RECALL FROM LAST WEDNESDAY:

A SMOOTH CURVE IS A FUNCTION $\underline{X}(t)$, $\underline{X}: \mathbb{R} \to \mathbb{R}^n$

$\underline{X}(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix}$ SUCH THAT $\underline{V}(t) = \underline{X}'(t) = \begin{pmatrix} x_1'(t) \\ \vdots \\ x_n'(t) \end{pmatrix}$

EXISTS AND $\|\underline{V}(t)\| \neq 0$ ALL t.

RECALL FROM LAST WEDNESDAY:

THEOREM: $\underline{x}(t), \underline{y}(t)$ DIFFERENTIABLE CURVES IN $\mathbb{R}^n$, $\varphi(t)$ SCALAR VALUED DIFF. FN. THEN

(1) $\frac{d}{dt}\left(\underline{x}(t)+\underline{y}(t)\right) = \underline{x}'(t)+\underline{y}'(t)$

(2) $\frac{d}{dt}\left(\varphi(t)\underline{x}(t)\right) = \varphi'(t)\underline{x}(t) + \varphi(t)\underline{x}'(t).$

(3) $\frac{d}{dt}\left(\underline{x}(t)\cdot\underline{y}(t)\right) = \underline{x}'(t)\cdot\underline{y}(t) + \underline{x}(t)\cdot\underline{y}'(t)$

(4) $\frac{d}{dt}\left(\underline{x}(\varphi(t))\right) = \underline{x}'(\varphi(t))\cdot\varphi'(t)$

(5) IN $\mathbb{R}^3$:

$\frac{d}{dt}\left(\underline{x}(t)\times\underline{y}(t)\right) = \underline{x}'(t)\times\underline{y}(t) + \underline{x}(t)\times\underline{y}'(t).$

(3): $\frac{d}{dt}\left(\underline{x}(t)\cdot\underline{y}(t)\right)$

$= \frac{d}{dt}\left(x_1(t)y_1(t) + \dots + x_n(t)y_n(t)\right)$

$= \Big((x_1'(t)y_1(t) + x_1(t)y_1'(t))$
$\quad + \dots + (x_n'(t)y_n(t) + x_n(t)y_n'(t)\Big)$

$= x_1'(t)y_1(t) + \dots + x_n'(t)y_n(t)$
$\quad + x_1(t)y_1'(t) + \dots + x_n(t)y_n'(t).$

$= \underline{x}'(t)\cdot\underline{y}(t) + \underline{x}(t)\cdot\underline{y}'(t).$

$$\frac{d}{dt}\left(\underline{x}(t) \times \underline{y}(t)\right).$$

$$\underline{x} \times \underline{y} = \det\begin{pmatrix} \underline{i} & \underline{j} & \underline{k} \\ x_1(t) & x_2(t) & x_3(t) \\ y_1(t) & y_2(t) & y_3(t) \end{pmatrix}$$

$$= \underline{i} \cdot \left(x_2(t)y_3(t) - x_3(t)y_2(t)\right)$$
$$- \underline{j} \cdot \left(x_1(t)y_3(t) - x_3(t)y_1(t)\right)$$
$$+ \underline{k}\left(x_1(t)y_2(t) - x_2(t)y_1(t)\right)$$

$$\frac{d}{dt}\underline{x} \times \underline{y} = \underline{i} \cdot \left(x_2'y_3 + x_2y_3' - x_3'y_2 - x_3y_2'\right)$$
$$+ \underline{j}\,(\ldots)$$
$$+ \underline{k}\,(\ldots)$$

NOTICE THE $\underline{i}$ COMPONENT IS THE SUM OF $\underline{i}$ COMPONENTS OF $\underline{x}' \times \underline{y} + \underline{x} \times \underline{y}'$. THE OTHER COMPONENTS ARE SIMILAR. $\square$

THE <u>GRAPH</u> OF A FUNCTION f IS DEFINED TO BE

$$\left\{ \begin{pmatrix} \underline{x} \\ f(\underline{x}) \end{pmatrix} : \underline{x} \in \text{DOMAIN} \right\}.$$

EXAMPLES OF QUADRIC SURFACES:

· HYPERBOLOID OF ONE SHEET:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-z^2=k, \qquad k>0$$

· HYPERBOLOID OF TWO SHEETS

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-z^2=k, \qquad k<0$$

· ELLIPTIC CONE:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-z^2=0.$$

HOMOGENEOUS, DEGREE 2.

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

· ELLIPSOID:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=k \qquad k>0.$$

DEFINE FURTHER PARTIAL

<u>DERIVATIVES:</u>

$$\frac{\partial^2}{\partial x \partial y} f(x,y) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y} f(x,y)\right)$$

$f(x,y) = x^2 y^3$;

$\frac{\partial f}{\partial y} = 3x^2y^2$;

$\frac{\partial^2 f}{\partial x \partial y} = 6xy^2$.

