D_\delta$

LET $\delta \downarrow 0$.

THE IMPROPER INTEGRAL IS THE LIMIT OF THE INTEGRALS IF IT EXISTS.

EXAMPLE:

$$\iint_{[0,1]^2} -\ln(xy)\, dx\, dy.$$

$$= \lim_{\delta \downarrow 0} \iint_{[\delta,1]^2} \underbrace{-\ln(x y)}_{\substack{\shortparallel \\ -\ln x \;-\ln y}}\, dx\, dy$$

$$= -2 \lim_{\delta \downarrow 0} \int_{\delta}^{1} \int_{\delta}^{1} \ln x \, dx\, dy.$$

$$= -2 \lim_{\delta \downarrow 0} (1-\delta) \cdot \Big[x \ln x - x \Big]_{\delta}^{1}$$

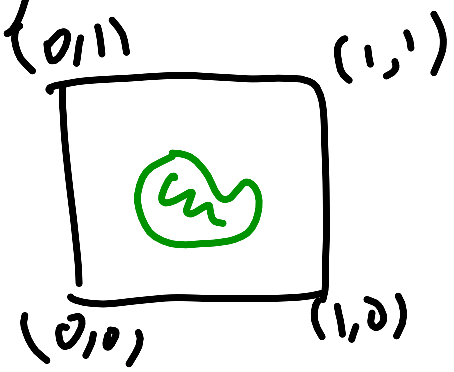
$$= -2 \lim_{\delta \downarrow 0} (1-\delta) \Big[-1 - \delta \ln \delta + \delta \Big]$$

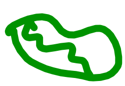
$$= \boxed{2.}$$

PROBABILITY DENSITIES:

THESE ARE FUNCTIONS $\rho(x)$, $\int \rho(x)\,dx = 1$,

$\rho(x) \geq 0$.



WITH UNIF. DENSITY 1, PROB A PT IN  IS THE AREA.