MAT 307: MULTIVAR. CALCULUS AND LINEAR ALGEBRA

ROBERT HOUGH

RECITATIONS: M 6:05 – 7 PM
Th 4:45 – 5:40 PM
EARTH & SPACE 69.

OFFICE HOURS: W 6-7 PM IN MLC
F 9–11 AM IN MATH 4-118

COURSE WEBSITE:
math.stonybrook.edu/~rdhough/mat307_fall20/math307.html

SEE THIS FOR HOMEWORK, CLASS SCHEDULE.
WEEKLY HW DUE IN CLASS MONDAYS.

GRADING: HW 20%
MIDTERMS I, II 20% EACH
FINAL EXAM 40%.

TOPICS: VECTOR SPACES, BASES AND CHANGE OF BASES, DIFFERENTIATION AND INTEGRATION IN SEVERAL DIMENSIONS, VECTOR FIELDS, GREEN'S THM, GAUSS'S THM, STOKE'S THM.

THE COURSE WILL GIVE MATHEMATICAL PROOFS, GOOD PRACTICE IN WRITING PROOFS.

VECTORS AND VECTOR SPACES:
FIRST EXAMPLE: $\mathbb{R}^n$
A VECTOR IN $\mathbb{R}^n$ IS AN n-TUPLE $\underline{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

VECTORS ARE ADDED COORDINATE WISE

$$\underline{x} + \underline{y} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{pmatrix}.$$

$r \in \mathbb{R}$ IS CALLED A SCALAR.

$$r \cdot \underline{x} = \begin{pmatrix} r x_1 \\ \vdots \\ r x_n \end{pmatrix}.$$

VECTOR SPACE PROPERTIES: $\in$ "BELONGS TO"

1. $r \cdot \underline{x} + s \cdot \underline{x} = (r+s) \cdot \underline{x}$, $r, s \in \mathbb{R}$, $\underline{x} \in \mathbb{R}^n$
2. $r \cdot (\underline{x} + \underline{y}) = r \cdot \underline{x} + r \cdot \underline{y}$, $r \in \mathbb{R}$, $\underline{x}, \underline{y} \in \mathbb{R}^n$
3. $r \cdot (s \underline{x}) = (rs) \cdot \underline{x}$ $r, s \in \mathbb{R}$, $\underline{x} \in \mathbb{R}^n$
4. $\underline{x} + \underline{y} = \underline{y} + \underline{x}$ $\underline{x}, \underline{y} \in \mathbb{R}^n$
5. $(\underline{x} + \underline{y}) + \underline{z} = \underline{x} + (\underline{y} + \underline{z})$
6. $\underline{x} + \underline{0} = \underline{x}$ $\underline{0} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$
7. $\underline{x} + -\underline{x} = \underline{0}$ $-x = \begin{pmatrix} -x_1 \\ \vdots \\ -x_n \end{pmatrix}$
8. $1 \cdot \underline{x} = \underline{x}$.
9. $0 \cdot \underline{x} = \underline{0}$.

PROOF OF (2) $\underline{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, $\underline{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$

$$r\underline{x} = \begin{pmatrix} r x_1 \\ \vdots \\ r x_n \end{pmatrix}, \quad r\underline{y} = \begin{pmatrix} r y_1 \\ \vdots \\ r y_n \end{pmatrix}$$

$$r\underline{x} + r\underline{y} = \begin{pmatrix} r(x_1 + y_1) \\ \vdots \\ r(x_n + y_n) \end{pmatrix} = r(\underline{x} + \underline{y}). \quad \square$$

THE SUM

$$a_1 \underline{x}_1 + a_2 \underline{x}_2 + \dots + a_n \underline{x}_n$$

WHERE $a_1, \dots, a_n$ ARE SCALARS AND $\underline{x}_1, \dots, \underline{x}_n$ ARE VECTORS IS CALLED A LINEAR COMBINATION.

STANDARD BASIS VECTORS.