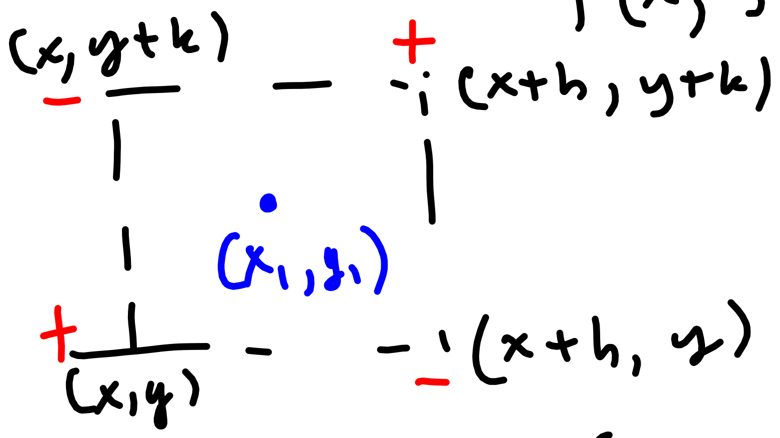
$\frac{\partial f}{\partial x} = 2xy^3$;

$\frac{\partial^2 f}{\partial y \partial x} = 6xy^2$; so

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}.$$

PROOF: DEFINE

$$F(h,k) = f(x+h, y+k) - f(x+h, y) - f(x, y+k) + f(x,y).$$



$$\Delta_{x,h} f(x,y) = f(x+h, y) - f(x,y).$$

$$\Delta_{y,k} f(x,y) = f(x, y+k) - f(x,y).$$

$$\Delta_{x,h}(\Delta_{y,k} f(x,y)) = \Delta_{x,h}\Big(\underbrace{f(x, y+k) - f(x,y)}_{G_k(x,y)}\Big)$$

$$= G_k(x+h, y) - G_k(x,y)$$

$$= f(x+h, y+k) - f(x, y+k) - f(x+h, y) + f(x,y)$$

$$= F_{h,k}(x,y).$$

BY SYMMETRY $= \Delta_{y,k}(\Delta_{x,h} f(x,y)).$

IN GENERAL WE CONSIDER

$f : \mathbb{R}^m \to \mathbb{R}^n$

$$f(\underline{x}) = \begin{pmatrix} f_1(\underline{x}) \\ \vdots \\ f_n(\underline{x}) \end{pmatrix}.$$

$$\frac{\partial f}{\partial x_i}(\underline{x}) = \begin{pmatrix} \frac{\partial f_1}{\partial x_i}(\underline{x}) \\ \vdots \\ \frac{\partial f_n}{\partial x_i}(\underline{x}) \end{pmatrix}$$

PARAMETRICALLY, THIS PLANE IS GIVEN BY

$$P(s,t) = \underline{X}\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} + s \cdot \frac{\partial \underline{x}}{\partial u}\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} + t \cdot \frac{\partial \underline{x}}{\partial v}\begin{pmatrix} u_0 \\ v_0 \end{pmatrix}.$$

E.G. AT $(u_0, v_0) = (1, \frac{\pi}{4})$

$\cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$

$\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$

$$f\begin{pmatrix} 1 \\ \pi/4 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ \pi/4 \end{pmatrix}; \quad \frac{\partial f}{\partial u}\begin{pmatrix} 1 \\ \pi/4 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}$$

$$\frac{\partial f}{\partial v}\begin{pmatrix} 1 \\ \frac{\pi}{4} \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 1 \end{pmatrix}$$

TANGENT PLANE:

$$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ \pi/4 \end{pmatrix} + s \cdot \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} + t \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 1 \end{pmatrix}.$$

TOPOLOGICAL NOTIONS IN HIGHER DIMENSIONS:

GIVEN $\delta > 0$, AND $\underline{x}_0 \in \mathbb{R}^n$, DEFINE BALL OF RADIUS δ AT $\underline{x}_0$:

$$B_\delta(\underline{x}_0) = \{ \underline{x} \in \mathbb{R}^n : \|\underline{x} - \underline{x}_0\| < \delta \}.$$

"δ NEIGHBORHOOD OF $\underline{x}_0$."

THEOREM: S IS CLOSED $\iff \partial S \subset S$.

PROOF: SUPPOSE S IS CLOSED. THEN S^c IS OPEN. LET $\underline{x} \in S^c$ AND LET $\delta > 0$ SO THAT $B_\delta(\underline{x}) \subset S^c$. THIS PROVES $\underline{x} \notin \partial S \Rightarrow \partial S \cap S^c = \emptyset$

$$\Rightarrow \partial S \subset S.$$

SUPPOSE INSTEAD, $\partial S \subset S$. WE MUST SHOW S^c IS OPEN.

GIVEN $\underline{x} \in S^c$, $\underline{x} \notin \partial S$, SO THERE IS SOME $\delta > 0$ SO THAT $B_\delta(\underline{x}) \subset S$ OR S^c.

SINCE $\underline{x} \in S^c$ IT FOLLOWS $B_\delta(\underline{x}) \subset S^c$.