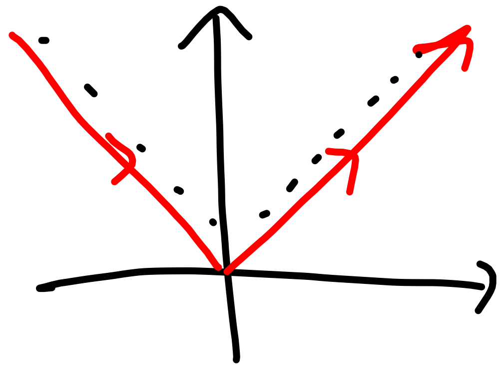
MAT 307: CURVES AND SURFACES.

EXAMPLE: $\underline{x}(t) = \begin{pmatrix} t \\ |t| \end{pmatrix}$



DOES NOT HAVE A DERIV. AT 0 AND A CORNER THERE.

$v(t) = \begin{pmatrix} 1 \\ \text{SIGN}(t) \end{pmatrix}$ IF $t \neq 0$,

$\Rightarrow \|\underline{v}(t)\| = \sqrt{2}$ CONSTANT $t \neq 0$.

TIME CHANGE: $\underline{x}_1(t) = \begin{pmatrix} t \cdot |t| \\ t^2 \end{pmatrix}$

$\underline{v}_1(t) = \dot{\underline{x}}_1(t) = \begin{cases} \text{IF } t \neq 0: \begin{pmatrix} 2|t| \\ 2t \end{pmatrix} \\ \text{IF } t = 0: \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{cases}$

$\lim_{t \to 0} \frac{t \cdot |t| - 0}{t} = \lim_{t \to 0} |t| = 0.$ DERIV AT $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ EXISTS, BUT $= 0$

STILL NOT SMOOTH!

PROOF: (1) EASY

(2) $$\varphi(t)\underline{x}(t) = \begin{pmatrix} \varphi(t)x_1(t) \\ \vdots \\ \varphi(t)x_n(t) \end{pmatrix}$$

$$\Longrightarrow \frac{d}{dt}\left(\varphi(t)\underline{x}(t)\right) = \begin{pmatrix} \frac{d}{dt}(\varphi(t)x_1(t)) \\ \vdots \\ \frac{d}{dt}(\varphi(t)x_n(t)) \end{pmatrix}$$

$$= \begin{pmatrix} \varphi'(t)x_1(t) + \varphi(t)x_1'(t) \\ \vdots \\ \varphi'(t)x_n(t) + \varphi(t)x_n'(t) \end{pmatrix}$$

$$= \varphi'(t)\begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix} + \varphi(t)\begin{pmatrix} x_1'(t) \\ \vdots \\ x_n'(t) \end{pmatrix}.$$

(v) $\frac{d}{dt}\left(\underline{x}\left(\varphi(t)\right)\right)$

$$= \frac{d}{dt}\begin{pmatrix} x_1(\varphi(t)) \\ x_2(\varphi(t)) \\ \vdots \\ x_n(\varphi(t)) \end{pmatrix} = \begin{pmatrix} \frac{d}{dt} x_1(\varphi(t)) \\ \vdots \\ \frac{d}{dt} x_n(\varphi(t)) \end{pmatrix}$$

$$= \begin{pmatrix} x_1'(\varphi(t)) \cdot \varphi'(t) \\ \vdots \\ x_n'(\varphi(t))\varphi'(t) \end{pmatrix} = \varphi'(t) \cdot \begin{pmatrix} x_1'(\varphi(t)) \\ \vdots \\ x_n'(\varphi(t)) \end{pmatrix}.$$

$$= \varphi'(t)\, \underline{x}'(\varphi(t)).$$

THE <u>VECTOR INTEGRAL</u> OF A VECTOR-VALUED FUNCTION $\underline{f}: \mathbb{R} \to \mathbb{R}^n$ IS DEFINED BY

$$\int \underline{f}(t)\,dt = \begin{pmatrix} \int f_1(t)\,dt \\ \vdots \\ \int f_n(t)\,dt \end{pmatrix}$$

<u>EXAMPLE:</u> $\int \begin{pmatrix} 1 \\ t^2 \end{pmatrix} dt = \begin{pmatrix} 1 \\ t^3/3 \end{pmatrix} + \underline{c}$

WHERE $\underline{c}$ IS A <u>CONSTANT VECTOR</u>.

