MAT 307 LECTURE 14:

COORDINATE SYSTEMS,

MULTIPLE INTEGRALS

MIDTERM 2 NEXT MONDAY

THEOREM: LET f_{xx}, f_{xy}, f_{yy} BE CONTINUOUS FOR $f: \mathbb{R}^2 \to \mathbb{R}$.

THEN, FOR $\underline{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$

$$\frac{\partial^2 f}{\partial u^2} = f_{xx}(x_0)u_1^2 + 2f_{xy}(x_0)u_1u_2 + f_{yy}(x_0)u_2^2.$$

THEOREM: LET

$$D = f_{xx} f_{yy} - f_{xy}^2 = \det(H(f)).$$

AT THE CRITICAL POINT x_0,

(1) IF $D > 0$ AND $f_{xx} < 0$ THEN LOCAL MAX

(2) IF $D > 0$ AND $f_{xx} > 0$ THEN LOCAL MIN

(3) IF $D < 0$ THEN SADDLE POINT.

EXAMPLE: $f(x,y) = 3x^2 - 6xy + 5y^2 + y^3.$

GRADIENT: $\nabla f = \begin{pmatrix} 6x - 6y \\ -6x + 10y + 3y^2 \end{pmatrix}$

CRITICAL PT: $x = y.$

$4y + 3y^2 = 0 \iff y(4+3y) = 0$

$y = 0$ OR $-\frac{4}{3}.$

HESSIAN: $\begin{bmatrix} 6 & -6 \\ -6 & 10+6y \end{bmatrix}$

AT $y=0$: $\begin{bmatrix} 6 & -6 \\ -6 & 10 \end{bmatrix}$, $D = 60 - 36 = 24 > 0$

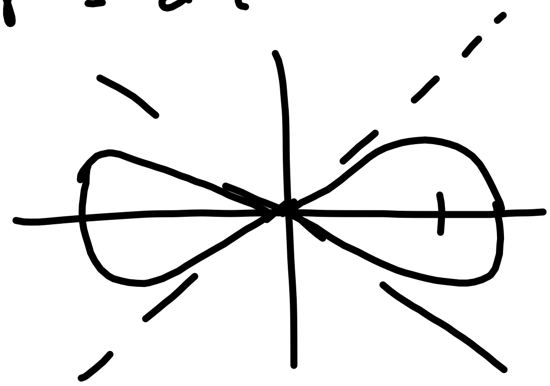
$f_{xx} = 6 \Rightarrow$ LOCAL MIN.

AT $y = -\frac{4}{3}$ $\begin{bmatrix} 6 & -6 \\ -6 & 2 \end{bmatrix}$, $D = 12 - 36 = -24 < 0$

SADDLE POINT.

EXAMPLE: LEMNISCATE

$r^2 = 2\cos 2\theta$



IN PASSING FROM SPHERICAL TO CARTESIAN COORDINATES,

$$S\begin{pmatrix}\rho\\ \theta\\ \varphi\end{pmatrix} = \begin{pmatrix}\rho\cos\theta\cos\varphi\\ \rho\sin\theta\cos\varphi\\ \rho\sin\varphi\end{pmatrix}$$

} PROJ. TO xy PLANE.

JACOBIAN MATRICES OF THE CHANGE OF COORDINATES:

$\boxed{\text{POLAR}}$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix} \qquad D = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

$\boxed{\text{SPHERICAL}}$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \rho\cos\theta\cos\varphi \\ \rho\sin\theta\cos\varphi \\ \rho\sin\varphi \end{pmatrix},$$

$$D = \begin{pmatrix} \cos\theta\cos\varphi & -\rho\sin\theta\cos\varphi & -\rho\cos\theta\sin\varphi \\ \sin\theta\cos\varphi & \rho\cos\theta\cos\varphi & -\rho\sin\theta\sin\varphi \\ \sin\varphi & 0 & \rho\cos\varphi \end{pmatrix}.$$

$\boxed{\text{CYLINDRICAL}}$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r\cos\theta \\ r\sin\theta \\ z \end{pmatrix}$$

$$D = \begin{pmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

REMARK: WHEN WE DISCUSS INTEGRATION, THE DETERMINANT OF THE JACOBIAN APPEARS IN THE CHANGE OF VARIABLE FORMULA FOR $\int$.

MULTIPLE INTEGRALS:

WE'RE INTERESTED IN EXTENDING THE ONE-DIMENSIONAL INTEGRAL TO $\mathbb{R}^n$. FIRST WE'LL CONSIDER ITERATED INTEGRALS.

