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CUT-OFF PHENOMENON IN THE UNIFORM PLANE KAC WALK

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We consider an analogue of the Kac random walk on the special orthogonal group $\mathrm{SO}(N)$, in which at each step a random rotation is performed in a randomly chosen 2-plane of $\mathbb{R}^N$. We obtain sharp asymptotics for the rate of convergence in total variance distance, establishing a cut-off phenomenon in the large N limit. In the special case where the angle of rotation is deterministic, this confirms a conjecture of Rosenthal [*Ann. Probab.* **22** (1994) 398–423]. Under mild conditions, we also establish a cut-off for convergence of the walk to stationarity under the L^2 norm. Depending on the distribution of the randomly chosen angle of rotation, several surprising features emerge. For instance, it is sometimes the case that the mixing times differ in the total variation and L^2 norms. Our estimates use an integral representation of the characters of the special orthogonal group together with saddle point analysis.

CONTENTS

1. Introduction 2249
List of notation 2253
2. Background on representation theory 2254
3. Deterministic θ: Proof of Theorem 1.2 2258
3.1. Character ratios of small representations 2260
3.2. Insights from the trivial character 2262
3.3. Character ratios of moderate representations 2266
3.4. Large representations 2274
3.5. The case of θ close to π 2279
4. Mixture of rotations 2280
4.1. Random θ in total variation: Proof of Theorem 1.1 2281
4.2. Random θ walk in L^2: Theorem 1.3 2282
5. L^∞ mixing time 2293
6. Open problems 2294
Appendix A: Bounds for dimensions 2295
Appendix B: Contour formula for characters of $\mathrm{SO}(2n+1)$ 2304
References 2307

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1. Introduction. Asymptotic analysis of the mixing time of Markov chains is an emerging field, and while discrete space Markov chains have a healthy growing literature, there are still comparatively few continuous state chains where the mixing time has been determined. Among the most natural, such chains is the random walk on the special orthogonal group introduced by Mark Kac [14], modeling the velocities of a large number N of particles making elastic collisions. Briefly, at each step of the walk a uniform angle $\theta \in \mathbb{R}/2\pi\mathbb{Z} = \mathbb{T}$ is chosen, and two velocities v_i and v_j are chosen at random and updated according to the rule

$$\begin{aligned} v_i' &= v_i \cos\theta + v_j \sin\theta, \\ v_j' &= -v_i \sin\theta + v_j \cos\theta. \end{aligned}$$

That is, the vector of velocities $(v_1, \ldots, v_N)$ is multiplied by a matrix from the special orthogonal group $\mathrm{SO}(N)$ which consists of a rotation by a randomly chosen angle, in the randomly chosen coordinate 2-plane, $e_i \wedge e_j$.

The asymptotic mixing time of Kac's walk has received quite a bit of attention, going back at least to the paper of Diaconis and Saloff-Coste [6], but remains only incompletely understood. Besides the evident difficulty that the steps do not commute, the most significant challenge is that the bulk of the spectrum of the transition kernel is not known (although the spectral gap has been determined, see [2, 12, 15]). Even granting the spectrum, the walk's density is not in L^2 after any finite number of steps, which complicates the use of standard spectral techniques. In fact, only recently the second author [13] has given the first polynomial bound for the mixing time in the total variation metric, but the bound $O(N^5(\log N)^2)$ it is still far from the expected bound $O(N^2)$.

The purpose of this article is to study a related but simplified model of the Kac walk for which we are able to prove precise results. In the *uniform plane Kac walk* or *Rosenthal's walk* at each step, we perform a rotation by a random angle θ, but now with the plane of rotation chosen uniformly from the space of 2-planes in $\mathbb{R}^N$, $\Lambda^2(S^{N-1})$. Formally, let ξ be a Borel probability measure on $\mathbb{T}$ and define the block diagonal matrix

$$R(1,2;\theta) := \begin{pmatrix} \cos\theta & -\sin\theta & \\ \sin\theta & \cos\theta & \\ & & I_{N-2} \end{pmatrix}.$$

The transition kernel of our walk is the map $P_\xi : L^2(\mathrm{SO}(N)) \to L^2(\mathrm{SO}(N))$,

$$P_\xi f(X) = \int_{\mathbb{T}} \int_{\mathrm{SO}(N)} f\big(UR(1,2;\theta)U^{-1}X\big)\, d\nu(U)\, d\xi(\theta),$$

where $d\nu$ denotes the probability Haar measure on $\mathrm{SO}(N)$. To complete the analogy, if we replace the integral over $U \in \mathrm{SO}(N)$ by an integral over the permutation matrices, we get exactly the Kac walk.

Throughout this work, we restrict attention to odd ($N = 2n + 1$) orthogonal groups. We expect that the results carry over to even orthogonal groups with straightforward modifications, but have not checked the details carefully. Note that $R(1, 2; \theta)$ and $R(1, 2; -\theta)$ are conjugate in $\mathrm{SO}(N)$ for $N \geq 3$, so henceforth we restrict attention to measures ξ supported in $\mathbb{T}_0 = [0, \pi]$.

