EXAMPLE: THE LAPLACE OPERATOR ON $\mathbb{R}^n$ IS GIVEN BY

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \dots + \frac{\partial^2}{\partial x_n^2}.$$

RECALL FROM 1-DIM'L CALC,

IF f IS MONOTONE,

BOTH, f, f^{-1} DIFF, THEN

$f^{-1} \circ f\,(x) = x$

THE CHAIN RULE IMPLIES

$(f^{-1})'(f(x)) \cdot f'(x) = 1$

OR $(f^{-1})'(f(x)) = \dfrac{1}{f'(x)}$.

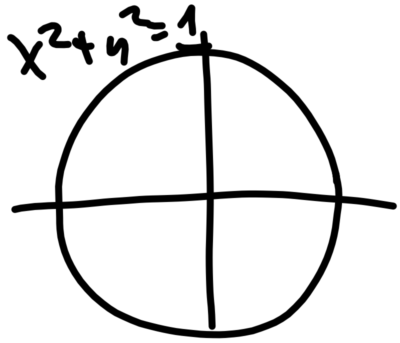
IF $f'(x) \neq 0$.

WE CALL THE SCALAR

$\det F'(x)$

IS THE <u>JACOBIAN DETERMINANT</u>.

EXAMPLE: LET $F(x,y) = x^2+y^2-1$.



$f(x)^2 = 1-x^2$

$f(x) = \pm\sqrt{1-x^2}$

TWO BRANCHES OF IMPLICITLY DEFINED FUNCTION.

$f_1(x) = \sqrt{1-x^2}$

$f_2(x) = -\sqrt{1-x^2}$.

THE IMPLICIT FUNCTION THEOREM GIVES CONDITIONS FOR THE EXISTENCE OF A DIFFERENTIABLE FUNCTION G SUCH THAT

$$F(\underline{x}, G(\underline{x})) = 0.$$

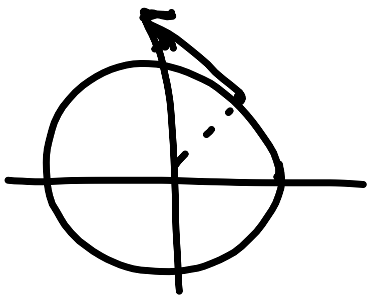
EXAMPLE:

IF $F(x,y) = x^2 + y^2 - 1 = 0$,

$2x + 2y\frac{dy}{dx} = 0$

$\Rightarrow \frac{dy}{dx} = \frac{-x}{y}$. AT THE POINT $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$

$\frac{dy}{dx} = -1$.



THEOREM: SUPPOSE

$$\mathbb{R}^{n+m} \xrightarrow{F} \mathbb{R}^m$$

AND $G : \mathbb{R}^n \to \mathbb{R}^m$

ARE DIFFERENTIABLE AND $\underline{y} = G(\underline{x})$ SATISFIES $F(x, G(x)) = 0$ ALL x IN AN OPEN SUBSET OF $\mathbb{R}^n$ THEN

$$G'(x) = -F_y(\underline{x}, G(\underline{x}))\, F_{\underline{x}}(\underline{x}, G(\underline{x})).$$

<u>DEFINITION</u>: A REAL VALUED FUNCTION f HAS AN <u>ABSOLUTE MAXIMUM</u> AT x IF $f(y) \leq f(x)$ FOR ALL y.

IT HAS A <u>LOCAL MAX</u> AT x IF THERE IS A $\delta > 0$ SO THAT $f(y) \leq f(x)$ FOR ALL $y \in B_\delta(x)$.

<u>THEOREM</u>: FOR A FUNCTION

$f: S \to \mathbb{R}$ SUCH THAT

(1) S IS CLOSED AND BOUNDED IN $\mathbb{R}^n$

(2) f IS CONTINUOUS.

f ACHIEVES ITS ABSOLUTE MAXIMUM AND MINIMUM ON S.

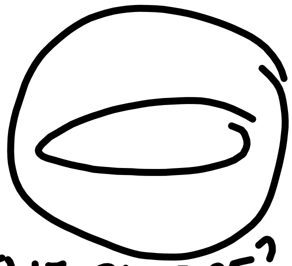
THEOREM: SUPPOSE $f : \mathbb{R}^n \to \mathbb{R}$ IS DIFFERENTIABLE, AND HAS A LOCAL EXTREMA AT $\underline{x}_0$. THEN $\nabla f(\underline{x}_0) = 0$.

DEFINITION: WE SAY A POINT $\underline{x}_0$ WHERE $Df(\underline{x}_0) = 0$ ARE **CRITICAL POINTS**.

LAGRANGE MULTIPLIERS:

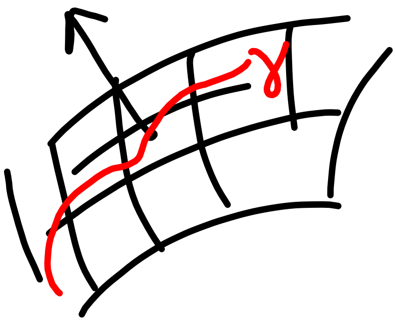
THIS IS A METHOD OF OPTIMIZATION FOR A FUNCTION SUBJECT TO A CONSTRAINT.