IN $\mathbb{R}^2$: $\underline{e}_1 = \begin{pmatrix}1\\0\end{pmatrix}$, $\underline{e}_2 = \begin{pmatrix}0\\1\end{pmatrix}$

IN $\mathbb{R}^3$: $\underline{e}_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}$, $\underline{e}_2 \begin{pmatrix}0\\1\\0\end{pmatrix}$, $\underline{e}_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}$.

IN $\mathbb{R}^n$: $\underline{e}_1 = \begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix}$, $\underline{e}_2 = \begin{pmatrix}0\\1\\0\\\vdots\\0\end{pmatrix}, \ldots, \underline{e}_n = \begin{pmatrix}0\\\vdots\\0\\1\end{pmatrix}$

$\underline{e}_j$ HAS 1 IN JTH COORDINATE

IN PHYSICS:

IN $\mathbb{R}^2$ FREQUENTLY

$\underline{i} = \underline{e}_1 = \begin{pmatrix}1\\0\end{pmatrix}$, $\underline{j} = \underline{e}_2 = \begin{pmatrix}0\\1\end{pmatrix}$

IN $\mathbb{R}^3$:

$\underline{i} = \begin{pmatrix}1\\0\\0\end{pmatrix}$, $\underline{j} = \begin{pmatrix}0\\1\\0\end{pmatrix}$, $\underline{k} = \begin{pmatrix}0\\0\\1\end{pmatrix}$.

THE LENGTH OF A VECTOR

$\underline{x} = \begin{pmatrix}x_1\\\vdots\\x_n\end{pmatrix} \in \mathbb{R}^n$ IS "EUCLIDEAN NORM".

$$|\underline{x}| = \sqrt{x_1^2 + \ldots + x_n^2}.$$

THE DISTANCE FROM $\underline{x}$ TO $\underline{y}$ IS

$$d(\underline{x}, \underline{y}) = |\underline{x} - \underline{y}| = \sqrt{(x_1-y_1)^2 + \ldots + (x_n - y_n)^2}$$

IF $r \in \mathbb{R}$, $\underline{x} \in \mathbb{R}^n$,

$$|r \cdot \underline{x}| = \sqrt{(r x_1)^2 + \ldots + (r x_n)^2}$$

$$= \sqrt{r^2 (x_1^2 + \ldots + x_n^2)}$$

$$= |r| \cdot \sqrt{x_1^2 + \ldots + x_n^2} = \boxed{|r| \cdot |\underline{x}|}$$

THE MIDPOINT OF TWO VECTORS $\underline{x}, \underline{y}$ IS

$$\underline{m} = \frac{1}{2}(\underline{x} + \underline{y}).$$

$$|\underline{m} - \underline{x}| = \left| \frac{1}{2}(\underline{x} + \underline{y}) - \underline{x} \right| = \left| -\frac{1}{2}\underline{x} + \frac{1}{2}\underline{y} \right|$$

$$= \frac{1}{2}|\underline{x} - \underline{y}|.$$

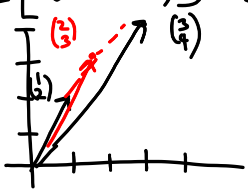
$$|\underline{m} - \underline{y}| = \left| \frac{1}{2}(\underline{x} + \underline{y}) - \underline{y} \right|$$

$$= \left| \frac{1}{2}(\underline{x} - \underline{y}) \right|$$

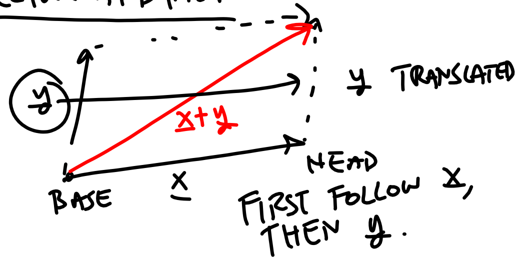
$$= \frac{1}{2}|\underline{x} - \underline{y}| = |\underline{m} - \underline{x}|.$$

EXAMPLE: $\underline{x} = \begin{pmatrix}1\\2\end{pmatrix}$, $\underline{y} = \begin{pmatrix}3\\4\end{pmatrix}$

$$\underline{m} = \frac{1}{2}\left[\begin{pmatrix}1\\2\end{pmatrix} + \begin{pmatrix}3\\4\end{pmatrix}\right] = \frac{1}{2}\left[\begin{pmatrix}4\\6\end{pmatrix}\right] = \begin{pmatrix}2\\3\end{pmatrix}.$$



