MAT 307 LECTURE 13

CONSERVATIVE VECTOR FIELDS

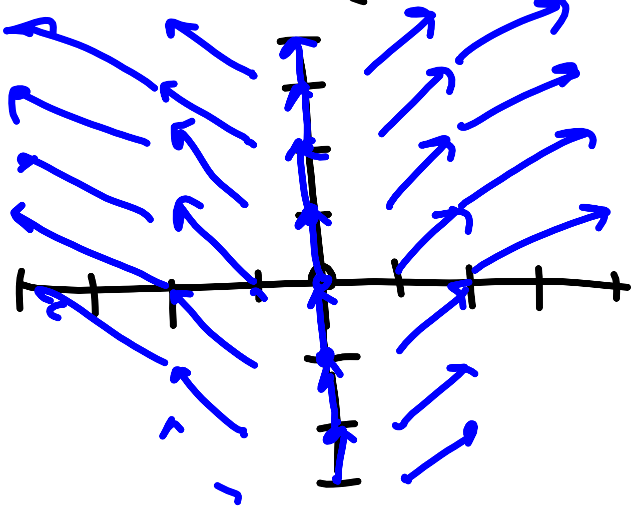
GRADIENT FIELD:

POTENTIAL FUNCTION

$f : \mathbb{R}^n \to \mathbb{R}.$

$F(\underline{x}) = \nabla f(\underline{x}).$ ← GRADIENT FIELD

$$F(x,y) = \begin{pmatrix} x \\ 1 \end{pmatrix}$$



EXAMPLE:

$f(x,y) = e^{xy}$, POTENTIAL FUNCTION

$$F(x,y) = \begin{pmatrix} y e^{xy} \\ x e^{xy} \end{pmatrix}$$

$$\nabla f(1,2) = \begin{pmatrix} 2e^2 \\ 1e^2 \end{pmatrix}$$

THE DIRECTION (OR UNIT VECTOR) OF MOST RAPID INCREASE IS $\underline{u} = \frac{\nabla f}{\|\nabla f\|} = \begin{pmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix}$

<u>THEOREM:</u> LET g BE REAL VALUED, CTSLY DIFF ON $D \subset \mathbb{R}^n$ WHICH IS OPEN, LET $f:(a,b) \to \mathbb{R}^n$ BE A DIFFENTIABLE CURVE, VALUES IN D.

LET $F(t) = g \circ f(t)$.

THEN
$$F'(t) = \nabla g(f(t)) \cdot f'(t)$$

DEFINITION: THE LEVEL CURVE OF LEVEL $k \in \mathbb{R}$ OF A POTENTENTIAL FN

$f: \mathbb{R}^n \to \mathbb{R}$

IS THE SET

$$S_k = \{\underline{x} \in \mathbb{R}^n : f(\underline{x}) = k\}.$$

EXAMPLE:

$$S = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 = 1\}$$

$\underline{n} \perp \underline{t}$

FOR ANY TANGENT VECTOR $\underline{t}$ TO A CURVE PASSING THROUGH $\underline{x}_0$.

PROOF: $\gamma : [a,b] \to S$,

$\gamma(0) = x_0$, A CURVE PASSING THROUGH x_0.

SINCE γ IS CONTAINED IN S,

$f(\gamma(t))$ IS CONSTANT $\Rightarrow$

$$\frac{d}{dt} f(\gamma(t)) = 0.$$

BY THE CHAIN RULE,

$$\frac{d}{dt} f(\gamma(t)) = \nabla f(\gamma(t)) \cdot \gamma'(t)$$

$$\Rightarrow \nabla f(x_0) \cdot \gamma'(0) = 0.$$

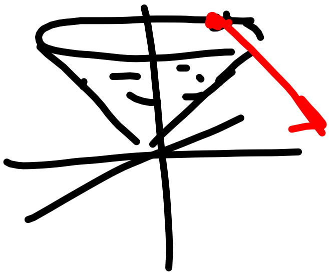
TANGENT VECTOR TO CURVE.

