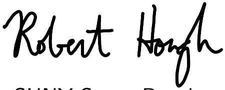
Random walk on unipotent groups



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Outline

Random walk on a group

Figure: Persi Diaconis

Set-up

- G a locally compact (finite) group
- $\mathscr{P}(G)$ the set of Borel probability measures on G
- For $\mu, \nu \in \mathscr{P}(G)$, $f \in C_c(G)$,

$$\langle f, \mu * \nu \rangle = \int_G \int_G f(xy) d\mu(x) d\nu(y)$$

- Consider, for $\mu \in \mathscr{P}(G)$, the large N behavior of μ^{*N} as a weak-* limit in one of several function spaces, e.g. $L^\infty(G)$, Lipschitz functions, Sobolev spaces, etc., and also the growth of $\text{supp}(\mu^{*N})$
- We seek quantitative statements, e.g. a rate of convergence.

Example: riffle shuffling

Let $N > 1$ and consider the following random walk on the symmetric group $\mathfrak{S}_N$ (Gilbert-Shannon-Reeds)

- μ is the distribution on $\mathfrak{S}_N$ given by
  - Choose $1 \leq n \leq N$ according to the binomial distribution $\mathbf{P}(n) = \frac{\binom{N}{n}}{2^N}$
  - Conditioned on the value of n, the measure is uniform over all permutations which preserve the relative order of the first n and last $N - n$ cards
- Convergence to uniform is observed after $\frac{3}{2}\log_2 N + O(1)$ steps in the total variation (L^1) metric [1], [2].

Example: groups of moderate growth

Let $m \geq 2$ and let $\mathbb{H}(\mathbb{Z}/m\mathbb{Z}) = \begin{pmatrix} 1 & \mathbb{Z}/m\mathbb{Z} & \mathbb{Z}/m\mathbb{Z} \\ 0 & 1 & \mathbb{Z}/m\mathbb{Z} \\ 0 & 0 & 1 \end{pmatrix}$. Let U be uniform measure on $\mathbb{H}(\mathbb{Z}/m\mathbb{Z})$ and let μ be uniform measure on

$$S = \left\{ I_3, \begin{pmatrix} 1 & \pm 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \pm 1 \\ 0 & 0 & 1 \end{pmatrix} \right\}.$$

Theorem (Diaconis-Saloff-Coste, '94)

There are constants a, b, a', b' such that

$$a' e^{-b' N/m^2} \leq \left\| \mu^{*N} - U \right\|_{\mathrm{TV}(\mathbb{Z}/m\mathbb{Z})} \leq a e^{-bN/m^2}.$$

Example: simple random walk on $\mathbb{Z}$

Let μ be the measure which assigns ± 1 equal probability $\frac{1}{2}$.

$$\mu^{*2N}(2k) = \frac{1}{2^{2N}}\binom{2N}{N+k}$$
$$= 2\frac{\exp\left(-\frac{k^2}{N}\right)}{\sqrt{2\pi(2N)}}\left(1 + O\left(N^{-\frac{1}{2}}\right)\right) + O_A\left(N^{-A}\right).$$

Outline

Upper triangular walks

Let $N_n(\mathbb{Z})$, $n \geq 3$ be the upper triangular group of $n \times n$ matrices

$$N_n(\mathbb{Z}) = \begin{pmatrix} 1 & \mathbb{Z} & \mathbb{Z} & \cdots & \mathbb{Z} \\ 0 & 1 & \mathbb{Z} & \cdots & \mathbb{Z} \\ \vdots & & \ddots & \ddots & \vdots \\ & & & 1 & \mathbb{Z} \\ 0 & \cdots & 0 & 0 & 1 \end{pmatrix}_{n \times n} .$$

- $Z_{1,n}$ denotes the upper right corner (central) coordinate.
- M_j is the matrix with 1 at jth position in the first super-diagonal, $M_j = I + e_j \otimes e_{j+1}$.
- Measure $\mu_n \in \mathscr{M}(N_n(\mathbb{Z}))$ is uniform on the set $\{I_n, M_1^{\pm 1}, \ldots, M_{n-1}^{\pm 1}\}$.

Central coordinate mixing

Theorem (Diaconis-H., 2015)

Let $n \geq 3$. There is $C = C(n) > 0$ such that, as prime $p \to \infty$,

$$\sum_{x \bmod p} \left| \mu_n^{*N}(Z_{1,n} \equiv x \bmod p) - \frac{1}{p} \right| \ll \exp\left(-C \frac{N}{p^{\frac{2}{n-1}}} \right).$$

See work of Peres and Sly [11] for results in the case p is fixed and $n \to \infty$.

