MAT 307 LECTURE 14:

THE INVERSE AND IMPLICIT FUNCTION THEOREMS

THE PROOFS ARE USUALLY GIVEN IN MAT 322 (MULTIVARIABLE ANALYSIS).

CHANGE OF COORDINATES:

ON $\mathbb{R}^2$, LET $z = x+y$, $w = x-y$

SO $x = \frac{z+w}{2}$, $y = \frac{z-w}{2}$.

LET $\bar{u}(z,w) = u\left(\frac{z+w}{2}, \frac{z-w}{2}\right)$

NEW FUNCTION IN z-w COORDINATES.

$u_x = \bar{u}_z z_x + \bar{u}_w w_x = \bar{u}_z + \bar{u}_w$.

$u_y = \bar{u}_z z_y + \bar{u}_w w_y = \bar{u}_z - \bar{u}_w$.

$$u_{xx} = (\bar{u}_z)_z z_x + (\bar{u}_z)_w w_x + (\bar{u}_w)_z z_x + (\bar{u}_w)_w w_x$$
$$= \bar{u}_{zz} + 2\bar{u}_{zw} + \bar{u}_{ww}.$$

$$u_{yy} = (\bar{u}_z)_z z_w + (\bar{u}_z)_w w_y - (\bar{u}_w)_z z_w - (\bar{u}_w)_w w_y$$
$$= \bar{u}_{zz} - 2\bar{u}_{zw} + \bar{u}_{ww}.$$

THUS $u_{xx} + u_{yy} = 2\bar{u}_{zz} + 2\bar{u}_{ww}$.

THE LAPLACIAN IS AN IMPORTANT DIFF OPERATOR IN GEOMETRY

INVARIANT UNDER RIGID MOTIONS
TRANSLATION / ROTATION.

IT APPEARS IN MANY IMPORTANT DIFF EQ. FROM SCIENCE
- HEAT EQ'N
- WAVE EQ'N.

INVERSE FUNCTION THEOREM:

LET $\mathbb{R}^n \xrightarrow{F} \mathbb{R}^n$ CTSLY DIFF. ON AN OPEN SET $S \subset \mathbb{R}^n$. LET $\underline{x}_0 \in S$. SUPPOSE $F'(x_0)$ IS INVERTIBLE. THEN THERE EXISTS A BALL $B_\delta(x_0)$ SUCH THAT F IS 1-1 FROM $B_\delta(x_0)$ TO THE RANGE. THE INVERSE FUNCTION F^{-1} IS DIFFERENTIABLE AT $F(x_0)$ AND ITS DERIVATIVE IS $(F^{-1})'(F(x_0)) = (F'(x_0))^{-1}$.

TO SUMMARIZE: F HAS A LOCAL INVERSE FUNCTION WHICH IS DIFFERENTIABLE.

DEFINING FUNCTIONS IMPLICITLY

GIVEN $\mathbb{R}^2 \xrightarrow{F} \mathbb{R}$, $\mathbb{R} \xrightarrow{f} \mathbb{R}$

$$F(x,y) = 0$$

THIS DEFINES A FUNCTION $f(x)$ IMPLICITLY BY

$$F(x, f(x)) = 0.$$

IN GENERAL, GIVEN

$\mathbb{R}^{n+m} \xrightarrow{F} \mathbb{R}^n$, THE

FUNCTION $\mathbb{R}^n \xrightarrow{G} \mathbb{R}^m$ IS

DEFINED IMPLICITLY BY

$$F(\underline{x}, G(\underline{x})) = \underline{0}$$

BY THE CHAIN RULE

$$F_{\underline{x}}(\underline{x}, G(\underline{x})) + F_{\underline{y}}(\underline{x}, G(x))\, G'(\underline{x}) = 0$$

$$\frac{dy}{dx} = -\frac{F_x(x,y)}{F_y(x,y)}, \quad \text{IF } F_y \neq 0$$

IMPLICIT DIFFERENTIATION:

EXAMPLE:

$$\begin{pmatrix} x^2 + y^2 + z^2 - 6 \\ xyz + 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

TAKING PARTIALS WITH RESPECT TO z:

$$2x\frac{dx}{dz} + 2y\frac{dy}{dz} + 2z = 0$$

$$yz\frac{dx}{dz} + xz\frac{dy}{dz} + xy = 0$$

$$\Rightarrow \begin{pmatrix} \frac{dx}{dz} \\ \frac{dy}{dz} \end{pmatrix} = \begin{pmatrix} \frac{x(y^2 - z^2)}{z(x^2 - y^2)} \\ \frac{y(z^2 - x^2)}{z(x^2 - y^2)} \end{pmatrix}$$

<u>EXAMPLE:</u> $F(x,y,z) = \begin{pmatrix} x^2y + xz \\ xz + yz \end{pmatrix}$

$$F_x(x,y,z) = \begin{pmatrix} 2xy + z \\ z \end{pmatrix}$$

$$F_{(y,z)}(x,y,z) = \begin{pmatrix} x^2 & x \\ z & x+y \end{pmatrix}$$

WHERE $\det F_{(y,z)} \neq 0$

THE DERIVATIVE OF x AS A FUNCTION OF y, z IS GIVEN BY

$$-\begin{pmatrix} x^2 & x \\ z & x+y \end{pmatrix}^{-1} \begin{pmatrix} 2xy+z \\ z \end{pmatrix}.$$

IF THE INEQUALITIES GO
THE OTHER DIRECTION THEN

ABSOLUTE MINIMUM OR

LOCAL MIN.