PARALLELOGRAM RULE FOR VECTOR ADDITION:



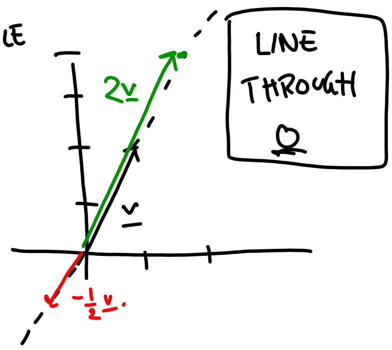
TWO VECTORS ARE CONSIDERED EQUIVALENT IF ONE CAN BE TRANSLATED ONTO THE OTHER.

THE COORDINATES IN $\mathbb{R}^n$ DESCRIBE THE DISPLACEMENT FROM BASE TO HEAD.

THE SCALAR MULTIPLES

$t \cdot \underline{v}$, $t \in \mathbb{R}$ FORM A LINE IN THE DIRECTION $\underline{v}$, IF $\underline{v} \neq \underline{0}$.

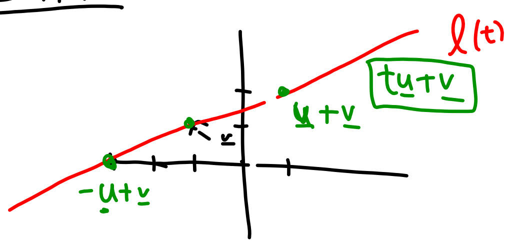
THE MULTIPLE $t\underline{v}$ IS IN THE SAME DIRECTION $\Leftrightarrow t > 0$.



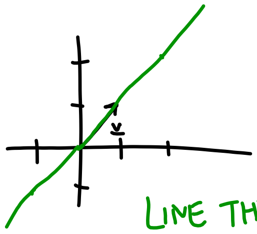
THE GENERAL PARAMETRIC FORM OF A LINE IS

$\ell(t) = t\underline{u} + \underline{v}$ WHERE $\underline{u}, \underline{v}$ VECTORS. $t \in \mathbb{R}$.

EXAMPLE: $\underline{u} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \underline{v} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$



EXAMPLE: $\underline{u} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \underline{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$



$t = -1$ GIVES $t\underline{u}+\underline{v} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

LINE THRU $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

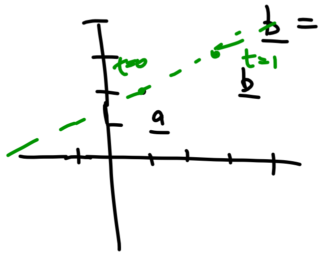
TWO POINTS DETERMINE A LINE.

TO OBTAIN A LINE THROUGH $\underline{a}, \underline{b}$, $\ell(t) = (1-t)\underline{a} + t \cdot \underline{b}$.

$\ell(0) = \underline{a}, \quad \ell(1) = \underline{b}$.

$$\boxed{\ell(t) = \underline{a} + t \cdot (\underline{b} - \underline{a})}.$$

$\underline{b} - \underline{a} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ $\quad \underline{a} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\underline{b} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$

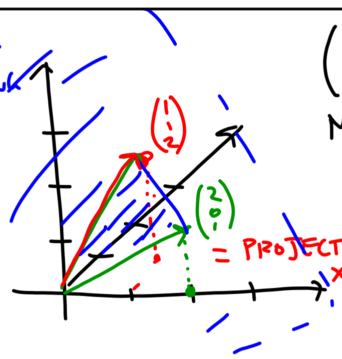


THE $\underline{\text{PLANE}}$ THROUGH THE ORIGIN CONTAINING VECTORS $\underline{v}_1$ AND $\underline{v}_2$ HAS PARAMETRIC FORM $s\underline{v}_1 + t\underline{v}_2$, $s, t \in \mathbb{R}$.