The central coordinate

Let $M : \mathbb{Z}^{n-1} \to N_n(\mathbb{Z})$ be the map

$$M : \mathbb{Z}^{n-1} \ni v = \begin{pmatrix} v^{(1)} \\ v^{(2)} \\ \vdots \\ v^{(n-1)} \end{pmatrix} \mapsto \begin{pmatrix} 1 & v^{(1)} & 0 & \cdots & 0 \\ 0 & 1 & v^{(2)} & 0 & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ & 0 & 0 & 1 & v^{(n-1)} \\ 0 & \cdots & & 0 & 1 \end{pmatrix}.$$

Given sequence of vectors $\underline{v} = \{v_i\}_{i=1}^N \in \left(\mathbb{Z}^{n-1}\right)^N$ the central coordinate satisfies the product rule

$$Z_{1,n}\left(\prod_{i=1}^N M(v_i)\right) = \sum_{1\le i_1<i_2<\ldots<i_{n-1}\le N} v_{i_1}^{(1)} v_{i_2}^{(2)} \cdots v_{i_{n-1}}^{(n-1)}.$$

The central coordinate

Write

$$Z_n^N = \sum_{1 \le i_1 < i_2 < \ldots < i_{n-1} \le N} e_{i_1}^{(1)} \otimes \cdots \otimes e_{i_{n-1}}^{(n-1)}.$$

$Z_{n,\mu}^N$ is the distribution of Z_n^N evaluated on N vectors v_i drawn i.i.d. from μ, which is uniform on $\{0; \pm e_j : 1 \le j \le n-1\}$.

The central coordinate

- Since the central coordinate is a polynomial of degree $n - 1$, it has a distribution at scale $N^{\frac{n-1}{2}}$, which suggests the mixing time of $p^{\frac{2}{n-1}}$ of the theorem.
- The challenge of the theorem is in demonstrating the uniformity of distribution at finer scales, as in a local limit theorem.
- We perform the decomposition on scales using Fourier analysis.

Cauchy-Schwarz and Plancherel give

$$\sum_{x \bmod p} \left| Z_{n,\mu}^N(x) - \frac{1}{p} \right| \leq \left(\sum_{0 \not\equiv \xi \bmod p} \left| \hat{Z}_{n,\mu}^N \left(\frac{\xi}{p} \right) \right|^2 \right)^{\frac{1}{2}}$$

where

$$\hat{Z}_{n,\mu}^N(\alpha) = \sum_{m \in \mathbb{Z}} e^{2\pi i \alpha m} Z_{n,\mu}^N(m).$$

We show there exists constant $C(n) > 0$ such that for all $N > 0$ and all $0 < |\xi| \leq \frac{1}{2}$

$$\left| \hat{Z}_{n,\mu}^N(\xi) \right| \ll \exp\left(-CN|\xi|^{\frac{2}{n-1}} \right).$$

The group action

- $C_2 = \mathbb{Z}/2\mathbb{Z}$. C_2^{n-2} acts on blocks of vectors of length 2^{n-2} with the jth factor from C_2^{n-2}, $j \geq 1$ switching the relative order of the first 2^{j-1} and second 2^{j-1} indices.
- Thus, for instance, in case $n = 5$, if $\underline{x} = x_1x_2x_3x_4x_5x_6x_7x_8$,

$$\begin{aligned} \tau_1\underline{x} &= x_2x_1x_3x_4x_5x_6x_7x_8 \\ \tau_2\underline{x} &= x_3x_4x_1x_2x_5x_6x_7x_8 \\ \tau_1\tau_3\underline{x} = \tau_3\tau_1\underline{x} &= x_5x_6x_7x_8x_2x_1x_3x_4 \\ \tau_1\tau_2\tau_3\underline{x} &= x_5x_6x_7x_8x_3x_4x_2x_1. \end{aligned}$$

The group action

- Given $\frac{1}{p} \leq |\xi| \leq \frac{1}{2}$, let $k^{\frac{n-1}{2}} \asymp \frac{1}{|\xi|}$.
- Let $N' = \left\lfloor \frac{N}{k2^{n-2}} \right\rfloor$.
- $G_k = (C_2^{n-2})^{N'}$ acts on sequences of vectors $\underline{v} \in (\mathbb{Z}^{n-1})^N$ with jth factor acting on the jth block of length $k2^{n-2}$ as in the previous slide, moving blocks of length k together.
- Set

$$\chi_k(\xi, \underline{v}) = \mathbf{E}_{\underline{\tau} \in G_k}\left[e^{2\pi i \xi Z_n^N(\underline{\tau} \cdot \underline{v})}\right].$$