THE LEVEL SETS OF A REAL VALUED FUNCTON $f: \mathbb{R}^n \to \mathbb{R}$ ARE THE SETS $S_k = \{ \underline{x} : f(\underline{x}) = k \}$.

THE GRAPH AND LEVEL SETS OF A REAL VALUED FUNCTION ARE TWO COMMON METHODS OF DEFINING A SURFACE.

PARTIAL DERIVATIVES: $f: \mathbb{R}^2 \to \mathbb{R}$.

$$\frac{\partial f}{\partial x}(x,y) = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}.$$

$$\frac{\partial f}{\partial y}(x,y) = \lim_{k \to 0} \frac{f(x,y+k) - f(x,y)}{k}.$$

TREAT ALL BUT ONE COORDINATE AS FIXED. DIFF. IN THE VARYING COORDINATE AS FOR A FUNCTION OF 1-VARIABLE.

CLAIRAUT'S THEOREM (MIXED PARTIALS COMMUTE):

LET $f: \mathbb{R}^2 \to \mathbb{R}$ BE CTS, AND SUCH THAT $f_x = \frac{\partial f}{\partial x}$, $f_y = \frac{\partial f}{\partial y}$, $f_{xy} = \frac{\partial^2 f}{\partial x \partial y}$, $f_{yx} = \frac{\partial^2 f}{\partial y \partial x}$ ARE CTS.

THEN $f_{xy} = f_{yx}$.

WE'LL SHOW:

$$\lim_{h\to 0}\lim_{k\to 0}\frac{F_{h,k}(x,y)}{h\cdot k} = f_{yx}$$

$$= f_{xy} = \lim_{k\to 0}\lim_{h\to 0}\frac{F_{h,k}(x,y)}{h\cdot k}.$$

BY THE MEAN VALUE THEOREM OF ONE VARIABLE CALCULUS,

$$\frac{1}{h}\Delta_{x,h}\Big(\underbrace{\frac{1}{k}\Delta_{y,k} f(x,y)}_{\frac{1}{k}G_k(x,y)}\Big)$$

FOR SOME x_1 BETWEEN $x, x+h$

$$\frac{1}{h}\left(\frac{1}{k}G_k(x+h,y) - \frac{1}{k}G_k(x,y)\right)$$

$$= \frac{1}{k}\frac{\partial}{\partial x}G_k(x_1,y)$$

$$= \frac{1}{k}\left(\frac{\partial}{\partial x}\left(f(x_1,y+k) - f(x_1,y)\right)\right)$$

$$= \frac{1}{k}\left(\frac{\partial f}{\partial x}(x_1,y+k) - \frac{\partial f}{\partial x}(x_1,y)\right).$$

BY THE MEAN VALUE THM APPLIED TO $\frac{\partial f}{\partial x}$ IN THE 2^{ND} VARIABLE, THERE IS y_2 BETWEEN $y, y+k$ SO THAT

$$\frac{1}{k}\left(\frac{\partial f}{\partial x}(x_1,y+k) - \frac{\partial f}{\partial x}(x_1,y)\right) = \frac{\partial^2 f}{\partial y\partial x}(x_1,y_1)$$

THUS $\frac{1}{hk}F_{h,k}(x,y) = \frac{\partial^2 f}{\partial y\partial x}(x_1,y_1).$

BY SYMMETRY, THERE IS ALSO A PT (x_2,y_2) IN THE SAME RECTANGLE SO THAT $\frac{1}{hk}F_{hk}(x,y) = \frac{\partial^2 f}{\partial x\partial y}(x_2,y_2).$