HERE WE REQUIRE $\underline{v}_1, \underline{v}_2$ TO BE LINEARLY INDEPENDENT

/// PLANE THROUGH $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ CONTAINING $\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$

$\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$ NEITHER SCALAR MULT OTHER



$\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$ = PROJECTION INTO xy PLANE.

TWO PARAMETERS SINCE PLANE IS 2 DIM'L, 2 VECTORS GIVE TWO DIRECTIONS.

THE GENERAL PARAMETRIC REPRESENTATION OF A PLANE IS

$$\{ t_1 \underline{u}_1 + t_2 \underline{u}_2 + \underline{v} \}$$

OFFSET.

$t_1, t_2 \in \mathbb{R}$, $\underline{u}_1, \underline{u}_2$ LIN. INDEPENDENT.

DOT PRODUCT:

GIVEN TWO VECTORS

$$\underline{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad \underline{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix},$$

$$\underline{x} \cdot \underline{y} = x_1 y_1 + x_2 y_2 + \dots + x_n y_n.$$

THE DOT PRODUCT SATISFIES THE FOLLOWING PROPERTIES:

1. POSITIVITY: $\underline{x} \cdot \underline{x} \geq 0$, $\underline{x} \cdot \underline{x} > 0$ IF $\underline{x} \neq 0$.
2. SYMMETRY: $\underline{x} \cdot \underline{y} = \underline{y} \cdot \underline{x}$
3. ADD: $(\underline{x} + \underline{y}) \cdot \underline{z} = \underline{x} \cdot \underline{z} + \underline{y} \cdot \underline{z}$
4. HOMOGENEOUS: $(r\underline{x}) \cdot \underline{y} = r \cdot (\underline{x} \cdot \underline{y})$.

PROOF: (1) $\underline{x} \cdot \underline{x} = x_1^2 + x_2^2 + \dots + x_n^2$.

THIS IS ≥ 0 SINCE $x_i^2 \geq 0$.

THIS $= 0$ IF AND ONLY IF $x_i = 0$ ALL i, SO $\underline{x} = \underline{0}$.

(2) SYMMETRY:

$$\underline{x} \cdot \underline{y} = x_1 y_1 + x_2 y_2 + \dots + x_n y_n$$
$$= y_1 x_1 + y_2 x_2 + \dots + y_n x_n$$
$$= \underline{y} \cdot \underline{x}.$$

(3.) ADDITIVITY: $(\underline{x}+\underline{y})\cdot\underline{z}$

$$= \begin{pmatrix} x_1+y_1 \\ \vdots \\ x_n+y_n \end{pmatrix} \cdot \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix}$$

$$= (x_1+y_1)z_1 + (x_2+y_2)z_2 + \dots + (x_n+y_n)z_n.$$

$$= x_1z_1 + y_1z_1 + x_2z_2 + y_2z_2 + \dots + x_nz_n + y_nz_n$$

$$= x_1z_1 + \dots + x_nz_n + y_1z_1 + \dots + y_nz_n$$

$$= \underline{x}\cdot\underline{z} + \underline{y}\cdot\underline{z}$$

(4.) HOMOGENEITY: $r \in \mathbb{R}$, $\underline{x},\underline{y} \in \mathbb{R}^n$.

$$(r\underline{x})\cdot\underline{y} = \begin{pmatrix} rx_1 \\ \vdots \\ rx_n \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

$$= rx_1y_1 + rx_2y_2 + \dots + rx_ny_n.$$

$$= r(x_1y_1 + \dots + x_ny_n)$$

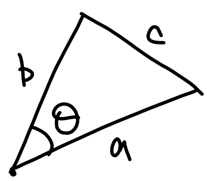
$$= r\cdot(\underline{x}\cdot\underline{y}). \quad \square$$