Given two Borel probability measures μ and ν defined on the common space $(\Omega = \mathrm{SO}(2n + 1), \mathscr{B})$ with $\mathscr{B}$ the Borel σ-algebra, recall that the total variation distance between them is induced from a norm $\|\cdot\|_{\mathrm{TV}} := \|\cdot\|_{\mathrm{TV}(\Omega)}$ on signed measures defined by

$$\|\mu - \nu\|_{\mathrm{TV}} = \sup_{B \in \mathscr{B}} \mu(B) - \nu(B).$$

This is also known as the L^1 distance since when $|\frac{d\mu}{d\nu}| < \infty$,

$$\|\mu - \nu\|_{\mathrm{TV}} = \frac{1}{2} \int_\Omega \left| \frac{d\mu}{d\nu} - 1 \right| d\nu.$$

From a probabilistic point of view, the total variation distance is a natural one, since it is well defined on all measures. Throughout, we take ν to be the Haar probability measure on $\mathrm{SO}(2n + 1)$. In the case that μ has an L^2 density with respect to ν, we define their L^2 distance

$$\|\mu - \nu\|_{L^2} = \left(\int_\Omega \left(\frac{d|\mu - \nu|}{d\nu} \right)^2 d\nu \right)^{\frac{1}{2}}.$$

Otherwise, we set the L^2 distance to be ∞.

THEOREM 1.1. *Let ξ be any Borel probability measure on $\mathbb{T}_0$ excluding the delta measure at* 0. *Define $\sigma(\theta) = \sin \frac{\theta}{2}$ and*

$$\xi(\sigma^2) = \int_{\mathbb{T}_0} \sigma^2(\theta)\, d\xi(\theta).$$

The uniform plane Kac walk P_ξ with initial state at the identity on $\mathrm{SO}(2n+1)$ *has total variation cut-off at $t = \frac{n \log n}{2\xi(\sigma^2)}$. Precisely, there exists $f : \mathbb{R} \to [0, 1]$ satisfying* $\lim_{|c| \to \infty} f(c) = 0$, *such that, uniformly in n, for $t = \frac{n(\log n + c)}{2\xi(\sigma^2)}$ we have*

$$\left| \left\| \delta_{\mathrm{Id}} \cdot P_\xi^t - \nu \right\|_{\mathrm{TV}} - \mathbf{1}(c < 0) \right| < f(c).$$

The above result establishes a cut-off phenomenon in total variation for the uniform plane Kac walk, in which the distance to uniform transitions in a window that is asymptotically small compared to the mixing time. For general Markov chains, cut-off phenomena were first proved in discrete models such as the nearest neighbor walk on the hypercube $\mathbb{Z}^n/(2\mathbb{Z})^n$ [1] and the random transposition card shuffling model [7]; see also [5] for an extensive survey. Note that there are also common random walks in which the cut-off phenomenon provably does not occur, for instance, in nearest neighbor random walk on the cycle $\mathbb{Z}/p\mathbb{Z}$. It is an open and open-ended question whether the cut-off phenomenon represents a special or generic behavior of Markov chains. This is partly difficult to answer because models in which the distance to uniform can be accurately analyzed are rare.

The chief simplifying feature of our walk is that the transition kernel is invariant under conjugation, so that the spectrum is completely described by the character theory of $\mathrm{SO}(2n+1)$, an observation that goes back to Diaconis and Shahshahani in [7]. Our analysis builds on that of Rosenthal [18], who studied the walk in the special case where ξ is a point mass at a fixed angle θ, proving the lower bound of our theorem in this case, and the more difficult upper bound for $\theta = \pi$; see also the work of Porod on a related walk [17]. The first stage in our proof of Theorem 1.1 is to complete Rosenthal's upper bound analysis for any fixed angle, with strong uniformity in the angle θ.

THEOREM 1.2. *Let $\theta = \theta(n)$ vary with n in such a way that $\frac{\log n}{\sqrt{n}} \le \theta \le \pi$. Let P_θ denote the transition kernel of our walk in the special case when ξ is a point mass at θ. There exists $f : \mathbb{R} \to [0,1]$ with $f(c)$ tending to 0 as $|c| \to \infty$ such that, for $t = \frac{n(\log n + c)}{2\sigma^2(\theta)}$, uniformly in n and $\theta(n)$,*

$$\left| \left\| \delta_{\mathrm{Id}} \cdot P_\theta^t - \nu \right\|_{\mathrm{TV}} - \mathbf{1}(c < 0) \right| < f(c).$$

As in the special case $\theta = \pi$ treated by Rosenthal, the upper bound of Theorem 1.2 is obtained by bounding the total variation norm with the L^2 norm, which translates the problem into bounding a sum of character ratios; see [4] for similar computations in the finite group setting. Although this analysis does not go through for the upper bound in Theorem 1.1 because the density need not be in L^2, we are able to sidestep this issue with a truncation argument, in which the extra symmetry of the walk is exploited a second time.