LET, SAY, $h=k$, AND LET $h\to 0$. THEN THE POINTS (x_1,y_1) AND (x_2,y_2) BOTH CONVERGE TO x,y. SINCE THE MIXED PARTIALS ARE CTS,

$$\lim_{h\to 0}\frac{1}{h^2}F_{h,h}(x,y) = f_{xy}(x,y) = f_{yx}(x,y)$$

$\square$

DEFINE THE TANGENT PLANE

PASSING THROUGH $\underline{X}\begin{pmatrix} u_0 \\ v_0 \end{pmatrix}$

TO BE THE PLANE THROUGH

THE POINT, CONTAINING

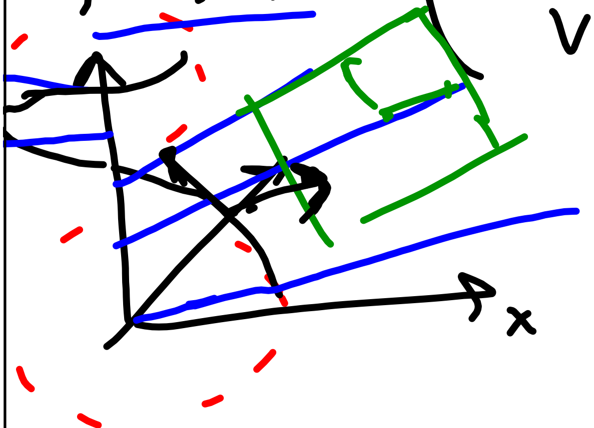
$$\frac{\partial \underline{x}}{\partial u}\begin{pmatrix} u_0 \\ v_0 \end{pmatrix}, \frac{\partial \underline{x}}{\partial v}\begin{pmatrix} u_0 \\ v_0 \end{pmatrix}$$

AS VECTORS.

EXAMPLE: A HELICOID SURFACE

IS PARAMETERIZED BY

$$f(u,v) = \begin{pmatrix} u\cos v \\ u\sin v \\ v \end{pmatrix}$$



TANGENT PLANE:

$$\frac{\partial f}{\partial u} = \begin{pmatrix} \cos v \\ \sin v \\ 0 \end{pmatrix}$$

$$\frac{\partial f}{\partial v} = \begin{pmatrix} -u\sin v \\ u\cos v \\ 1 \end{pmatrix}$$

MORE EXAMPLES OF QUADRIC SURFACES (PARAMETRIC FORM):

ELLIPTIC CONE:

$$\underline{X}\begin{pmatrix} u \\ v \end{pmatrix} = u\,(a\cos v,\ b\sin v,\ 1)$$

ELLIPSOID:

$$\underline{X}(u,v) = \begin{pmatrix} a \cos u \sin v \\ b \cos u \cos v \\ c \sin u \end{pmatrix}$$

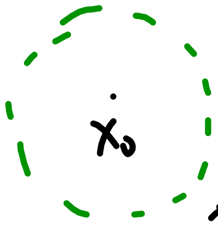
$$\left(\frac{a\cos u \sin v}{a}\right)^2 + \left(\frac{b\cos u \cos v}{b}\right)^2 + \left(\frac{c\sin u}{c}\right)^2$$

$$= \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2$$

$$= \underbrace{\cos^2 u \sin^2 v + \cos^2 u \cos^2 v} + \sin^2 u$$

$$= \cos^2 u + \sin^2 u$$

$$= 1.$$



$\|\underline{x} - \underline{x}_0\| < \delta$

$B_\delta(\underline{x}_0)$

<u>DEFINITION:</u> A SET S WITH A POINT $\underline{x} \in S$ HAS $\underline{x}$ AS AN <u>INTERIOR POINT</u>

IF THERE IS A $\delta > 0$ SUCH THAT $B_\delta(\underline{x}) \subset S$.

<u>EX:</u> $\underline{0}$ IS AN INTERIOR POINT OF $\{\underline{x} : \|\underline{x}\| < 1\}$.

WE SAY S IS <u>OPEN</u> IF EVERY $\underline{x} \in S$ IS AN INTERIOR POINT.

WE SAY S IS <u>CLOSED</u> IF $S^c = \{\underline{x} \in \mathbb{R}^n : \underline{x} \notin S\}$ IS OPEN.

WE SAY $\underline{x}$ IS A <u>BOUNDARY POINT</u> OF S IF, FOR EVERY $\delta > 0$,

$B_\delta(\underline{x}) \cap S \neq \emptyset$, <u>AND</u>

$B_\delta(\underline{x}) \cap S^c \neq \emptyset$.

∂S = BOUNDARY POINTS.

PART OF BALL OUTSIDE, INSIDE S.