The local zeta function in enumerating quartic fields

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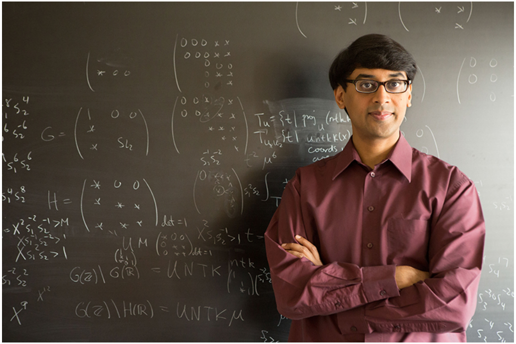

Quartic fields

The motivation for this talk is the following theorem of Bhargava.

Theorem (Bhargava)

Let $N_4^{(i)}(\xi, \eta)$ denote the number of S_4-quartic fields K having $4 - 2i$ real embeddings such that $\xi < \mathrm{Disc}(K) < \eta$. Then

$$\lim_{X \to \infty} \frac{N_4^{(0)}(0, X)}{X} = \frac{1}{48} \prod_p \left(1 + p^{-2} - p^{-3} - p^{-4}\right),$$

$$\lim_{X \to \infty} \frac{N_4^{(1)}(-X, 0)}{X} = \frac{1}{8} \prod_p \left(1 + p^{-2} - p^{-3} - p^{-4}\right),$$

$$\lim_{X \to \infty} \frac{N_4^{(2)}(0, X)}{X} = \frac{1}{16} \prod \left(1 + p^{-2} - p^{-3} - p^{-4}\right).$$

Parameterizing quartic fields

Theorem (Bhargava)

There is a canonical bijection between the set of $\mathbb{GL}_3(\mathbb{Z}) \times \mathbb{GL}_2(\mathbb{Z})$-orbits on the space $(\mathrm{Sym}^2 \mathbb{Z}^3 \otimes \mathbb{Z}^2)^*$ of pairs of integral ternary quadratic forms and the set of isomorphism classes of pairs (Q, R), where Q is a quartic ring and R is a cubic resolvent ring of Q.

In the action, $g = (g_3, g_2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix})$ acts by

$$g \cdot (A, B) = (ag_3 A g_3^t + bg_3 B g_3^t, cg_3 A g_3^t + dg_3 B g_3^t).$$

Maximal quartic rings

Lemma

If Q is any quartic ring not maximal at p, then there exists a $\mathbb{Z}$-basis $1, \alpha_1, \alpha_2, \alpha_3$ of Q such that at least one of the following forms a ring

1. $\mathbb{Z} + \mathbb{Z} \cdot (\alpha_1/p) + \mathbb{Z} \cdot \alpha_2 + \mathbb{Z} \cdot \alpha_3$
2. $\mathbb{Z} + \mathbb{Z} \cdot (\alpha_1/p) + \mathbb{Z} \cdot (\alpha_2/p) + \mathbb{Z} \cdot \alpha_3$
3. $\mathbb{Z} + \mathbb{Z} \cdot (\alpha_1/p) + \mathbb{Z} \cdot (\alpha_2/p) + \mathbb{Z} \cdot (\alpha_3/p)$.

At the level of forms, this condition is determined modulo p^2. Irreducible forms which are maximal at all primes p correspond to the ring of integers in quartic number fields.

The cubic case

Theorem (Taniguchi-Thorne)

Let $N_3^{\pm}(X)$ be the number of cubic fields K with $0 < \pm \operatorname{Disc}(K) < X$. There are constants $C^{\pm}, K^{\pm}$ such that

$$N_3^{\pm}(X) = C^{\pm}\frac{1}{12\zeta(3)}X + K^{\pm}\frac{4\zeta\left(\frac{1}{3}\right)}{5\Gamma\left(\frac{2}{3}\right)^2\zeta\left(\frac{5}{3}\right)}X^{\frac{5}{6}} + O\left(X^{\frac{7}{9}+\epsilon}\right).$$

This theorem used an exact formula for the Fourier transform of the indicator function that a cubic ring is maximal at p.