Although G_k is a product group, for $n \geq 5$, $\chi_k(\xi, \underline{v})$ does not factor through the group product structure. To correct this we apply the Gowers-Cauchy-Schwarz inequality, writing $G_k = \left(C_2^{n-2}\right)^{N'} = \left(C_2^{N'}\right)^{n-2}$.

$$\begin{aligned} &\left|\chi_k(\xi, \underline{v})\right|^{2^{n-2}} \\ &\leq \mathbf{E}_{\underline{\tau}_1, \underline{\tau}_1' \in C_2^{N'}} \cdots \mathbf{E}_{\underline{\tau}_{n-2}, \underline{\tau}_{n-2}' \in C_2^{N'}} \left[e_{-\xi}\left(\sum_{S \subset [n-2]} (-1)^{n-2-|S|} Z_n^N(\underline{\tau}_S \cdot \underline{v}) \right) \right] \\ &= \mathbf{E}_{\underline{\tau}, \underline{\tau}' \in G_k} \left[e_{-\xi}\left(\sum_{S \subset [n-2]} (-1)^{n-2-|S|} Z_n^N(\underline{\tau}_S \cdot \underline{v}) \right) \right] \\ &=: F_k(\xi, \underline{v}), \end{aligned}$$

where

$$\underline{\tau}_S = (\underline{\tau}_{S,1}, \ldots, \underline{\tau}_{S,n-2}), \qquad \underline{\tau}_{S,i} = \begin{cases} \underline{\tau}_i & i \in S \\ \underline{\tau}_i' & i \notin S \end{cases}.$$

Gowers-Cauchy-Schwarz

Lemma

$F_k(\xi, \underline{v})$ factors as the product

$$F_k(\xi, \underline{v}) = \prod_{j=1}^{N'} \left(1 - \frac{1}{2^{n-2}} + \frac{F_{k,j}(\xi, \underline{v})}{2^{n-2}}\right)$$

where $F_{k,j}(\xi, \underline{v})$ is a function of only the jth block of length $k \cdot 2^{n-2}$ in $\underline{v}$.

Final steps

Averaging over $\underline{v}$ and applying Hölder's inequality, then the independence,

$$\begin{aligned}\left|\hat{Z}^N_{n,\mu}(\xi)\right|^{2^{n-2}} &= \left|\mathbf{E}_{\underline{v}}\left[\chi_k(\xi,\underline{v})\right]\right|^{2^{n-2}} \\ &\leq \mathbf{E}_{\underline{v}}\left[\left|\chi_k(\xi,\underline{v})\right|^{2^{n-2}}\right] \\ &\leq \mathbf{E}_{\underline{v}}\left[\prod_{j=1}^{N'}\left(1-\frac{1}{2^{n-2}}+\frac{F_{k,j}(\xi,\underline{v})}{2^{n-2}}\right)\right] \\ &= \prod_{j=1}^{N'}\mathbf{E}_{\underline{v}}\left[\left(1-\frac{1}{2^{n-2}}+\frac{F_{k,j}(\xi,\underline{v})}{2^{n-2}}\right)\right].\end{aligned}$$

The choice of $k = k(\xi)$, $k^{\frac{n-1}{2}} \asymp \frac{1}{|\xi|}$ is such that $\mathbf{E}\left[F_{k,j}(\xi,\underline{v})\right]$ is bounded by a fixed constant less than 1, uniformly in ξ, which suffices to complete the proof.

Related results

- We have obtained an optimal rate in Breuillard's local limit theorem on the real Heisenberg group

$$\mathbb{H}(\mathbb{R}) = \begin{pmatrix} 1 & \mathbb{R} & \mathbb{R} \\ 0 & 1 & \mathbb{R} \\ 0 & 0 & 1 \end{pmatrix}.$$

- The argument combines the group action method with a Lindeberg-type replacement scheme, in which the steps in the walk are replaced with Gaussians of the same covariance matrix.

Outline

1 Overview

2 Upper triangular groups

3 Abelian sandpiles

4 Cycle walks

Sandpiles on the square lattice

(a) Daniel Jerison

(b) Lionel Levine

Sandpiles on the square lattice

- A *sandpile* on the square lattice $\mathbb{Z}^2$ is a sand allocation

$$\sigma : \mathbb{Z}^2 \to \mathbb{Z}_{\geq 0}.$$

- If $\sigma(x) \geq 4$ the pile at x can *topple*, passing one grain of sand to each of its neighbors. This toppling procedure is known to be abelian.
- An allocation is *stable* if $\sigma \leq 3$, and *unstable* otherwise.

