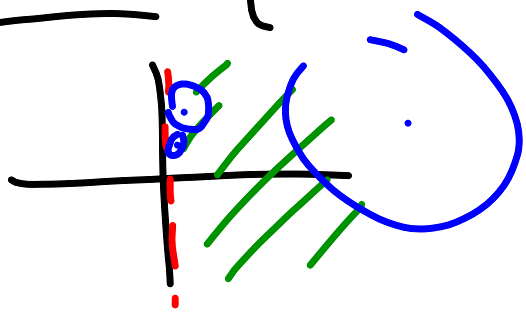
MAT 307 LECTURE 11:

THE DERIVATIVE IN SEVERAL DIMENSIONS

WE DEFINED:

OPEN SETS: ANY PT $\underline{x}$ IN THE SET S HAS A $\delta > 0$ SO THAT $B_\delta(\underline{x}) \subset S$.

Ex: $\{\underline{x} \in \mathbb{R}^2 : x_1 > 0\}$



CLOSED SET S: THIS MEANS S^c IS OPEN

THE BOUNDARY ∂S OF S.

IF THE LIMIT EXISTS, IT IS UNIQUE, SINCE, IF BOTH $\underline{y}_0, \underline{y}_1$ ARE LIMITS AT $\underline{x}_0$, THEN. FOR ANY $\varepsilon > 0$ WE CAN FIND $\delta > 0$ SO THAT IF $0 < \|\underline{x} - \underline{x}_0\| < \delta$ THEN

$$\|f(\underline{x}) - \underline{y}_0\| < \varepsilon, \quad \|f(\underline{x}) - \underline{y}_1\| < \varepsilon$$

SO, BY THE TRIANGLE INEQUALITY,

$$\|\underline{y}_0 - \underline{y}_1\| \leq \|\underline{y}_0 - f(\underline{x})\| + \|f(\underline{x}) - \underline{y}_1\| < 2\varepsilon.$$

SINCE THIS HOLDS FOR ALL $\varepsilon > 0$, $\underline{y}_0 = \underline{y}_1$. $\square$

PROOF:

$$\|f(\underline{x}) - \underline{y}\| = \sqrt{(f_1(x) - y_1)^2 + \dots + (f_m(\underline{x}) - y_m)^2}.$$

SUPPOSE THAT $\lim_{\underline{x} \to \underline{x}_0} f(\underline{x}) = \underline{y}$.

GIVEN $\varepsilon > 0$, CHOOSE $\delta > 0$ SO THAT,

IF $0 < \|\underline{x} - \underline{x}_0\| < \delta$ THEN

$\|f(\underline{x}) - \underline{y}\| < \varepsilon$. THIS IMPLIES

FOR EACH i, $|f_i(x) - y_i| < \varepsilon$

SINCE $|f_i(x) - y_i| \leq \sqrt{\sum (f_j(x) - y_j)^2}$.

SUPPOSE, INSTEAD, FOR EACH i THAT $\lim_{\underline{x} \to x_0} f_i(x) = y_i$.

CONTINUITY: A FUNCTION f IS CTS AT $\underline{x}_0$ IF

(a) $\underline{x}_0$ IS IN THE DOMAIN OF f

(b) $\lim_{\underline{x} \to \underline{x}_0} f(\underline{x}) = f(\underline{x}_0)$.

RMK: IF $D \subset \mathbb{R}^n$, $f: D \to \mathbb{R}^m$

THEN WE STILL DEFINE A NOTION OF LIMIT, FOR $\underline{x}_0 \in$ LIMIT PTS OF D

$\lim_{\underline{x} \to \underline{x}_0} f(\underline{x}) = \underline{y}$

THIS MEANS FOR ALL $\varepsilon > 0$ THERE EXISTS $\delta > 0$ SO THAT

IF $\underline{x} \in D$ AND

$0 < \|\underline{x} - \underline{x}_0\| < \delta$

THEN $\|f(\underline{x}) - \underline{y}\| < \varepsilon$.

BY A LIMIT POINT OF D WE MEAN A POINT x_0 SO THAT FOR ANY $\delta > 0$,

$B_\delta(x_0)$ CONTAINS A POINT OF D OTHER THAN $\underline{x}_0$.

THEOREM: A VECTOR FUNCTION IS CONTINUOUS AT A POINT IF AND ONLY IF THE COORDINATE FUNCTIONS ARE CONTINUOUS THERE.

PROOF: f CTS AT $\underline{x}_0 \iff$

$$\lim_{\underline{x} \to \underline{x}_0} f(\underline{x}) = f(\underline{x}_0)$$

$\iff$ FOR ALL $i = 1, 2, \ldots, m$,

$$\lim_{\underline{x} \to \underline{x}_0} f_i(\underline{x}) = f_i(\underline{x}_0)$$

$\iff f_i$ IS CONTINUOUS AT $\underline{x}_0$ FOR ALL $i = 1, 2, \ldots, m$.

$\square$

THEOREM: THE FUNCTIONS

$\mathbb{R}^2 \xrightarrow{S} \mathbb{R}$, $\mathbb{R}^2 \xrightarrow{M} \mathbb{R}$

$S(x,y) = x+y$, $M(x,y) = xy$

ARE CTS.