The shape of the ring of integers

$K/\mathbb{Q}$ r_1 real, r_2 complex embeddings. The canonical embedding is

$$\sigma(x) = (\sigma_1(x), ..., \sigma_{r_1+r_2}(x)) \in \mathbb{R}^{r_1} \times \mathbb{C}^{r_2}.$$

The lattice shape of the ring of integers is

$$\Lambda_K = \mathbb{R} \cdot \mathrm{Proj}_{\sigma(1)^\perp} \sigma(\mathcal{O}_\mathrm{K}),$$

which is a point in $\mathrm{SL}_{n-1}(\mathbb{Z}) \backslash \mathrm{SL}_{n-1}(\mathbb{R})$.

The shape of cubic fields

Theorem (H., 2019)

Let ϕ be a cuspidal automorphic form on $\mathrm{SL}_2(\mathbb{Z})\backslash \mathrm{SL}_2(\mathbb{R})$ of right K-type $2k$, which is an eigenfunction of the Casimir operator and the Hecke algebra. Let $F \in C_c^\infty(\mathbb{R}^+)$ be a smooth test function. For any $\epsilon > 0$, as $X \to \infty$,

$$N_{3,\pm}(\phi, F, X) := \sum_{[K:\mathbb{Q}]=3} \phi(\Lambda_K) F\left(\frac{\pm \mathrm{Disc}(K)}{X}\right) \ll_{\phi,\epsilon} X^{\frac{2}{3}+\epsilon}.$$

A version of this theorem treating the real analytic Eisenstein series is in progress, joint with Eun Hye Lee.

The space of pairs of ternary quadratic forms

Identify $\mathrm{Sym}^2(R^3)$ with 3×3 symmetric matrices

$$A = \begin{pmatrix} a & \frac{b}{2} & \frac{c}{2} \\ \frac{b}{2} & d & \frac{e}{2} \\ \frac{c}{2} & \frac{e}{2} & f \end{pmatrix}.$$

There is a natural bi-linear pairing $[A, A'] = \mathrm{Tr}[A(A')^t]$. $\mathbb{GL}_3$ acts by $g \cdot A = gAg^t$. The pairing is invariant,

$$[g \cdot A, (g^{-1})^t \cdot A'] = \mathrm{Tr}(gA(A')^t g^{-1}) = [A, A'].$$

Given pairs of forms, define $[(A, B), (A', B')] = [A, A'] + [B, B']$.

Mod p orbits and exponential sums

Taniguchi and Thorne identified the 20 orbits of $\mathrm{Sym}^2(\mathbf{F}_p^3) \otimes \mathbf{F}_p^2$ under $\mathbb{GL}_3(\mathbf{F}_p) \times \mathbb{GL}_2(\mathbf{F}_p)$.