Sandpiles on the square lattice

- We consider parallel toppling dynamics in which time progresses in discrete steps, and at each time step, every site that can topple topples once.
- A configuration σ is said to be *stabilizable* if under parallel toppling, each vertex topples finitely many times. The stabilized sandpile is written σ^{∞}.

Sandpiles on the square lattice

Theorem (H., Jerison, Levine, '17)

Let $(\sigma_x)_{x\in\mathbb{Z}^2}$, distributed i.i.d. on $\mathbb{Z}_{\geq 0}$, be a sandpile configuration. If $(\sigma_x)_{x\in\mathbb{Z}^2}$ is stabilizable almost surely then

$$\mathbf{E}[\sigma_0] \leq 3 - \epsilon$$

$$\epsilon \gg \min\left(1, \int\int |z_1 - z_2|^{\frac{2}{3}} d\sigma_0(z_1) d\sigma_0(z_2)\right).$$

Graph Laplacian and Green's function

- Let Δ denote the graph Laplacian on $\mathbb{Z}^2$,

$$\Delta f(x) = 4f(x) - \sum_{y:\|y-x\|_1=1} f(y)$$

- Denote $G(x)$ the Green's function, which satisfies $\Delta G = \delta_0$.

A function harmonic modulo 1

- Write $D_1 f(x) = f(x + (1,0)) - f(x)$.
- The argument uses that $\xi = D_1^3 G$ satisfies
  1. $\xi \in L^1(\mathbb{Z}^2)$
  2. ξ is 'harmonic modulo 1', that is, $\Delta\xi \equiv 0 \bmod 1$
  3. For $A > 1$,
  $$\sum_{x:|\xi_x|<\frac{1}{A}} |\xi_x|^2 \asymp A^{-\frac{4}{3}}.$$

Several lemmas

Lemma ([9])

Let $(\sigma_x)_{x\in\mathbb{Z}^2} \in \mathbb{Z}_{\geq 0}^{\mathbb{Z}^2}$ *be an i.i.d. sandpile which stabilizes a.s.. Then*

$$\mathbf{E}[\sigma_0] = \mathbf{E}[\sigma_0^\infty].$$

Lemma

Let $(\sigma_x)_{x\in\mathbb{Z}^2} \in \mathbb{Z}_{\geq 0}^{\mathbb{Z}^2}$ *be an i.i.d. sandpile which stabilizes a.s.. Then*

$$\langle \xi, \sigma \rangle \equiv \langle \xi, \sigma^\infty \rangle \bmod 1, \qquad a.s..$$

Proof.

- Let $u^n(x)$ denote the number of times that site x topples in the first n rounds of toppling, and let $\sigma^n = \sigma - \Delta u^n$ be the sandpile after n topplings.
- We have $u^n \leq n$ and $\sigma^n(x) \leq \max(\sigma^{n-1}(x), 7)$.
- $\langle \xi, \sigma \rangle$ converges absolutely a.s. by the weak law of large numbers.
- For each n, a.s. $\langle \xi, \sigma \rangle - \langle \xi, \sigma^n \rangle = -\langle \xi, \Delta u^n \rangle = -\langle \Delta\xi, u^n \rangle \in \mathbb{Z}$.
- Thus a.s. $\langle \xi, \sigma \rangle - \langle \xi, \sigma^\infty \rangle \in \mathbb{Z}$, by dominated convergence.

□

Proof sketch of theorem

- The previous lemma implies that the equality

$$\chi(\xi,\sigma) = \mathbf{E}\left[e^{-2\pi i\langle \xi,\sigma\rangle}\right] = \mathbf{E}\left[e^{-2\pi i\langle \xi,\sigma^\infty\rangle}\right] = \chi(\xi,\sigma^\infty).$$

- By independence,

$$|\mathrm{LHS}| = \prod_{x\in\mathbb{Z}^2}\left|\mathbf{E}\left[e^{-2\pi i \xi_x \sigma_0}\right]\right|$$

while $\sigma^\infty \le 3$ implies

$$|\,\mathrm{RHS}\,| = 1 - O\left(3 - \mathbf{E}[\sigma_0]\right).$$

- The theorem follows on comparing the LHS and RHS, we omit the details.

Torus sandpiles

Consider sandpile dynamics on the torus $\mathbb{T}_m = (\mathbb{Z}/m\mathbb{Z})^2$, given as follows.

- The point $(0, 0)$ is designated 'sink' and is special, in that any grain of sand which falls on the sink is lost from the model.
- Each non-sink point on the torus has a sand allocation indicated by

$$\sigma : \mathbb{T}_m \setminus \{(0, 0)\} \to \mathbb{Z}_{\geq 0}.$$

- A move in the model consists of dropping a grain of sand on a uniformly chosen vertex v, then performing topplings until the configuration is stable.