When the measure ξ of Theorem 1.1 has support bounded away from zero, the resulting walk converges in L^2 and, under the additional assumption that the support of ξ is bounded away from π, we are also able to determine the L^2 mixing time, and establish a cut-off phenomenon.

THEOREM 1.3. *Let ξ be a probability measure on $\mathbb{T}_0$ having support bounded away from 0 and π, and let q be the point in the support of ξ that is closest to 0. Let $\sigma(\theta) := \sin\frac{\theta}{2}$ and set $\xi(\sigma^2) = \int_0^\pi \sigma^2(\theta)\xi(d\theta)$. Define for $k \in (0,\infty)$ the threshold*

$$T_{2,k}(n) := \inf\left\{t \in \mathbb{Z}_{\ge 0} : \left\| \delta_{\mathrm{Id}} \cdot P_\xi^t - \nu \right\|_{L^2} < k\right\}.$$

We have

$$T_{2,k}(n) \sim \frac{n \log n}{4\sigma^2(q) \wedge 2\xi(\sigma^2)}.$$

Furthermore, there exists $0 < c_k < \infty$ *such that*

$$T_{2,k}(n) \le \max\left(\frac{n \log n}{2\xi(\sigma^2)}, \frac{n(\log n + 2\log\log n)}{4\sigma^2(q)}\right) + c_k n.$$

With the additional assumption that ξ *has positive one-sided lower derivative w.r.t. Lebesgue measure at* q, *we have the lower bound*

$$T_{2,k}(n) \ge \max\left(\frac{n \log n}{2\xi(\sigma^2)}, \frac{n(\log n - 3\log\log n)}{4\sigma^2(q)}\right) - c_k n.$$

REMARK 1. When $\sigma^2(q) \le \frac{1}{2}\xi(\sigma^2)$ we exhibit a cut-off window of size $O(n \log\log n)$. It is likely that $O(n)$ is the true size.

Together, Theorems 1.1 and 1.3 have several surprising features. For one, the proof of Theorem 1.3 exhibits a competition between the natural representation, which produces the spectral gap, and some moderately sized representations for which the spectrum is smaller, but of higher multiplicity. We are not familiar with another such naturally occurring walk that has been studied. Also, for measures ξ such that $\sigma^2(q) < \frac{\xi(\sigma^2)}{2}$ we have a situation in which there is a cut-off phenomenon in both the total variation and L^2 norms, the two cut-off points are not the same, and both may be precisely analyzed. Again, we are not aware of comparable examples.

We record several further consequences of Theorems 1.1 and 1.3. First, from Theorem 1.1 one can also derive near optimal mixing time results under the Wasserstein distance. Recall that for two probability measures μ, ν defined on a metric space (Ω, d), the L^2 Wasserstein distance is defined by

$$W^2(\mu, \nu) = \inf_{(X,Y)\in\mathscr{M}(\mu,\nu)} \left[\mathbb{E} d(X,Y)^2\right]^{\frac{1}{2}},$$

where $\mathscr{M}(\mu, \nu)$ denotes the set of all couplings of μ and ν. We obtain the following.

COROLLARY 1.1. *Let* ξ *denote the uniform measure on* $\mathbb{T}_0$. *For* $t = 2n(\log n + c)$, *the Wasserstein distance of the uniform plane Kac walk* $\delta_{\mathrm{Id}} \cdot P_\xi^t$ *from Haar measure is bounded by*

$$W^2(\delta_{\mathrm{Id}} \cdot P_\xi^t, \nu) = o(1), \qquad c \to \infty.$$

PROOF SKETCH. The optimal transport inequality [20] gives

$$W^2(\mu, \nu) \ll \sqrt{d^2 \|\mu - \nu\|_{\mathrm{TV}}},$$

where d is the diameter of the state space. For our walk on $\mathrm{SO}(n)$, the diameter is of order $\sqrt{n}$. It follows in a straightforward way from our analysis that $\|\delta_{\mathrm{Id}} \cdot P^{2n(\log n + c)} - \nu\|_{\mathrm{TV}} = o(\frac{1}{n})$ as $c \to \infty$, since the second largest eigenvalue is of size $1 - \frac{1}{n} + O(n^{-2})$. $\square$

REMARK 2. Oliviera [16] proved that the Kac walk on $\mathrm{SO}(2n+1)$ converges in at most $t = n^2 \log n$ steps under L^2 Wasserstein distance. A direct adaptation of his method would give the same upper bound for the uniform plane Kac walk defined above. Our result does better than this by an order of n.

The argument toward Theorem 1.3 also implies a cut-off phenomenon at twice the L^2 mixing time in the "L^∞ norm with respect to Haar measure". Note that in the continuous state space setting there is some subtlety in working with L^∞ norms. This is explained in Section 5.