$$s(a, b, c, d) = (p-1)^a p^b (p+1)^c (p^2+p+1)^{\frac{d}{2}}.$$

Mod p orbits and exponential sums

Orbit	Representative	Orbit size	Stabilizer size
$\mathcal{O}_0$	$(0,0)$	1	$s(5,4,2,2)$
$\mathcal{O}_{D1^2}$	$(0,w^2)$	$s(1,0,1,2)$	$s(4,4,1,0)$
$\mathcal{O}_{D11}$	$(0,vw)$	$s(1,1,2,2)/2$	$2s(4,3,0,0)$
$\mathcal{O}_{D2}$	$(0,v^2-\ell w^2)$	$s(2,1,1,2)/2$	$2s(3,3,1,0)$
$\mathcal{O}_{Dns}$	$(0,u^2-vw)$	$s(2,2,1,2)$	$s(3,2,1,0)$
$\mathcal{O}_{Cs}$	(w^2,vw)	$s(2,1,2,2)$	$s(3,3,0,0)$
$\mathcal{O}_{Cns}$	(vw,uw)	$s(2,3,1,2)$	$s(3,1,1,0)$
$\mathcal{O}_{B11}$	(w^2,v^2)	$s(2,2,2,2)/2$	$2s(3,2,0,0)$
$\mathcal{O}_{B2}$	$(vw,v^2+\ell w^2)$	$s(3,2,1,2)/2$	$2s(2,2,1,0)$
$\mathcal{O}_{1^4}$	$(w^2,uw+v^2)$	$s(3,2,2,2)$	$s(2,2,0,0)$
$\mathcal{O}_{1^31}$	$(vw,uw+v^2)$	$s(3,3,2,2)$	$s(2,1,0,0)$
$\mathcal{O}_{1^21^2}$	(w^2,uv)	$s(2,4,2,2)/2$	$2s(3,0,0,0)$
$\mathcal{O}_{2^2}$	$(w^2,u^2-\ell v^2)$	$s(3,4,1,2)/2$	$2s(2,0,1,0)$
$\mathcal{O}_{1^211}$	(v^2-w^2,uw)	$s(3,4,2,2)/2$	$2s(2,0,0,0)$
$\mathcal{O}_{1^22}$	$(v^2-\ell w^2,uw)$	$s(3,4,2,2)/2$	$2s(2,0,0,0)$
$\mathcal{O}_{1111}$	$(uw-vw,uv-vw)$	$s(4,4,2,2)/24$	$24s(1,0,0,0)$
$\mathcal{O}_{112}$	$(vw,u^2-v^2-\ell w^2)$	$s(4,4,2,2)/4$	$4s(1,0,0,0)$
$\mathcal{O}_{22}$	$(vw,u^2-\ell v^2-\ell w^2)$	$s(4,4,2,2)/8$	$8s(1,0,0,0)$
$\mathcal{O}_{13}$	$(uw-v^2,B_3)$	$s(4,4,2,2)/3$	$3s(1,0,0,0)$
$\mathcal{O}_4$	$(uw-v^2,B_4)$	$s(4,4,2,2)/4$	$4s(1,0,0,0)$

The items B_3 and B_4 indicate $B_3=uv+a_3v^2+b_3vw+c_3w^2$ and $B_4=u^2+a_4uv+b_4v^2+c_4vw+d_4w^2$ where $X^3+a_3X^2+b_3X+c_3$ and $X^4+a_4X^3+b_4X^2+c_4X+d_4$ are irreducible over $\mathbb{Z}/p\mathbb{Z}$.

Fourier transform theorem

Theorem (H., 2019)

For $p > 3$ the Fourier transform

$$\widehat{\mathbf{1}_{\text{non-max}}}(\xi) = \sum_{x \in V(\mathbb{Z}/p^2\mathbb{Z})} \mathbf{1}_{\text{non-max}}(x) e_{p^2}([x, \xi])$$

is supported on the $\bmod p$ *orbits* $\mathcal{O}_0$, $\mathcal{O}_{D1^2}$, $\mathcal{O}_{D11}$ *and* $\mathcal{O}_{D2}$. *It satisfies*

$$\begin{aligned}
\left\|\widehat{\mathbf{1}_{\text{non-max}}}\right\|_1 &= 2p^{29} + 2p^{28} + 4p^{27} - 8p^{26} - 19p^{25} - 2p^{24} + 20p^{23} + 24p^{22} - 5p^{21} \\
&\quad - 17p^{20} - 5p^{19} + 3p^{18} + 2p^{17} - 2p^{16} + p^{15} + p^{14}, \\
\left\|\widehat{\mathbf{1}_{\text{non-max}}}\right\|_2^2 &= p^{46} + 2p^{45} + 2p^{44} - 3p^{43} - 4p^{42} - p^{41} + 3p^{40} + 3p^{39} - p^{38} - p^{37}, \\
\left|\operatorname{supp} \widehat{\mathbf{1}_{\text{non-max}}}\right| &= 2p^{15} + p^{14} - 2p^{13} - p^{12} + 2p^{10} - p^{8}.
\end{aligned}$$