Torus sandpiles

- Those states $\mathscr{S}_m$ for which $\sigma \leq 3$ are *stable*, while those states $\mathscr{R}_m$ which may be reached from the maximal state $\sigma \equiv 3$ are recurrent.
- The stationary distribution of the model is the uniform distribution on recurrent states.

Torus sandpiles

Theorem (H., Jerison, Levine, 2017)

There is a constant $c_0 > 0$ *and* $t_m^{\mathrm{mix}} = c_0 m^2 \log m$ *such that the following holds. For each fixed* $\epsilon > 0$,

$$\lim_{m\to\infty} \min_{\sigma\in\mathscr{S}_m} \left\| P^{\lceil (1-\epsilon)t^{\mathrm{mix}}\rceil}\delta_\sigma - \mathbb{U}_{\mathscr{R}} \right\|_{\mathrm{TV}(\mathscr{S}_m)} = 1, \quad (1)$$

$$\lim_{m\to\infty} \max_{\sigma\in\mathscr{S}_m} \left\| P^{\lfloor (1+\epsilon)t^{\mathrm{mix}}\rfloor}\delta_\sigma - \mathbb{U}_{\mathscr{R}} \right\|_{\mathrm{TV}(\mathscr{S}_m)} = 0.$$

We say that sandpile dynamics on the torus exhibits a cut-off phenomenon with mixing time asymptotic to $c_0 m^2 \log m$.

Ideas in the argument

- A simple coupon collector type argument shows that, started from an arbitrary state, a recurrent state is reached in $O(m^2\sqrt{\log m})$ steps with probability $1 + o(1)$.
- We thus reduce to considering the dynamics started from a recurrent state. These have the structure of an abelian group, isomorphic to $\mathscr{G} = \mathbb{Z}^{\mathbb{T}_m \setminus \{(0,0)\}} / \overline{\Delta}\mathbb{Z}^{\mathbb{T}_m \setminus \{(0,0)\}}$, where $\overline{\Delta}$ is the reduced graph Laplacian, found by omitting the row and column corresponding to the sink.
- In this identification, the dynamics are given by convolution with the measure μ which is uniform on the standard basis vectors of $\mathbb{Z}^{\mathbb{T}_m \setminus \{(0,0\}}$ and 0.

Ideas in the argument

- Denote the dual group $\hat{\mathscr{G}} = \overline{\Delta}^{-1}\mathbb{Z}^{\mathbb{T}_m \backslash \{(0,0)\}} / \mathbb{Z}^{\mathbb{T}_m \backslash \{(0,0)\}}$
- The spectrum is given by the Fourier coefficients

$$\hat{\mu}(\xi) = \mathbf{E}_{x \in \mathbb{T}_m}[e^{2\pi i \xi_x}] : \qquad \xi \in \hat{\mathscr{G}}.$$

- Identify $\xi \in \hat{\mathscr{G}}$ with prevector $v = \overline{\Delta}\xi \in \mathscr{G}$. Choose representative v with $\|v\|_\infty \leq 3$.

Ideas in the argument

- The Fourier coefficients of high frequencies for which $\|v\|_1 > m^{2-\theta}$ are bounded by using that $\overline{\Delta}$ is bounded $L^2 \to L^2$.
- The remaining frequencies have v which are sparse. An agglomeration scheme is performed to decompose v into clustered components.
- The local nature of $\overline{\Delta}$ is used to show that cancellation in $\hat{\mu}(\xi)$ is essentially additive from separated clusters. This is the most technical part of the argument, since the inverse map, given by convolution with the Green's function, only satisfies a decay condition on derivatives.
- Strong additivity at small frequencies is used to demonstrate the cut-off phenomenon via second moment methods.

Torus sandpiles

We also evaluate the spectral gap as follows.

Theorem (H., Jerison, Levine, 2017)

Let $m \geq 1$. Restricted to recurrent states, the spectral gap of sandpile dynamics on $\mathbb{T}_m$ is given by $\mathrm{gap}_m = \frac{\gamma + o(1)}{m^2}$ where

$$\gamma = \inf \left\{ \sum_{x \in \mathbb{Z}^2} (1 - \cos(2\pi \xi_x)) : \xi \in \mathbb{R}^{\mathbb{Z}^2}, \xi \not\equiv 0 \bmod 1, \Delta \xi \equiv 0 \bmod 1. \right\}$$

Torus sandpiles

The value of γ (and also c_0) is determined as follows.