COROLLARY 1.2. *The conclusions of Theorem 1.3 are valid with the mixing time doubled and the L^2 norm replaced by the L^∞ norm with respect to Haar measure.*

The main new technique in our paper is an integral and corresponding differential method of evaluating the character ratios of the orthogonal group at a rotation, which may be of independent interest. Let $\rho_{\mathbf{a}}$ be an irreducible representation of $\mathrm{SO}(2n+1)$ indexed by dominant weight $\mathbf{a}$, of dimension $d_{\mathbf{a}}$ and character $\chi_{\mathbf{a}}$. Rosenthal derives from the Weyl character formula an expression for the ratios

$$r_{\mathbf{a}}(\theta) = \frac{\chi_{\mathbf{a}}(R(\theta))}{d_{\mathbf{a}}}$$

as a trigonometric polynomial, which he could bound successfully when $\theta = \pi$. We observe that in fact the character ratio is equal to the sum of the residues of a meromorphic function attached to the representation, which allows us to use a contour integral and the method of stationary phase to give strong estimates for the character; see Section 2 for our differential and integral formulas. It is a surprising feature of the method that the special case $\theta = \pi$ that was resolved by Rosenthal using combinatorial arguments is most difficult for us to handle using the integral approach.

While for this investigation we needed to understand the characters of $\mathrm{SO}(2n+1)$ evaluated only at a single rotation, in Appendix A we prove a corresponding multiple integral formula for the characters of $\mathrm{SO}(2n+1)$ evaluated at an arbitrary conjugacy class. Thus, in principle, the techniques developed here could be used to study a random walk generated by elements of $\mathrm{SO}(k) \subset \mathrm{SO}(2n+1)$ instead of by rotations in a single plane. A similar formula for character ratios on the symmetric group appears in [11].

List of notation. We collect together symbols used in the course of our argument.

- $N = 2n + 1$ is the dimension of the ambient space of the special orthogonal group $\mathrm{SO}(2n+1)$.

- $\mathbb{T}_0 = [0, \pi]$.
- θ is the angle of rotation defining the fixed-angle Rosenthal walk in Theorem 1.2.
- ξ is the probability measure on $\mathbb{T}_0$ defining the random angle Rosenthal walk.
- P_θ or P_ξ is the corresponding Markov kernel.
- $\sigma = \sin \frac{\theta}{2}$.
- For ξ-integrable $f : \mathbb{T}_0 \to \mathbb{R}$, $\xi(f) = \int_{\theta \in \mathbb{T}_0} f(\theta)\, d\xi(\theta)$.
- $\mathbf{a} \in \mathbb{N}^n$ is the index of an irreducible representation of $\mathrm{SO}(2n+1)$. $\mathbf{0}$ is the 0 string indicating the trivial representation.
- $\tilde{\mathbf{a}}$ has $\tilde{a}_j = a_j + j - \frac{1}{2}$ is the Rosenthal index for irreps. $\tilde{\mathbf{0}}$ is the shifted 0 string.
- $\boldsymbol{\alpha}$ with $\alpha_j = \frac{\tilde{a}_j}{n}$ denotes the rescale index component. $\boldsymbol{\omega}$ with $\omega_j = \frac{j - \frac{1}{2}}{n}$ is the rescaled shifted 0 string.
- $\mathbf{s} \in \mathbb{N}^n$, $s_1 = a_1$, $s_i = a_i - a_{i-1}$ for $i > 1$, is the index of an irrep in shift notation.
- $d_{\mathbf{a}}$ is the dimension of the irrep $\rho_{\mathbf{a}}$.
- $r_{\mathbf{a}}(\theta) = \frac{\mathrm{Tr}\rho_{\mathbf{a}}(\mathrm{R}(1,2;\theta))}{d_{\mathbf{a}}}$ is the character ratio at $\mathbf{a}$.
- $g_{\mathbf{a}}(z) = \theta z - \frac{1}{n}\sum_{j=1}^{n} \log(z^2 + \alpha_j^2)$ is the logarithm of the contour integrand for $r_{\mathbf{a}}(\theta)$.
- $\omega \sim \cot \frac{\theta}{2}$ is the saddle point of $g_{\mathbf{0}}$ on the positive real axis.
- We write $\oint_{\mathscr{C}} f(z)\, dz$ for the integral $\frac{1}{2\pi i}\int_{z \in \mathscr{C}} f(z)\, dz$ of meromorphic function f about contour $\mathscr{C}$. For real ω, $\oint_{(\omega)} f(z)\, dz$ indicates integration on the contour $(\omega - i\infty, \omega + i\infty)$.

2. Background on representation theory.

Our results make use of harmonic analysis on $\mathrm{SO}(2n+1)$; a reference for the theory of harmonic analysis on compact groups is [10]; see also [19] and [8]. In addition, Rosenthal's pioneering paper [18] on the special case of the present investigation has an excellent coverage of the needed tools from representation theory. Thus, our treatment below is somewhat brief.