Orbital exponential sums

Definition

Given $x, \xi \in V(\mathbb{Z}/p^2\mathbb{Z})$, define their *orbital exponential sum*

$$S_{p^2}(x,\xi) = \sum_{g \in G(\mathbb{Z}/p^2\mathbb{Z})} e_{p^2}\left([g \cdot x, \xi]\right).$$

p-adic tangent space

Definition

Given a form $x \in V(\mathbb{Z}/p^2\mathbb{Z})$, define the *annihilator subspace*, or *p-adic tangent space* V_x associated to x to be a subspace of $V(\mathbb{Z}/p\mathbb{Z})$ defined by

$$(I + pM_3(\mathbb{Z}/p\mathbb{Z}), I + pM_2(\mathbb{Z}/p\mathbb{Z})) \cdot x = x + pV_x.$$

The orbital exponential sums

Lemma

For $x, \xi \in V(\mathbb{Z}/p^2\mathbb{Z})$ the orbital exponential sum may be expressed

$$S_{p^2}(x,\xi) = \sum_{g \in G(\mathbb{Z}/p^2\mathbb{Z})} e_{p^2}\left([g \cdot x, \xi]\right) \mathbf{1}(g \cdot x \in V_\xi^\perp \bmod p).$$

The orbital exponential sums

Proof.

Since $(I + pM_3(\mathbb{Z}/p\mathbb{Z}), I + pM_2(\mathbb{Z}/p\mathbb{Z}))$ is a subgroup

$$S_{p^2}(x,\xi) = \sum_{g \in G(\mathbb{Z}/p^2\mathbb{Z})} e_{p^2}\left([g \cdot x, \xi]\right)$$
$$= \sum_{g \in G(\mathbb{Z}/p^2\mathbb{Z})} \frac{1}{p^{13}} \sum_{g_1 \in M_3(\mathbb{Z}/p\mathbb{Z}) \times M_2(\mathbb{Z}/p\mathbb{Z})} e_{p^2}\left([g \cdot x, (I + pg_1)^t \cdot \xi]\right)$$

Since $(I + pg_1)^t \cdot \xi$ is uniform on $\xi + pV_\xi$,

$$S_{p^2}(x,\xi) = \sum_{g \in G(\mathbb{Z}/p^2\mathbb{Z})} e_{p^2}\left([g \cdot x, \xi]\right) \mathbf{1}(g \cdot x \in V_\xi^\perp \bmod p).$$

□

Action sets

Lemma

For any $x, \xi \in V(\mathbb{Z}/p^2\mathbb{Z})$ *and* $g \in G(\mathbb{Z}/p^2\mathbb{Z})$, $g \cdot x \in V_\xi^\perp$ *if and only if* $g^t \cdot \xi \in V_x^\perp$.

Action sets

Definition

Given forms $x, \xi \in V(\mathbb{Z}/p\mathbb{Z})$, define the *action set*

$$G_{x,\xi} = \{g \in G(\mathbb{Z}/p\mathbb{Z}) : g^t \cdot \xi \in V_x^\perp \bmod p\}.$$

Action sets

Lemma

The following table lists all pairs $(\mathcal{O}_x, \mathcal{O}_\xi)$ of non-zero orbits modulo p such that $\mathcal{O}_x$ contains both maximal and non-maximal elements, and such that $x \in \mathcal{O}_x$, $\xi \in \mathcal{O}_\xi$ have $G_{x,\xi} \neq \emptyset$.