- Let $\xi \in (-1/2, 1/2]^{\mathbb{Z}^2}$ and write $\Delta\xi = v \in \mathbb{Z}^{\mathbb{Z}^2}$.
- Given a subset $S \subset \mathbb{Z}^2$, define $N(S) = \{x \in \mathbb{Z}^2 : d(x, S) \leq 1\}$ it's distance-1 enlargement.
- Define $P(S; v)$ to be the program:

$$\begin{aligned} \text{minimize:} \quad & \sum_{n \in N} (1 - \cos(2\pi x_n)) \\ \text{subject to:} \quad & (x_n)_{n \in N} \subset \left[0, \frac{1}{2}\right)^N, \\ & \forall s \in S, \; 4x_s + \sum_{t : \|t-s\|_1 = 1} x_t \geq |v_s|. \end{aligned}$$

Thus $\sum_{x \in \mathbb{Z}^2} 1 - \cos(2\pi\xi_x) \geq P(S; v)$.

In practice this search is of a reasonable size.

Outline

Cycle walks

- The total variation distance between two probability measures μ, ν on $\mathbb{Z}/p\mathbb{Z}$ is
$$\|\mu - \nu\|_{\mathrm{TV}(\mathbb{Z}/p\mathbb{Z})} = \sup_{B \subset \mathbb{Z}/p\mathbb{Z}} |\mu(B) - \nu(B)| .$$
- The total variation mixing time for random walk driven by probability measure μ is
$$t_1^{\mathrm{mix}} = \inf \left\{ N : \left\| \mu^{*N} - \mathbb{U}_{\mathbb{Z}/p\mathbb{Z}} \right\|_{\mathrm{TV}(\mathbb{Z}/p\mathbb{Z})} < \frac{1}{e} \right\} .$$
- Given set $A \subset \mathbb{Z}/p\mathbb{Z}$, write μ_A for its uniform measure.

Cayley graphs

- Given symmetric generating set $A \subset \mathbb{Z}/p\mathbb{Z}$ denote $\mathscr{C}(A, p)$ the Cayley graph with vertices $V = \mathbb{Z}/p\mathbb{Z}$ and edge set

$$E = \left\{(n_1, n_2) \in (\mathbb{Z}/p\mathbb{Z})^2 : n_1 - n_2 \in A\right\}.$$

- Write $\mathrm{diam}(\mathscr{C}(A, p))$ for the graph-theoretic diameter of $\mathscr{C}(A, p)$.
- Since $\mathbb{Z}/p\mathbb{Z}$ is abelian there is a more geometric notion of diameter

$$\begin{aligned}&\mathrm{diam}_{\mathrm{geom}}(\mathscr{C}(A, p))\\ &= \max_{x \in \mathbb{Z}/p\mathbb{Z}} \min\left(\|\underline{n}\|_2 : \underline{n} \in \mathbb{Z}^k,\ \exists \underline{a} \in A^k,\ \underline{n} \cdot \underline{a} \equiv x \bmod p\right).\end{aligned}$$

Note $\mathrm{diam}_{\mathrm{geom}}(\mathscr{C}(A, p)) \leq \mathrm{diam}(\mathscr{C}(A, p))$.

Geometric mixing bound

Theorem (H., 2015)

Let p be an odd prime, let $1 \le k \le \frac{\log p}{\log \log p}$ and let $A, |A| = 2k+1$ be a symmetric generating set. The mixing time t_1^{mix} of random walk driven by μ_A satisfies

$$t_1^{\mathrm{mix}} \ll k \cdot \mathrm{diam}_{\mathrm{geom}}(\mathscr{C}(A, p))^2.$$

This extends to growing generating sets a previous result of Diaconis and Saloff-Coste [8].