EXAMPLE:

$$\int_0^1 \left[\int_1^2 (x^2+y)\, dy \right] dx$$

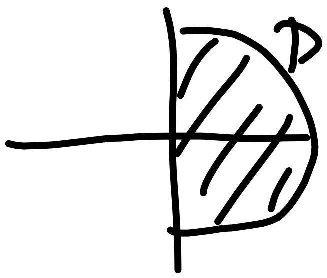
$$= \int_0^1 \left[x^2 y + \frac{y^2}{2} \right]_1^2 dx$$

$$= \int_0^1 \left[x^2 + \frac{3}{2} \right] dx = \left[\frac{x^3}{3} + \frac{3}{2}x \right]_0^1$$

$$= \frac{1}{3} + \frac{3}{2} = \frac{11}{6}.$$

EXAMPLE: LET $D \subset \mathbb{R}^2$,

$0 \le x$, $x^2+y^2 \le 1$. $f(x,y)=x$



$$\int_{-1}^{1}\left[\int_0^{\sqrt{1-y^2}} x\,dx\right]dy.$$

$$=\int_{-1}^{1}\left[\frac{x^2}{2}\right]_0^{\sqrt{1-y^2}} dy$$

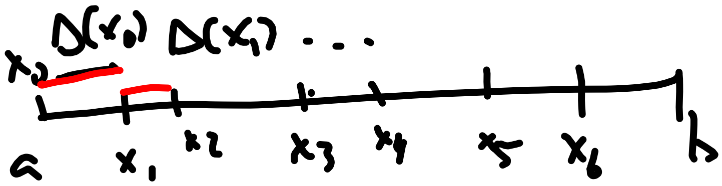
$$=\int_{-1}^{1}\frac{1-y^2}{2}\,dy = \frac{1}{2}\left[y-\frac{y^3}{3}\right]_{-1}^{1}$$

$$=1-\frac{1}{3}=\frac{2}{3}.$$

DEFINITION: RECALL IN 1 DIMENSION, A FUNCTION f IS **RIEMANN INTEGRABLE** IF

$$\int_a^b f(x)\,dx = \lim_{\Delta(x_k)\to 0} \sum_{j=0}^{K} f(x_j)\,\Delta(x_j)$$

EXISTS.



$\Delta(x_i) = x_{i+1} - x_i$.

WE WISH TO EXTEND THIS DEFINITION TO HIGHER DIMENSIONS.

THE <u>n-DIMENSIONAL VOLUME</u>

OF THE RECTANGLE R IS

$$V(R) = (b_1 - a_1)(b_2 - a_2)\dots(b_n - a_n)$$

= PRODUCT OF LENGTHS OF SIDES.

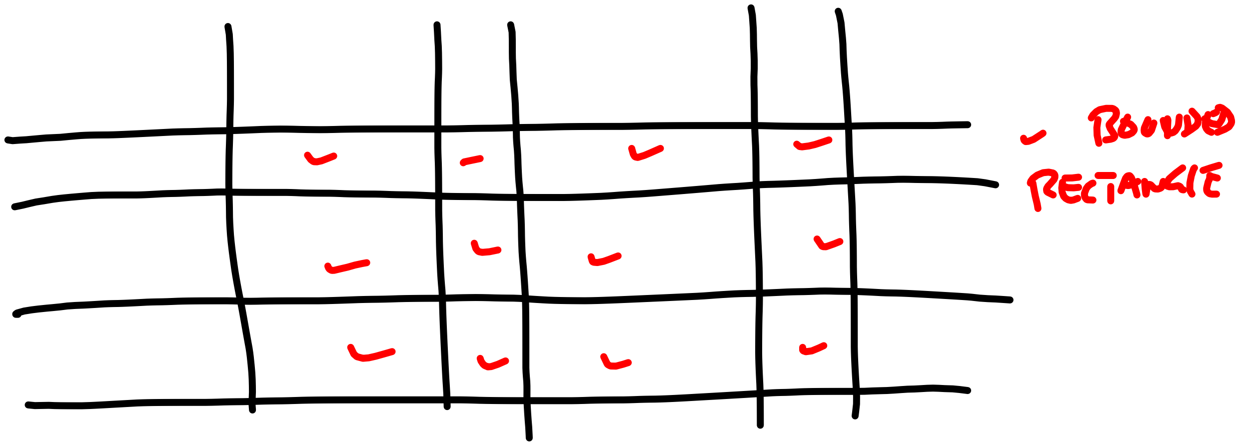
AN $\underline{n-1 \text{ - DIMENSIONAL HYPERPLANE}}$

IN $\mathbb{R}^n$ FIXES ONE COORDINATE,

E.G. $\{\underline{x} \in \mathbb{R}^n : x_i = a_i\}$

SINGLE CONSTRAINT.

EXAMPLE: A GRID OF $\mathbb{R}^2$



FOR EACH BOUNDED RECTANGLE R_i CONTAINING A POINT IN S, CHOOSE A POINT x_i.

THE RIEMANN SUM OF A FUNCTION f OF THE MESH WITH POINTS $\hat{x}_i$ IS

$$\boxed{\sum_{R_i} f(x_i)\, \mathrm{VOL}(R_i)}$$

DEFINITION: WE SAY A SET $S \subset \mathbb{R}^n$ HAS ZERO CONTENT IF

$$\int_S 1 \, dV$$

EXISTS AND IS EQUAL TO 0.