$\mathcal{O}_x$	$\mathcal{O}_\xi$
$\mathcal{O}_{1^4}$	$\mathcal{O}_{D1^2}, \mathcal{O}_{D11}, \mathcal{O}_{1^4}$
$\mathcal{O}_{1^31}$	$\mathcal{O}_{D1^2}, \mathcal{O}_{Cs}$
$\mathcal{O}_{1^21^2}$	$\mathcal{O}_{D1^2}, \mathcal{O}_{D11}, \mathcal{O}_{D2}$
$\mathcal{O}_{2^2}$	$\mathcal{O}_{D11}, \mathcal{O}_{D2}$
$\mathcal{O}_{1^211}$	$\mathcal{O}_{D1^2}$
$\mathcal{O}_{1^22}$	$\mathcal{O}_{D1^2}$

Action sets

In particular, many of the orbital exponential sums vanish. The remaining ones are calculated in coordinates after classifying the modulo p^2 orbits.

Modulo p^2 orbits

Theorem (H., 2019)

For each standard orbit representative x of an orbit

$$\mathcal{O}_{D1^2}, \mathcal{O}_{D11}, \mathcal{O}_{D2}, \mathcal{O}_{Cs}, \mathcal{O}_{1^4}, \mathcal{O}_{1^31}, \mathcal{O}_{1^21^2}, \mathcal{O}_{2^2}, \mathcal{O}_{1^211}, \mathcal{O}_{1^22}$$

of $V(\mathbb{Z}/p\mathbb{Z})$, the stabilizer subgroup G_x acts on $V(\mathbb{Z}/p\mathbb{Z})/V_x$. The orbits of $V(\mathbb{Z}/p^2\mathbb{Z})$ under $G(\mathbb{Z}/p^2\mathbb{Z})$ above $\mathcal{O}_x \bmod p$ are in bijection with the orbits of $V(\mathbb{Z}/p\mathbb{Z})/V_x$ under G_x.

Exact formula

Theorem (H., 2019)

The Fourier transform of the maximal set is supported on the mod p orbits $\mathcal{O}_0, \mathcal{O}_{D1^2}, \mathcal{O}_{D11}$ and $\mathcal{O}_{D2}$. It is given explicitly in the following tables.

Exact formula

Case $\mathcal{O}_0$, $\xi = p\xi_0$.