Probability measure μ has a Fourier development in terms of the finite dimensional irreducible representations of $\mathrm{SO}(2n+1)$. These are indexed by weakly increasing integer "highest weights"

$$\mathbf{a} = (a_1, \dots, a_n), \qquad 0 \le a_1 \le a_2 \le \cdots \le a_n.$$

We also frequently use strongly increasing half-integer weights (Rosenthal's convention, convenient for the Weyl character formula)

$$\tilde{\mathbf{a}} = (\tilde{a}_1, \dots, \tilde{a}_n), \qquad \tilde{a}_k = a_k + k - \frac{1}{2},$$

normalized weights,

$$\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n), \qquad \alpha_k = \frac{\tilde{a}_k}{n}$$

and the shift-indices

$$\mathbf{s} = (s_1, \dots, s_n), \qquad s_1 = a_1, \qquad s_i = a_i - a_{i-1}, \qquad 1 < i \le n.$$

Recall a representation ρ of a group G on a finite dimensional (complex) vector space V is a group homomorphism from G to $\mathrm{Hom}(V)$, the group of linear transformations on V. ρ is irreducible if its image does not lie in a nontrivial split $\mathrm{Hom}(V_1) \oplus \mathrm{Hom}(V_2)$ with $V_1 \oplus V_2 = V$. Every group G has a trivial representation ρ_0 that takes each $g \in G$ to $\mathrm{Id} \in \mathrm{Hom}(0)$, the identity map on the 0-dimensional vector space. Since $\mathrm{SO}(2n+1)$ is defined as the group of orthogonal transformations on $\mathbb{R}^{2n+1}$ with the standard inner product, we get a representation on $\mathbb{R}^{2n+1}$ for free, called the natural representation.

The representation corresponding to $\mathbf{a} = \mathbf{0}$ is the trivial representation while $\mathbf{a}$ with $a_n = 1$, $a_k = 0, k < n$ indicates the natural (matrix) representation. We set $d_{\mathbf{a}}$ for the dimension of irreducible representation $\rho_{\mathbf{a}}$, and let $\chi_{\mathbf{a}}(g) = \mathrm{Tr}(\rho_{\mathbf{a}}(\mathrm{g}))$ be the character.

We consider measures μ that are invariant under conjugation. Such measures have a Fourier series in terms of the irreducible characters on $\mathrm{SO}(2n+1)$:[2]

$$\mu \sim \sum_{\mathbf{a}} \hat{\mu}(\chi_{\mathbf{a}}) \chi_{\mathbf{a}}, \qquad \hat{\mu}(\chi_{\mathbf{a}}) = \frac{1}{d_{\mathbf{a}}} \int_{\mathrm{SO}(2n+1)} \chi_{\mathbf{a}}(g) \mu(dg).$$

The Fourier map carries convolution to point-wise multiplication:

$$\widehat{\mu * \mu'}(\chi_{\mathbf{a}}) = \hat{\mu}(\chi_{\mathbf{a}}) \cdot \hat{\mu}'(\chi_{\mathbf{a}}).$$

Plancherel's identity takes the form

$$\|\mu - \mu'\|_{L^2}^2 = \sum_{\mathbf{a}} d_{\mathbf{a}}^2 |\hat{\mu}(\chi_{\mathbf{a}}) - \hat{\mu}'(\chi_{\mathbf{a}})|^2,$$

with the interpretation that the right-hand side is finite if and only if $|\mu - \mu'|$ has an L^2 density with respect to Haar measure. The Fourier coefficients of Haar measure are given by $\hat{\nu}(\chi_{\mathbf{0}}) = 1$, and $\hat{\nu}(\chi_{\mathbf{a}}) = 0$ if $\mathbf{a} \ne \mathbf{0}$.

In the special case that $\mu_\theta = \delta_{\mathrm{Id}} \cdot P_\theta$ is the probability measure generating the fixed-θ Rosenthal walk, set $r_{\mathbf{a}}(\theta) = \hat{\mu}_\theta(\chi_{\mathbf{a}}) = \frac{\chi_{\mathbf{a}}(R_\theta)}{d_{\mathbf{a}}}$ for the character ratio at rotation R_θ. Rosenthal derives the dimension and character ratio formulae

$$d_{\mathbf{a}} = \frac{2^n}{1!3! \cdots (2n-1)!} \prod_{q=1}^{n} \tilde{a}_q \prod_{1 \le s < r \le n} (\tilde{a}_r^2 - \tilde{a}_s^2), \tag{1}$$

$$r_{\mathbf{a}}(\theta) = \frac{(2n-1)!}{(2\sin\frac{\theta}{2})^{2n-1}} \sum_{j=1}^{n} \frac{\sin(\tilde{a}_j \theta)}{\tilde{a}_j \prod_{r \ne j} (\tilde{a}_r^2 - \tilde{a}_j^2)} \tag{2}$$

[2]The characters of $\mathrm{SO}(2n+1)$ are real. Note that our normalization for $\hat{\mu}(\chi_{\mathbf{a}})$ differs from Rosenthal's in the factor of $\frac{1}{d_{\mathbf{a}}}$.

from the Weyl character formula. We note that his character ratio formula is exactly the sum of the residues at $\{\pm\tilde{a}_j\}_{j=1}^n$ of the meromorphic function

$$f_{\mathbf{a},\theta}(z) = \frac{(2n-1)!}{(2\sin\frac{\theta}{2})^{2n-1}} \frac{\sin(\theta z)}{\prod_{j=1}^n (\tilde{a}_j^2 - z^2)}.$$

This fact leads to the following integral formula.