THE LENGTH $|\underline{x}|$ OF A VECTOR CAN BE EXPRESSED

$$|\underline{x}| = \sqrt{x_1^2 + \dots + x_n^2}.$$

$$= \sqrt{\underline{x}\cdot\underline{x}}.$$

ANGLES: RECALL FROM HIGH SCHOOL TRIGONOMETRY

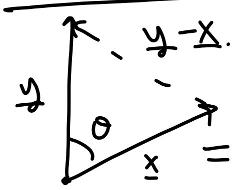


LAW OF COSINES:

$$c^2 = a^2 + b^2 - 2ab\cos\theta$$

0 IF $\theta = 90°$.

USING THE DOT PRODUCT: ①



$$\boxed{|\underline{y}-\underline{x}|^2 = |\underline{x}|^2 + |\underline{y}|^2 - 2|\underline{x}||\underline{y}|\cos\theta.}$$

$$= (\underline{y}-\underline{x})\cdot(\underline{y}-\underline{x})$$

$$= \underline{y}\cdot\underline{y} - \underline{y}\cdot\underline{x} - \underline{x}\cdot\underline{y} + \underline{x}\cdot\underline{x}$$

②

$$= \boxed{|\underline{y}|^2 + |\underline{x}|^2 - 2\,\underline{x}\cdot\underline{y}.}$$

$$\underline{x}\cdot\underline{y} = |\underline{x}|\cdot|\underline{y}|\cos\theta.$$

NOTE $|\cos\theta| \le 1$ SO

$$\underline{x}\cdot\underline{y} = |\underline{x}|\cdot|\underline{y}|\cdot\cos\theta$$

SATISFIES THE CAUCHY-SCHWARZ INEQUALITY $|\underline{x}\cdot\underline{y}| \le |\underline{x}|\cdot|\underline{y}|$

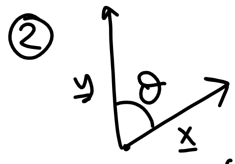
EQUALITY IF AND ONLY IF $\underline{x},\underline{y}$ ON SAME LINE $\iff$ $\cos\theta = \pm 1$.

FURTHER CONCLUSIONS:

① IF $\underline{x}\cdot\underline{y}=0$ THEN THE PYTHAGOREAN THM HOLDS:

$$|\underline{x}-\underline{y}|^2 = |\underline{x}|^2 + |\underline{y}|^2.$$

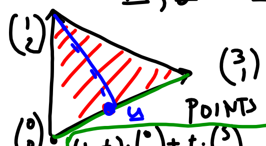
$$\Leftrightarrow \theta = 90^\circ.$$

② 

$$\theta = \cos^{-1}\left(\frac{\underline{x}\cdot\underline{y}}{|\underline{x}|\cdot|\underline{y}|}\right)$$

IF $\underline{x},\underline{y} \neq \underline{0}$.

EXAMPLE:



$(1-s)\cdot\begin{pmatrix}1\\2\end{pmatrix}+s\cdot\underline{u}$, $0\le s\le 1$

POINTS IN A:

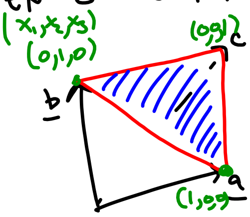
$\boxed{(1-t)\cdot\begin{pmatrix}0\\0\end{pmatrix}+t\cdot\begin{pmatrix}3\\1\end{pmatrix},\ 0\le t\le 1}$

ALL POINTS IN TRIANGLE:

$$(1-s)\cdot\begin{pmatrix}1\\2\end{pmatrix}+s\cdot\underline{u}$$
$$=(1-s)\cdot\begin{pmatrix}1\\2\end{pmatrix}+s\left((1-t)\cdot\begin{pmatrix}0\\0\end{pmatrix}+t\cdot\begin{pmatrix}3\\1\end{pmatrix}\right)$$
$$=(1-s)\begin{pmatrix}1\\2\end{pmatrix}+s(1-t)\cdot\begin{pmatrix}0\\0\end{pmatrix}+t\cdot\begin{pmatrix}3\\1\end{pmatrix}$$

$0\le s\le 1,\ 0\le t\le 1$

GIVEN 3 VECTORS $\underline{a},\underline{b},\underline{c}$



$/\!/\!/ = x_1\underline{a} + x_2\underline{b} + x_3\underline{c}$

WHERE $0\le x_i\le 1$

$x_1+x_2+x_3=1$

$= x_1\underline{a} + x_2\underline{b} + (1-x_1-x_2)\underline{c}.$