Orbit	$p^{-12}\widehat{\mathbf{1}_{\max}}(p\xi_0)$	Orbit size
$\mathcal{O}_0$	$(p-1)^4 p(p+1)^2(p^5+2p^4+4p^3+4p^2+3p+1)$	1
$\mathcal{O}_{D1^2}$	$-(p-1)^3 p(p+1)^4$	$(p-1)(p+1)(p^2+p+1)$
$\mathcal{O}_{D11}$	$-(p-1)^3 p(2p^3+6p^2+4p+1)$	$(p-1)p(p+1)^2(p^2+p+1)/2$
$\mathcal{O}_{D2}$	$(p-1)^2 p(2p^2+3p+1)$	$(p-1)^2 p(p+1)(p^2+p+1)/2$
$\mathcal{O}_{Dns}$	$(p-1)^2 p(2p^2+3p+1)$	$(p-1)^2 p^2(p+1)(p^2+p+1)$
$\mathcal{O}_{Cs}$	$-p^7+5p^5-3p^4-3p^3+p^2+p$	$(p-1)^2 p(p+1)^2(p^2+p+1)$
$\mathcal{O}_{Cns}$	$(p-1)^2 p(2p^2+3p+1)$	$(p-1)^2 p^3(p+1)(p^2+p+1)$
$\mathcal{O}_{B11}$	$(p-1)^2 p(2p^2+3p+1)$	$(p-1)^2 p^2(p+1)^2(p^2+p+1)/2$
$\mathcal{O}_{B2}$	$(p-1)^2 p(2p^2+3p+1)$	$(p-1)^3 p^2(p+1)(p^2+p+1)/2$
$\mathcal{O}_{1^4}$	$p(p^3-3p^2+p+1)$	$(p-1)^3 p^2(p+1)^2(p^2+p+1)$
$\mathcal{O}_{1^3 1}$	$p(p^3-3p^2+p+1)$	$(p-1)^3 p^3(p+1)^2(p^2+p+1)$
$\mathcal{O}_{1^2 1^2}$	$(p-1)^2 p(3p+1)$	$(p-1)^2 p^4(p+1)^2(p^2+p+1)/2$
$\mathcal{O}_{2^2}$	$-(p-1)p(p+1)^2$	$(p-1)^3 p^4(p+1)(p^2+p+1)/2$
$\mathcal{O}_{1^2 11}$	$p(p^3-3p^2+p+1)$	$(p-1)^3 p^4(p+1)^2(p^2+p+1)/2$
$\mathcal{O}_{1^2 2}$	$p(p^3-3p^2+p+1)$	$(p-1)^3 p^4(p+1)^2(p^2+p+1)/2$
$\mathcal{O}_{1111}$	$-p^3+p^2+p$	$(p-1)^4 p^4(p+1)^2(p^2+p+1)/24$
$\mathcal{O}_{112}$	$-p^3+p^2+p$	$(p-1)^4 p^4(p+1)^2(p^2+p+1)/4$
$\mathcal{O}_{22}$	$-p^3+p^2+p$	$(p-1)^4 p^4(p+1)^2(p^2+p+1)/8$
$\mathcal{O}_{13}$	$-p^3+p^2+p$	$(p-1)^4 p^4(p+1)^2(p^2+p+1)/3$
$\mathcal{O}_4$	$-p^3+p^2+p$	$(p-1)^4 p^4(p+1)^2(p^2+p+1)/4$

Exact formula

Case $\mathcal{O}_{D1^2}$.

Orbit	$p^{-12}\widehat{\mathbf{1}_{\max}}(\xi)$	Orbit size
1.	$-(p-1)^3p(p+1)^3$	$(p-1)p^4(p+1)(p^2+p+1)$
2.	$(p-1)^2p(2p+1)$	$(p-1)^2p^4(p+1)^2(p^2+p+1)$
3.	$(p-1)^2p(2p+1)$	$(p-1)^2p^5(p+1)^2(p^2+p+1)/2$
4.	$-(p-1)p(p+1)$	$(p-1)^3p^5(p+1)(p^2+p+1)/2$
5.	$p(p^3-2p^2+1)$	$(p-1)^3p^4(p+1)^2(p^2+p+1)$
6.	$-(p-1)p(p+1)$	$(p-1)^3p^5(p+1)^2(p^2+p+1)$
7.	$-(p-1)p(p+1)$	$(p-1)^3p^5(p+1)^2(p^2+p+1)$
8.	p	$(p-1)^4p^5(p+1)^2(p^2+p+1)/2$
9.	p	$(p-1)^4p^5(p+1)^2(p^2+p+1)/2$
10.	$(p-1)^2p(p+1)^2$	$(p-1)^2p^4(p+1)^2(p^2+p+1)$
11.	$(p-1)^2p(p+1)$	$(p-1)^2p^6(p+1)^2(p^2+p+1)$
12.	$-(p-1)p$	$(p-1)^3p^6(p+1)^2(p^2+p+1)$
13.	$-(p-1)p$	$(p-1)^3p^7(p+1)^2(p^2+p+1)$
14.	$-(p-1)p(p+1)$	$(p-1)^3p^6(p+1)^2(p^2+p+1)$
15.	p	$(p-1)^4p^6(p+1)^2(p^2+p+1)$
16.	p	$(p-1)^4p^7(p+1)^2(p^2+p+1)$
17.	0	$(p-1)^2p^8(p+1)^2(p^2+p+1)/2$
18.	0	$(p-1)^3p^8(p+1)^2(p^2+p+1)$
19.	0	$(p-1)^4p^8(p+1)^2(p^2+p+1)/4$
20.	0	$(p-1)^4p^8(p+1)^2(p^2+p+1)/4$
21.	0	$(p-1)^3p^8(p+1)(p^2+p+1)/2$
22.	0	$(p-1)^4p^8(p+1)^2(p^2+p+1)/4$
23.	0	$(p-1)^4p^8(p+1)^2(p^2+p+1)/4$

Exact formula

Case $\mathcal{O}_{D11}$.