Geometric mixing bound

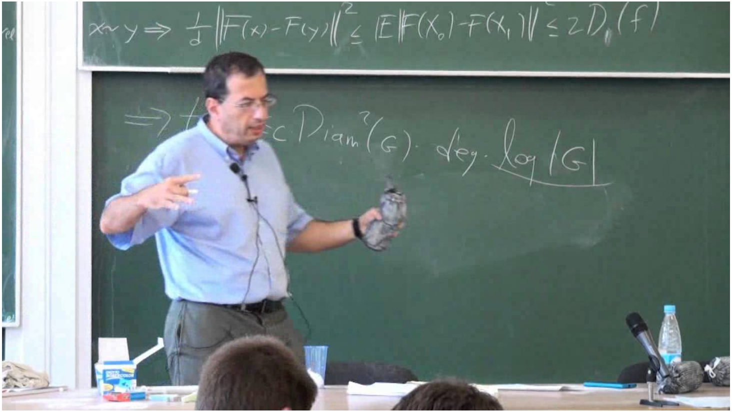

Figure: Yuval Peres

Steps in the proof

- Let $A = \{0, \pm a_1, ..., \pm a_k\}$, $\underline{a} = (a_1, ..., a_k)$ and let $\Lambda < \mathbb{Z}^k$ defined by
$$\Lambda = \{\lambda \in \mathbb{Z}^k : \lambda \cdot \underline{a} \equiv 0 \bmod p\}.$$
This is an index p lattice of $\mathbb{Z}^k$.
- Let $\nu_k = \frac{1}{2k+1}\left(\delta_0 + \sum_{j=1}^k (\delta_{\mathbf{e}_j} + \delta_{-\mathbf{e}_j})\right)$.
- For each $n \geq 1$, μ_A^{*n} on $\mathbb{Z}/p\mathbb{Z}$ and ν_k^{*n} on $\mathbb{Z}^k/\Lambda$ are equal in law.

Steps in the proof

Denote $\eta_k(\sigma, \underline{x})$ the spherically symmetric Gaussian on $\mathbb{R}^k$,

$$\eta_k(\sigma, \underline{x}) = \left(\frac{1}{2\pi\sigma^2}\right)^{\frac{k}{2}} \exp\left(-\frac{\|\underline{x}\|_2^2}{2\sigma^2}\right).$$

Lemma

Let $n, k(n)$ be parameters satisfying $k^2 = o(n)$ for large n. As $n \to \infty$,

$$\left\| \nu_k^{*n} * 1_{(-\frac{1}{2}, \frac{1}{2}]^k} - \eta_k\left(\sqrt{\frac{2n}{2k+1}}, \cdot\right) \right\|_{\mathrm{TV}(\mathbb{R}^k)} = o(1).$$

This permits working with a Gaussian diffusion on $\mathbb{R}^k/\Lambda$.

Steps in the proof

- At step t the Gaussian diffusion has density on $\mathbb{R}^k/\Lambda$ given by

$$\Theta(\underline{x};t) = \sum_{\lambda\in\Lambda} \eta_k\left(\sqrt{\frac{2t}{2k+1}}, \underline{x}-\lambda\right)$$

- Let $\mathscr{F}$ be a norm-minimal fundamental domain (Voronoi cell) for $\mathbb{R}^k/\Lambda$. Since $\overline{\Theta}(\underline{x};t) = \mathbf{E}_{\underline{y}\in\mathscr{F}}[\Theta(\underline{x};t)] = \mathbb{U}_{\mathbb{R}^k/\Lambda}$, the goal is to approximate $\Theta(\underline{x};t) \approx \overline{\Theta}(\underline{x};t)$.

Steps in the proof

- Estimate

$$\left|\eta_k\left(\sqrt{\frac{2t}{2k+1}},\underline{x}\right) - \mathbf{E}_{\underline{y}\in\mathscr{F}}\left[\eta_k\left(\sqrt{\frac{2t}{2k+1}},\underline{x}-\underline{y}\right)\right]\right|$$
$$\leq \eta_k\left(\sqrt{\frac{2t}{2k+1}},\underline{x}\right)\mathbf{E}_{\underline{y}\in\mathscr{F}}\left[\exp\left(\frac{2k+1}{4t}\left(\|\underline{y}\|_2^2+2|\langle \underline{x},\underline{y}\rangle|\right)\right)-1\right].$$

- The proof is completed by combining the integral over $\mathbb{R}^k/\Lambda$ with the sum over Λ, to average over $\underline{x}\in\mathbb{R}^k$. Note that $\mathscr{F}$ is contained in the ball of radius the geometric diameter.

The cut-off phenomenon

- Let $2 < p_1 < p_2 < \ldots$ be a sequence of primes. For each $n \geq 1$, let $A_n \subset \mathbb{Z}/p_n\mathbb{Z}$ be a symmetric subset. The sequence of random walks $(\mathbb{Z}/p_n\mathbb{Z}, \mu_{A_n})$ exhibits the 'cut-off phenomenon' if, for each $0 < \epsilon < 1$,

$$\lim_{n\to\infty} \left\| \mu_{A_n}^{*\lceil (1+\epsilon) t_{1,n}^{\mathrm{mix}} \rceil} - \mathbb{U}_{\mathbb{Z}/p_n\mathbb{Z}} \right\|_{\mathrm{TV}(\mathbb{Z}/p_n\mathbb{Z})} = 0$$

$$\lim_{n\to\infty} \left\| \mu_{A_n}^{*\lfloor (1-\epsilon) t_{1,n}^{\mathrm{mix}} \rfloor} - \mathbb{U}_{\mathbb{Z}/p_n\mathbb{Z}} \right\|_{\mathrm{TV}(\mathbb{Z}/p_n\mathbb{Z})} = 1.$$

- A choice of $\epsilon = \epsilon_n$ tending to zero with n for which this holds is a 'cut-off profile'.