THUS $\nabla f(x_0)$ IS $\perp$ TO

ANY TANGENT VECTOR PASSING THROUGH $\underline{x_0}$.

<u>EXAMPLE:</u> $f(x,y,z) = x^2 + y^2 - z^2$.

$$C = \{ x^2 + y^2 - z^2 = 0 \}.$$

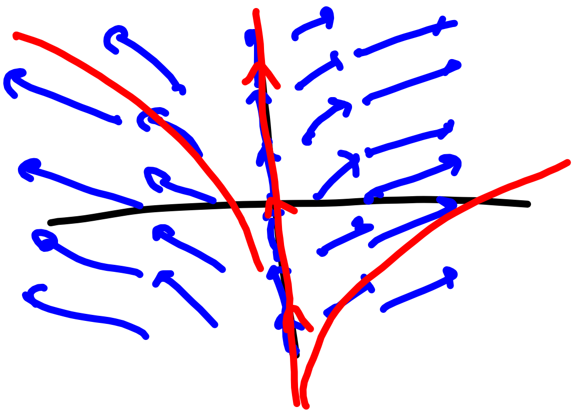


$$\underline{x}_0 = \begin{pmatrix} 1 \\ 1 \\ \sqrt{2} \end{pmatrix}$$

$$\nabla f = \begin{pmatrix} 2x \\ 2y \\ -2z \end{pmatrix}$$

$$\nabla f(x_0) = \begin{pmatrix} 2 \\ 2 \\ -2\sqrt{2} \end{pmatrix}$$

<u>FLOW LINES:</u> GIVEN A C^1 DIFFERENTIABLE VECTOR FIELD $F(\underline{x})$, A <u>FLOW LINE</u> IS A CURVE $\underline{x}(t) = g(t)$, SUCH THAT THE VELOCITY IS GIVEN BY $\underline{v}(t) = \underline{x}'(t) = F(\underline{x}(t))$



CHAIN RULE:

$$\mathbb{R}^n \xrightarrow{f} \mathbb{R}^m \xrightarrow{g} \mathbb{R}^p$$

ASSUME f IS DIFF. AT x, g IS DIFF AT $f(\underline{x})$. THEN $g \circ f$ IS DIFF. AT $\underline{x}$, AND

$$(g \circ f)'(\underline{x}) = g'(f(\underline{x})) \cdot f'(\underline{x})$$

MATRIX MULTIPLICATION.

EXAMPLE: SUPPOSE

$$f(x,y) = \begin{pmatrix} x^2+y^2 \\ x^2-y^2 \end{pmatrix}$$

$$g(u,v) = \begin{pmatrix} uv \\ u+v \end{pmatrix}$$

$$g'(u,v) = \begin{pmatrix} v & u \\ 1 & 1 \end{pmatrix}$$

$$f'(x,y) = \begin{pmatrix} 2x & 2y \\ 2x & -2y \end{pmatrix}.$$

$$f(g(u,v)) = \begin{pmatrix} u^2v^2 + (u+v)^2 \\ u^2v^2 - (u+v)^2 \end{pmatrix}$$

$$D(f\circ g) = \begin{pmatrix} 2uv^2 + 2(u+v) & 2u^2v + 2(u+v) \\ 2uv^2 - 2(u+v) & 2u^2v - 2(u+v) \end{pmatrix}$$

$$f'(g(u,v)) = \begin{pmatrix} 2uv & 2(u+v) \\ 2uv & -2(u+v) \end{pmatrix}.$$

$$g'(u,v) = \begin{pmatrix} v & u \\ 1 & 1 \end{pmatrix}$$

CHAIN RULE GIVES:

$$f'(g(u,v)) \cdot g'(u,v) = \begin{pmatrix} 2uv & 2(u+v) \\ 2uv & -2(u+v) \end{pmatrix} \cdot \begin{pmatrix} v & u \\ 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2uv^2 + 2(u+v) & 2u^2v + 2(u+v) \\ 2uv^2 - 2(u+v) & 2u^2v - 2(u+v) \end{pmatrix} \checkmark$$