Orbit	$p^{-12}\widehat{\mathbf{1}_{\max}}(\xi)$	Orbit size
1.	0	$(p-1)^2p^{10}(p+1)^2(p^2+p+1)/2$
2.	0	$(p-1)^3p^{10}(p+1)^2(p^2+p+1)/2$
3.	0	$(p-1)^4p^{10}(p+1)^2(p^2+p+1)/8$
4.	0	$(p-1)^3p^{10}(p+1)^2(p^2+p+1)/2$
5.	0	$(p-1)^4p^{10}(p+1)^2(p^2+p+1)/4$
6.	0	$(p-1)^4p^{10}(p+1)^2(p^2+p+1)/8$
7.	0	$(p-1)^2p^8(p+1)^2(p^2+p+1)$
8.	0	$(p-1)^3p^9(p+1)^2(p^2+p+1)/2$
9.	0	$(p-1)^3p^8(p+1)^2(p^2+p+1)$
10.	0	$(p-1)^3p^8(p+1)^2(p^2+p+1)$
11.	0	$(p-1)^4p^9(p+1)^2(p^2+p+1)/2$
12.	0	$(p-1)^4p^8(p+1)^2(p^2+p+1)$
13.	0	$(p-1)^2p^7(p+1)^2(p^2+p+1)$
14.	$(p-1)p^2$	$(p-1)^3p^7(p+1)^2(p^2+p+1)/4$
15.	$-(p-1)p^2$	$(p-1)^3p^7(p+1)^2(p^2+p+1)/4$
16.	0	$(p-1)^3p^7(p+1)^2(p^2+p+1)$
17.	$-p^2$	$(p-1)^4p^7(p+1)^2(p^2+p+1)/4$
18.	p^2	$(p-1)^4p^7(p+1)^2(p^2+p+1)/4$
19.	0	$(p-1)^2p^7(p+1)^2(p^2+p+1)/2$
20.	0	$(p-1)p^7(p+1)^2(p^2+p+1)/2$

Exact formula

Case $\mathcal{O}_{D2}$.

Orbit	$p^{-12}\widehat{\mathbf{1}_{\max}}(\xi)$	Orbit size
1.	0	$(p-1)^3p^{10}(p+1)(p^2+p+1)/2$
2.	0	$(p-1)^4p^{10}(p+1)^2(p^2+p+1)/4$
3.	0	$(p-1)^4p^{10}(p+1)^2(p^2+p+1)/4$
4.	0	$(p-1)^3p^9(p+1)^2(p^2+p+1)/2$
5.	0	$(p-1)^4p^9(p+1)^2(p^2+p+1)/2$
6.	$(p-1)p^2$	$(p-1)^3p^7(p+1)^2(p^2+p+1)/4$
7.	$-p^2$	$(p-1)^4p^7(p+1)^2(p^2+p+1)/4$
8.	$-(p-1)p^2$	$(p-1)^3p^7(p+1)^2(p^2+p+1)/4$
9.	p^2	$(p-1)^4p^7(p+1)^2(p^2+p+1)/4$
10.	0	$(p-1)^3p^7(p+1)(p^2+p+1)/2$
11.	0	$(p-1)^2p^7(p+1)(p^2+p+1)/2$

Applications

The exact formula for the Fourier transform can be combined with a sieve to produce predictions and asymptotic formulae with lower order terms for the count of quartic fields ordered by discriminant.

Thanks

Thanks for coming!