LEMMA 2.1. *Let $\theta \in \mathbb{T}_0$ and let $\mathbf{a}$ index an irrep of* SO$(2n+1)$. *For any $\alpha > 0$,*

$$r_{\mathbf{a}}(\theta) = \frac{(2n-1)!}{(2n\sin\frac{\theta}{2})^{2n-1}} \oint_{\Re(z)=\alpha} \frac{e^{n\theta z}}{\prod_{j=1}^n (\alpha_j^2 + z^2)}\, dz. \tag{3}$$

PROOF. We may express the sum of the residues of $f_{\mathbf{a},\theta}$ as

$$r_{\mathbf{a}}(\theta) = \oint_{\mathscr{R}} f_{\mathbf{a},\theta}(z)\, dz,$$

where $\mathscr{R}$ is any rectangle with corners $\pm B \pm in\alpha$, $B > \tilde{a}_n$, oriented counterclockwise. As $B \to \infty$, the integral over the vertical segments go to 0 leaving two horizontal line integrals at $\Im(z) = \pm in\alpha$. Exchanging $z \to -z$, the two integrals are seen to be equal, so we keep twice the top one. Now split $\sin(\theta z)$ as $\frac{e^{i\theta z} - e^{-i\theta z}}{2i}$. The contribution from $e^{i\theta z}$ vanishes by shifting the contour upward to $i\infty$. For the final result, replace z by $\frac{iz}{n}$. $\square$

We also obtain the following differential formula.

LEMMA 2.2. *Let $\theta \in \mathbb{T}_0$ and let $\rho_{\mathbf{a}}$, $\mathbf{a} = (a_1, \dots, a_n)$ index a representation of* SO$(2n+1)$. *Set $m = a_n$. Let $\tilde{b}_1 > \tilde{b}_2 > \cdots > \tilde{b}_m$ complement $\{\tilde{a}_1, \dots, \tilde{a}_n\}$ in $\{\frac{1}{2}, \frac{3}{2}, \dots, m+n-\frac{1}{2}\}$. The character ratio at $\mathbf{a}$ is given by*

$$r_{\mathbf{a}}(\theta) = \frac{(2n-1)!}{(2n+2m-1)!(2\sin\frac{\theta}{2})^{2n-1}} \left(\prod_{k=1}^m (\tilde{b}_k^2 + \partial_\theta^2)\right) \left(2\sin\frac{\theta}{2}\right)^{2n+2m-1}. \tag{4}$$

PROOF. Since we know that $\chi_{\mathbf{0}}(\theta) = 1$ and this holds uniformly for all θ and all orthogonal groups SO$(2m+1)$, we obtain the integral identity

$$1 = \frac{(2m-1)!}{(2\sin\frac{\theta}{2})^{2m-1}} \oint \frac{\sin(z\theta)}{\prod_{j=1}^m ((j-\frac{1}{2})^2 - z^2)}\, dz,$$

valid for $m = 0, 1, 2, \dots$, $0 < \theta \leq \pi$.

Now taking any contour that encloses the real axis between $\pm(m+n-\frac{1}{2})$ we have

$$r_{\mathbf{a}}(\theta) = \frac{(2n-1)!}{(2\sin\frac{\theta}{2})^{2n-1}} \oint \frac{\sin(z\theta)}{\prod_{j=1}^{n}(\tilde{a}_j^2 - z^2)}\,dz$$
$$= \frac{(2n-1)!}{(2\sin\frac{\theta}{2})^{2n-1}} \oint \frac{(\prod_{j=1}^{m}(\tilde{b}_j^2 + \partial_\theta^2))\sin(z\theta)}{\prod_{j=1}^{n+m}((j-\frac{1}{2})^2 - z^2)}\,dz.$$

Passing the differential operator $\prod_{j=1}^{m}(\tilde{b}_j^2 + \partial_\theta^2)$ outside the integral, we obtain the required expression. $\square$

We also give the differential formula a combinatorial expression in rising powers of $\sigma = \sin\frac{\theta}{2}$.

LEMMA 2.3. *Keep the notation of the previous lemma. We have*

$$r_{\mathbf{a}}(\theta) = \sum_{s=0}^{m} \frac{(-4)^s(2n-1)!}{(2(n+s)-1)!} E_s \sigma^{2s}, \tag{5}$$

where

$$E_s = \sum_{1\le j_1<j_2<\cdots<j_s\le m} \prod_{i=1}^{s}\left(\left(m+n+i-j_i-\frac{1}{2}\right)^2 - \tilde{b}_{j_i}^2\right).$$

REMARK 3. Note that $m+n+\frac{1}{2}-j-\tilde{b}_j \ge 1$ and is increasing in j. In particular, each term in each E_s is positive.