<u>REMARK</u>: A SET WHICH IS CLOSED AND BOUNDED IN $\mathbb{R}^n$ IS <u>COMPACT</u>.

PROOF: WE CAN ASSUME THAT f HAS A LOCAL MIN, OR REPLACE f WITH $-f$.

AS A (1-VARIABLE) FUNCTION OF EACH COORDINATE SEPARATELY, f HAS A LOCAL MIN AT x_0, SO

$$\frac{\partial f}{\partial x_i}(x_0) = 0.$$

THUS $\nabla f = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{pmatrix} = \underline{0}$. $\square$

EXAMPLE:

$f(x,y,z) = y^2 - z - x$

$$Df = \begin{pmatrix} -1 \\ 2y \\ -1 \end{pmatrix}$$

DOES NOT HAVE ANY CRITICAL POINTS.

LAGRANGE MULTIPLIERS: $\mathbb{R}^n \xrightarrow{f} \mathbb{R}$

DIFFERENTIABLE ON A SET $S \subset \mathbb{R}^n$, WITH LOCAL EXTREME POINT $x_0 \in S$.

SUPPOSE NEAR $\underline{x}_0$ THE SET S IS THE LEVEL SET OF A FUNCTION $G: \mathbb{R}^n \to \mathbb{R}^m$

$$G(\underline{x}) = \begin{pmatrix} G_1(\underline{x}) \\ \vdots \\ G_m(\underline{x}) \end{pmatrix}.$$

THEN AT $\underline{x}_0$, $\nabla f(\underline{x}_0) \in \text{SPAN}\{\nabla G_1(x_0), \ldots, \nabla G_m(x_0)\}$

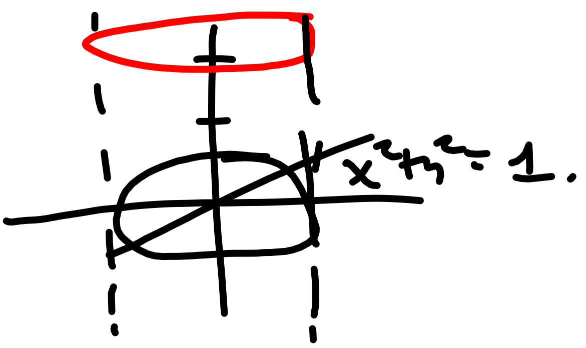
EXAMPLE: GIVEN

$f(x,y,z) = x + y + z$

TWO CONSTRAINTS:

$G_1(x,y,z) = x^2 + y^2 - 1$

$G_2(x,y,z) = z - 2.$



EXAMPLE: OPTIMIZATION FUNCTION

$f(x,y,z) = x - y + z$

CONSTRAINT: $G(x,y,z) = x^2 + y^2 + z^2 - 1$
$= 0.$

$\nabla f = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \quad \nabla G = \begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix}$

$\nabla f = \lambda \cdot \nabla G \Rightarrow$
$\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \lambda \cdot \begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix}$

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = c \cdot \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ $\qquad x^2 + y^2 + z^2 = 1.$

$3c^2 = 1$
$c = \pm \frac{1}{\sqrt{3}}$

MAX: $\begin{pmatrix} \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}$, MIN: $\begin{pmatrix} -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \end{pmatrix}$

VALUE: $3/\sqrt{3}$ $\qquad -3/\sqrt{3}.$

EXAMPLE: THE FUNCTION

$F(x,y) = x^2 + y^2$ HAS

$\nabla F(x,y) = \begin{pmatrix} 2x \\ 2y \end{pmatrix}$ CRITICAL PT. AT 0.

HESSIAN: $\begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$

QUANTITY CONSIDERING:

$(\underline{u}) \cdot \begin{pmatrix} 2x & 0 \\ 0 & 2y \end{pmatrix} \underline{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \cdot \begin{pmatrix} 2u_1 \\ 2u_2 \end{pmatrix}$

$= 2u_1^2 + 2u_2^2 > 0$

THUS $\underline{0}$ IS A LOCAL MIN.

INSTEAD: $G(x,y) = x^2 - y^2.$

$\nabla G = \begin{pmatrix} 2x \\ -2y \end{pmatrix}$ AGAIN, CRITICAL PT AT $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

HESSIAN: $\begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$

$u^T \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} u = 2u_1^2 - 2u_2^2$

THIS TAKES POSITIVE AND NEGATIVE VALUES, SO SADDLE POINT.