The cut-off phenomenon

- The cut-off phenomenon is a widely studied feature of Markov chains, although it typically is difficult to prove.
- Examples of sequences of Markov chains which have cut-off include nearest neighbor walk on the hypercube, the Gilbert-Shannon-Reeds model of riffle shuffling, and random walk on Ramanujan graphs of bounded degree.
- A necessary condition for cut-off is

$$\lim_{n\to\infty} \frac{t_{1,n}^{\mathrm{mix}}}{t_n^{\mathrm{rel}}} = \infty.$$

Random theorem

Theorem (H., 2015)

Let $k : \mathscr{P} \to \mathbb{Z}_{>0}$ *tend to* ∞ *with* p *in such a way that* $k(p) \leq \frac{\log p}{\log \log p}$. *Choose* $\{A_p \bmod p\}_{p \in \mathscr{P}}$ *independently with* A_p *uniform from symmetric sets of size* $2k+1$. *The following hold with probability 1.*

1. Worst case behavior:

$$\liminf \frac{t_1^{\mathrm{mix}}(p)}{t^{\mathrm{rel}}(p)} \ll 1.$$

In particular, the cut-off phenomenon does not occur for $(\mathbb{Z}/p\mathbb{Z}, \mu_{A_p}, \mathbb{U}_{\mathbb{Z}/p\mathbb{Z}})_{p \in \mathscr{P}}$. *Also*

$$\limsup \frac{t_1^{\mathrm{mix}}(p)}{p^{\frac{4}{k(p)}}} \gg 1, \qquad \limsup \frac{t_1^{\mathrm{mix}}(p)}{p^{\frac{4}{k(p)}} (\log p)^{\frac{2}{k(p)}}} \ll 1.$$

Random theorem, cont'd

Theorem (H., 2015)

2. Typical behavior: *For any sequence* $\{\epsilon(p)\}_{p\in\mathscr{P}} \subset \mathbb{R}_{>0}$ *satisfying* $\epsilon(p)\sqrt{k(p)} \to \infty$ *there is a density 1 subset* $\mathscr{P}_0 \subset \mathscr{P}$ *such that in the family* $(\mathbb{Z}/p\mathbb{Z}, \mu_{A_p}, \mathbb{U}_{\mathbb{Z}/p\mathbb{Z}})_{p\in\mathscr{P}_0}$ *we have*

$$t_1^{\mathrm{mix}}(p) \sim \frac{k(p)}{2\pi e} p^{\frac{2}{k(p)}},$$

and as $p \to \infty$ *running through* $\mathscr{P}_0$

$$\lim \left\| \mu_{A_p}^{(1\pm\epsilon)t_1^{\mathrm{mix}}(p)} - \mathbb{U}_{\mathbb{Z}/p\mathbb{Z}} \right\|_{\mathrm{TV}(\mathbb{Z}/p\mathbb{Z})} = \begin{cases} 0 & + \\ 1 & - \end{cases}.$$

Random theorem

- The theorem shows that random-random walk on the cycle with small generating sets has a generic behavior with a very sharp mixing cut-off. When $k \sim \frac{\log p}{\log \log p}$, the mixing time is of order

$$kp^{\frac{2}{k}} \asymp \frac{(\log p)^3}{\log \log p},$$

  while the window is shown to be smaller by a factor of essentially $\sqrt{k}$, which is a power of the mixing time.
- This is a much sharper window than in well-studied examples of random walk on abelian groups such as random walk on the hypercube, where the window is smaller than the mixing time by a factor of log.
- Rare walks exhibit a bounded-dimensional behavior, and have a transition to uniformity on the scale of the mixing time.

Remarks about the arguments

- The more refined estimates regarding cut-off use concentration properties of the high dimensional Gaussian to partition the theta function into a concentrated piece plus a small L^1 error. Removing the L^1 error permits the problem to be studied on L^2.
- The geometry of the fundamental domain for $\mathbb{R}^k/\Lambda$ is studied. A random fundamental domain behaves like the Euclidean 2-ball in a statistical sense. Combined with the spherical symmetry of the Gaussian, this explains the sharp transition window.
- The unfolding method of analytic number theory is used to study the theta function.

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