PROOF OF LEMMA 2.3. We have

$$(\tilde{b}_k^2 + \partial_\theta^2)\sigma^{2r-1} = \left(r-\frac{1}{2}\right)(r-1)\sigma^{2r-3} + \left(\tilde{b}_k^2 - \left(r-\frac{1}{2}\right)^2\right)\sigma^{2r-1}.$$

Iterating this in (4) we obtain

$$\begin{aligned}
&\prod_{k=1}^{m}(\tilde{b}_k^2 + \partial_\theta^2)\sigma^{2n+2m-1} \\
&\quad = \sum_{S\subseteq[m]} \frac{(2(n+m)-1)!}{(2(n+|S|)-1)!} 2^{2|S|-2m}\sigma^{2(n+|S|)-1} \\
&\qquad \times \prod_{j\in S}\left[\tilde{b}_j^2 - \left(n+m-m_S(j)-\frac{1}{2}\right)^2\right],
\end{aligned} \tag{6}$$

where $m_S(j) = \#([j-1]\setminus S)$. The claim now follows on grouping terms according to $s=|S|$. $\square$

As an example of the previous two lemmas, we now calculate the character ratio of several low-dimensional representations. These calculations may be used to prove the lower bound of Theorem 1.1, although as this proof appears in Rosenthal's work, we do not reproduce it here.

EXAMPLE 1. The trivial representation $\rho_{\mathbf{0}}$ has dimension 1 and character ratio 1.

The lowest dimensional nontrivial irrep is the natural representation $\rho_{(\mathbf{0},1)}$. Its dimension is $2n+1$. Using the previous lemma, we may easily calculate its character ratio. We have $m=1$ and $\tilde{b}_1 = n - \frac{1}{2}$ so that $E_1 = 2n$. Thus,

$$r_{(\mathbf{0},1)}(\theta) = 1 - \frac{4\sigma^2}{2n+1}. \tag{7}$$

The tensor product $\rho_{(\mathbf{0},1)} \otimes \rho_{(\mathbf{0},1)}$ decomposes as a direct sum of the trivial representation, the adjoint square and the symmetric square:

$$\rho_{(\mathbf{0},1)} \otimes \rho_{(\mathbf{0},1)} = \rho_{\mathbf{0}} \oplus \rho_{(\mathbf{0},1,1)} \oplus \rho_{(\mathbf{0},2)}.$$

The adjoint square $\rho_{(\mathbf{0},1,1)}$ is a representation of dimension $n(2n+1)$. For this representation, $m=1$ and $\tilde{b}_1 = n - \frac{3}{2}$ so that $E_1 = 4n-2$. Thus, its character ratio is given by

$$r_{(\mathbf{0},1,1)}(\theta) = 1 - \frac{4(2n-1)\sigma^2}{n(2n+1)}. \tag{8}$$

The symmetric square $\rho_{(\mathbf{0},2)}$ is a representation of dimension $n(2n+3)$. For this representation, $m=2$ and $\tilde{b}_1 = n + \frac{1}{2}$, $\tilde{b}_2 = n - \frac{1}{2}$. Thus, $E_1 = 4n+2$ and $E_2 = 8n^2 + 12n + 4$. Thus, the character ratio is given by

$$r_{(\mathbf{0},2)}(\theta) = 1 - \frac{4\sigma^2}{n} + \frac{16\sigma^4}{n(2n+3)}. \tag{9}$$

3. Deterministic $\boldsymbol{\theta}$: Proof of Theorem 1.2. The starting point for our random walk analysis is the upper bound lemma as discussed in [18].

LEMMA (Upper bound lemma). *Let μ be a conjugation invariant probability measure on* $\mathrm{SO}(2n+1)$ *and let ν be Haar measure. The total variation distance between μ and ν is bounded by*

$$\|\mu - \nu\|_{\mathrm{TV}} \le \frac{1}{2}\|\mu - \nu\|_{L^2} = \frac{1}{2}\left(\sum_{\mathbf{a}\neq\mathbf{0}} d_{\mathbf{a}}^2 |\hat{\mu}(\chi_{\mathbf{a}})|^2\right)^{\frac{1}{2}}.$$

In particular, the total variation distance to Haar measure for the Rosenthal walk of fixed angle θ and step t is bounded by

$$\left\|\delta_{\mathrm{Id}} \cdot P_\theta^t - \nu\right\|_{\mathrm{TV}} \le \frac{1}{2}\left(\sum_{\mathbf{a}\neq\mathbf{0}} d_{\mathbf{a}}^2 r_{\mathbf{a}}(\theta)^{2t}\right)^{\frac{1}{2}}. \tag{10}$$

Recall that for the fixed θ walk we set $\sigma = \sin\frac{\theta}{2}$, and that for Theorem 1.2 we need to prove, uniformly in n, that for $t = \frac{n(\log n + c)}{2\sigma^2}$, $\|\delta_{\mathrm{Id}} \cdot P_\theta^t - \nu\| \to 0$ as $c \to \infty$. The key estimate that we prove is the following.