PROOF: $|x_1+y_1 - (x_2+y_2)|$

$\leq |x_1-x_2| + |y_1-y_2|$

$\leq \sqrt{2}\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$,

BY CAUCHY-SCHWARZ.

THUS IF $\|(x_1,y_1)-(x_2,y_2)\| < \frac{\varepsilon}{\sqrt{2}}$,

THEN $|S(x_1,y_1) - S(x_2,y_2)| < \varepsilon$.

THIS SHOWS S IS CTS.

THEOREM: IF $f: \mathbb{R}^n \to \mathbb{R}^m$, $g: \mathbb{R}^m \to \mathbb{R}^t$ ARE CTS THEN $g \circ f: \mathbb{R}^n \to \mathbb{R}^t$ IS CTS.

PROOF: LET $\underline{x}_0 \in \mathbb{R}^n$. LET $\underline{y}_0 = f(\underline{x}_0)$. GIVEN $\varepsilon > 0$ THERE IS A $\delta > 0$ SO THAT, IF

$\|\underline{y} - \underline{y}_0\| < \delta$ THEN

$\|g(\underline{y}) - g(\underline{y}_0)\| < \varepsilon$, BY CTY OF g.

BY CTY OF f AT $\underline{x}_0$, THERE IS A NUMBER $\gamma > 0$ SO THAT IF $\|\underline{x} - \underline{x}_0\| < \gamma$, THEN

$\|f(\underline{x}) - f(\underline{x}_0)\| < \delta$.

THUS, $\|\underline{x} - \underline{x}_0\| < \gamma \Rightarrow$

$\|f(\underline{x}) - f(\underline{x}_0)\| < \delta$

$\Longrightarrow \|g(f(\underline{x})) - g(f(\underline{x}_0))\| < \varepsilon$.

$\square$

COMBINING THE ABOVE THMS,

$\Pi(\underline{x})$ WHICH IS A POLYNOMIAL IN $\underline{x}$ IS CTS.

REMARK: IF $f: \mathbb{R}^1 \to \mathbb{R}^1$, f IS DIFF AT x_0 WITH DERIV. $f'(x_0)$ IF

$$\lim_{x \to x_0} \frac{f(x_0) - f(x)}{x_0 - x} = f'(x_0)$$

THERE IS NOT A WAY OF DIVIDING BY A VECTOR. IN HIGHER DIM., $x_0 - x$ IS A VECTOR, SO WE REWRITE THE LIMIT AS

$$\boxed{\lim_{x \to x_0} \frac{f(x) - f(x_0) - f'(x_0)(x - x_0)}{|x - x_0|} = 0.}$$

THIS EXPRESSION IS WHAT GENERALIZES TO HIGHER DIMENSIONS.

WE DON'T NEED A VECTOR LIMIT.

$f'(x_0)\cdot(x - x_0)$ IS CALLED "THE BEST LINEAR APPROX TO f AT x_0", IN THE SENSE THAT THE DIFF BETWEEN

$f(x) - f(x_0)$ AND $f'(x_0)(x - x_0)$

TENDS TO 0 FASTER THAN $x - x_0$.

THEOREM: IF A FUNCTION $f: \mathbb{R}^n \to \mathbb{R}$ IS DIFF. AT $\underline{x}_0$ THEN THE KTH COORDINATE OF ∇f IS $\frac{\partial f}{\partial x_k}(\underline{x}_0)$.

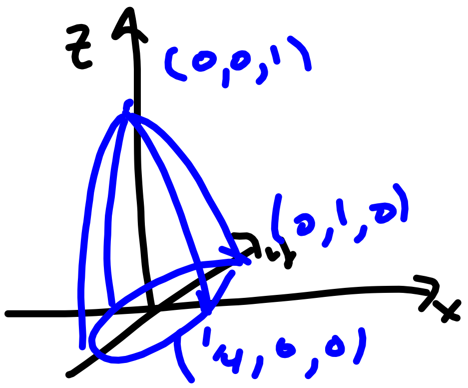
DEF'N: WE SAY $f: \mathbb{R}^n \to \mathbb{R}$ IS CONTINUOUSLY DIFFERENTIABLE ON AN OPEN SET D IF THE ENTRIES IN ∇f ARE CTS ON D $\Leftrightarrow$ IF THE PARTIAL DERIVS ARE CTS.

RECALL: IN 1 DIMENSION, THE TANGENT LINE TO $\begin{pmatrix} x \\ f(x) \end{pmatrix}$ AT $\begin{pmatrix} x_0 \\ f(x_0) \end{pmatrix}$ IS GIVEN BY THE BEST LINEAR APPROXIMATION:

$$\begin{pmatrix} x \\ f(x_0) + f'(x_0)(x - x_0) \end{pmatrix}.$$

EXAMPLE:

$$f(x,y) = 1 - 2x^2 - y^2$$



AT $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$, $z_0 = \frac{1}{4}$.

$$\nabla f = \begin{pmatrix} -4x \\ -2y \end{pmatrix}$$

$$\nabla f \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \end{pmatrix}$$

TANGENT PLANE: $\begin{pmatrix} x_0 \\ y_0 \\ \frac{1}{4} + \begin{pmatrix} -2 \\ -1 \end{pmatrix} \cdot (x - x_0, y - y_0) \end{pmatrix}$