WHERE IS THE MAX OF THE FN ON THE SPHERE?



$x^2+y^2+z^2=1$
CONSTRAINT.

$f(x)=x+y+z$.



$G(\underline{x}) = \underline{k}$.

$\nabla G(\underline{x})$ IS PERPENDICULAR TO THE TANGENT OF THE SURFACE.

$\gamma(t_0) = x_0$.

$f(\gamma(t_0))$ HAS A LOCAL MAX/MIN AT t_0

$\Rightarrow \nabla f(\gamma(t_0)) \cdot \gamma'(t_0) = 0$.

THIS SAYS THAT ∇f IS $\perp$ TO ANY VECTOR IN THE TANGENT SPACE OF THE SURFACE.

IF THERE IS JUST A SINGLE CONSTRAINT, THEN THE ONLY DIRECTION ORTHOG. TO THE SURFACE IS ∇G

AND SO $\nabla f(x_0) = \lambda \cdot \nabla G(x_0)$.

$$\nabla f = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\nabla G_1 = \begin{pmatrix} 2x \\ 2y \\ 0 \end{pmatrix}$$

$$\nabla G_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

$$\nabla f = \lambda_1 \nabla G_1 + \lambda_2 \nabla G_2 .$$

$$\lambda_2 = 1.$$

$$\lambda_1 \begin{pmatrix} 2x \\ 2y \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$\Rightarrow x = y$$

$$x^2 + y^2 = 1 \Rightarrow x = \pm \frac{1}{\sqrt{2}},\ y = \pm \frac{1}{\sqrt{2}}$$

$$z = 2.$$

<u>MAX:</u> $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 2 \end{pmatrix} \to 2 + \frac{2}{\sqrt{2}} = 2 + \sqrt{2}$

<u>MIN:</u> $\begin{pmatrix} -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 2 \end{pmatrix} \to 2 - \sqrt{2}.$

2ND DERIVATIVE TEST:

THEOREM: LET $f: \mathbb{R}^n \to \mathbb{R}$ TWICE CONTINUOUSLY DIFFERENTIABLE ON AN OPEN SET U WITH CRITICAL POINT AT $\underline{x}_0$.

HESSIAN MATRIX: $\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)_{i,j=1}^{n} = \frac{\partial^2 f}{\partial x^2}$

(i) IF $\underline{u} \cdot \left(\frac{\partial^2 f}{\partial x^2}\right) \cdot \underline{u} > 0$ ALL UNIT VECTORS $\underline{u}$ WE SAY THE MATRIX IS POSITIVE DEFINITE, $\underline{x}_0$ IS A LOCAL MIN.

(ii) IF $\underline{u} \cdot \left(\frac{\partial^2 f}{\partial x^2}\right) \cdot \underline{u} < 0$ ALL UNIT VECTORS $\underline{u}$, WE SAY $\frac{\partial^2 f}{\partial x^2}$ IS NEGATIVE DEFINITE, $\underline{x}_0$ IS A LOCAL MAX.

(iii) IF $\underline{u} \cdot \left(\frac{\partial^2 f}{\partial x^2}\right) \underline{u}$ TAKES POSITIVE AND NEGATIVE VALUES, SADDLE POINT, NEITHER A LOCAL MIN/MAX.

PROOF OF THEOREM:

DEFINE $g(t) = f(\underline{x} + t\underline{u})$.

$$g'(t) = \nabla f(\underline{x} + t\underline{u}) \cdot \underline{u}.$$
$$= u_1 \frac{\partial f}{\partial x_1}(\underline{x} + t\underline{u}) + \dots + u_n \frac{\partial f}{\partial x_n}(\underline{x} + t\underline{u})$$

$$g''(t) = \boxed{u_1 \nabla\left(\frac{\partial f}{\partial x_1}\right)(\underline{x} + t\underline{u}) \cdot \underline{u} + \dots + u_n \nabla\left(\frac{\partial f}{\partial x_n}\right)(\underline{x} + t\underline{u}) \cdot \underline{u}.}$$

$$= \sum_{i,j} u_i u_j \frac{\partial^2 f}{\partial x_i \partial x_j}$$

SINCE $g'(0) = \nabla f(x_0) \cdot \underline{u} = 0$.

WE HAVE

$$f(\underline{x}_0 + x \cdot \underline{u}) = f(x_0) + \int_0^x (x-t) \frac{\partial^2 f}{\partial u^2}(\underline{x}_0 + t\underline{u})\, dt$$

WHERE $\frac{\partial^2 f}{\partial u^2} = \underline{u} \cdot \left(\frac{\partial^2 f}{\partial x^2}\right) \underline{u}$.

IF $\frac{\partial^2 f}{\partial x^2}$ IS CTSLY VARYING,

AND $u \cdot \left(\frac{\partial^2 f}{\partial x^2} u\right) > 0$ ALL u

AT x_0 THEN THIS IS TRUE IN A NEIGHBORHOOD OF x_0

SO THE INTEGRAL IS POSITIVE

$\Rightarrow$ LOCAL MIN. SIMILARLY MAX / SADDLE POINT. $\square$