<u>EXAMPLE:</u> $x = u^2 + v^2$

$y = e^{uv}.$

$u = t+1$

$v = e^t.$

$$\mathbb{R} \xrightarrow{f} \mathbb{R}^2 \xrightarrow{g} \mathbb{R}^2$$

$$t \mapsto \begin{pmatrix} t+1 \\ e^t \end{pmatrix}$$

$$\begin{pmatrix} u \\ v \end{pmatrix} \mapsto \begin{pmatrix} u^2+v^2 \\ e^{uv} \end{pmatrix}.$$

$$f'(t) = \begin{pmatrix} 1 \\ e^t \end{pmatrix}, \quad g' = \begin{pmatrix} 2u & 2v \\ ve^{uv} & ue^{uv} \end{pmatrix}$$

$$g'(f(t)) = \begin{pmatrix} 2(t+1) & 2e^t \\ e^t e^{(t+1)e^t} & (t+1)e^{(t+1)e^t} \end{pmatrix}$$

$$g'(f(t)) \cdot f'(t) = \begin{pmatrix} 2(t+1) & 2e^t \\ e^{t+(t+1)e^t} & (t+1)e^{(t+1)e^t} \end{pmatrix} \begin{pmatrix} 1 \\ e^t \end{pmatrix}$$

$$= \begin{pmatrix} 2(t+1) + 2e^{2t} \\ (t+2)e^{t+(1+t)e^t} \end{pmatrix}. \quad \square$$

SKETCH PROOF OF CHAIN RULE:

NOTE: WE DIDN'T CHECK THAT THE COMPOSITE FN. IS DIFF. AND JUST CALCULATED THE MATRIX OF PARTIAL DERIVATIVES OF THE COMPOSITE.

$g \circ f(x)$

$f(x+h) = f(x) + f'(x)\cdot h + \varepsilon(h)$

WHERE $\lim_{h\to 0} \frac{|\varepsilon(h)|}{|h|} = 0.$ $\Leftrightarrow$ f DIFF AT $\underline{x}$.

$g(y+\underline{k}) = g(y) + g'(y)\cdot k + \delta(\underline{k})$

WHERE $\lim_{k\to 0} \frac{|\delta(k)|}{|\underline{k}|} = 0$ $\Leftrightarrow$ g DIFF AT $y = f(x)$.

$$g(f(x+h)) = g(\underbrace{f(x)}_{y} + \underbrace{f'(x)\cdot h + \varepsilon(h)}_{k})$$

FIRST ORDER TAYLOR APPROX

$$= \boxed{g\circ f(x) + g'(f(x))\boxed{f'(x)\cdot h}} + \varepsilon(h) + \delta(f'(x)\cdot h + \varepsilon(h)).$$

ERROR TENDING TO 0 FASTER THAN h.

AS $h \to 0$, $\varepsilon(h) \to 0$ AND

$$\lim_{h\to 0} \frac{|\varepsilon(h)|}{|h|} = 0 \Rightarrow \lim_{h\to 0} \frac{|g'(f(x))\cdot \varepsilon(h)|}{|h|} = 0$$

BECAUSE $\|g'(f(x))\cdot \varepsilon(h)\|_2 \le \|g'(f(x))\|_2 \|\varepsilon(h)\|_2$

JUST A FIXED MATRIX, SO EXPANDS LENGTH OF VECTOR BY BDED AMOUNT.

MEANWHILE,

$$\delta(g'(f(x))\cdot h + \varepsilon(h)) = o(\|g'(f(x))\cdot h + \varepsilon(h)\|)$$
$$= o(\|h\|)$$

WHERE $A = o(B)$ IF A IS $\frac{A}{B} \to 0$ AS THE PARAMETER TENDS TO 0.

THIS SHOWS THE PROOF OF THE CHAIN RULE, SINCE $g\circ f(x)$ IS GIVEN BY ITS DEG. 2 TAYLOR APPROX + ERROR WHICH TENDS TO 0 FASTER THAN LINEAR. $\square$