PROPOSITION 3.1. *Let $\theta = \theta(n) \in [\frac{\log n}{\sqrt{n}}, \pi]$ and set, as usual, $\sigma = \sin\frac{\theta}{2}$. For all sufficiently large n, there exists a fixed constant C independent of n, θ such that for each nontrivial representation $\rho_{\mathbf{a}}$,*

$$\left|r_{\mathbf{a}}(\theta)\right|^{\frac{n(\log n + C)}{\sigma^2}} < d_{\mathbf{a}}^{-2}.$$

We approach this estimate in one of several ways depending upon the size of the representation. We say that a representation $\rho_{\mathbf{a}}$ is *small* if $\sum_{i=1}^n a_i < \frac{n}{\sigma \log n}$. Those which are not small, but satisfy $a_n \ll \frac{n}{\sigma}$ are *moderate* and the remaining representations are *large*. The large representations are further split into ones of *controlled growth*—those for which there is an index $k \asymp n$ for which $a_k = O(n)$—and ones that are *giant*. For the most sensitive small representations, we treat the character ratio using the more exact differential character formula. For the moderate representations, we estimate the character ratio by treating its integral formula as a perturbation of the integral formula for the trivial character. For the controlled growth representations, we are able to effectively compare the character ratio with another character ratio on a smaller orthogonal group. For the giant representations, trivial estimates of the integral formula for the character ratio suffice.

We have collected a number of estimates for dimensions of representations in Appendix A, including the proof of the following estimate.

PROPOSITION 3.2. *Uniformly in $c \geq 1$ and n,*

$$\sum_{\mathbf{a} \neq \mathbf{0}} d_{\mathbf{a}}^{\frac{-c}{\log n}} = O\left(e^{-\frac{c}{8}}\right).$$

Combining these propositions, we deduce Theorem 1.2.

DEDUCTION OF THEOREM 1.2. The lower bound was proven by Rosenthal. For the upper bound, set $t = \frac{n(\log n + c)}{2\sigma^2}$. By the upper bound lemma,

$$\left\|\delta_{\mathrm{Id}} \cdot P_\theta^t - \nu\right\|_{\mathrm{TV}}^2 \leq \frac{1}{4} \sum_{\mathbf{a} \neq \mathbf{0}} d_{\mathbf{a}}^2 \left|r_{\mathbf{a}}(\theta)\right|^{2t},$$

and substituting the bounds of Propositions 3.1 and 3.2, for $c > C + 1$ this is bounded by

$$\frac{1}{4} \sum_{\mathbf{a} \neq \mathbf{0}} d_{\mathbf{a}}^{\frac{(C-c)}{\log n}} = O\left(e^{\frac{-c}{8}}\right),$$

as required. $\square$

3.1. *Character ratios of small representations*. We first treat the small characters using an extension of our combinatorial formula, Lemma 2.3 above before moving on to the analysis of the integral for larger representations. The estimates in this section apply to θ in the full range $\frac{\log n}{\sqrt{n}} < \theta = \theta(n) < \pi$.

LEMMA 3.3. *We have the exact evaluation*

$$E_1 = \sum_{i=1}^{n} a_i(a_i + 2i - 1) \tag{11}$$

and the bounds

$$|E_1| \le (m+2n)\sum a_n, \qquad |E_s| \le s|E_1|\Big[(3m+2n)\sum a_n\Big]^{s-1}.$$

In particular, for $C > 0$ *and for* $\mathbf{a}$ *satisfying* $\sum a_i \le \frac{Cn}{\sigma \log n}$ *we have*

$$\log r_{\mathbf{a}}(\theta) = -\frac{E_1\sigma^2}{n^2}\left(1 + O\left(\frac{C}{\log n}\right)\right). \tag{12}$$

PROOF. Since $\{\tilde{a}_i\}_{i=1}^n$ and $\{\tilde{b}_i\}_{i=1}^m$ form a partition of $\{\frac{1}{2}, \frac{3}{2}, \dots, m+n-\frac{1}{2}\}$ we have

$$E_1 = \sum_{i=1}^{m}\left(\left(m+n+\frac{1}{2}-i\right)^2 - \tilde{b}_i^2\right) = \sum_{i=1}^{n}\left(\tilde{a}_i^2 - \left(i-\frac{1}{2}\right)^2\right).$$

Since $\tilde{a}_i = a_i + i - \frac{1}{2}$, the evaluation of E_1 follows.

The bound for $|E_1|$ is immediate, since $a_i + 2i - 1 < a_n + 2n = m + 2n$.

To bound $|E_s|$, split off the sum over j_1 to obtain

$$\begin{aligned} E_s = {} & \sum_{j_1=1}^{m}\left(\left(m+n+\frac{1}{2}-j_1\right)^2 - \tilde{b}_{j_1}^2\right) \\ & \times \sum_{j_1<j_2<\cdots<j_s\le m}\prod_{i=2}^{s}\left(\left(m+n+i-j_i-\frac{1}{2}\right)^2 - \tilde{b}_{j_i}^2\right). \end{aligned}$$

In the inner sum bound,

$$\tilde{b}_{j_i} + m + n + i - j_i - \frac{1}{2} \le 3m + 2n,$$

and

$$\frac{m+n+i-j_i-\frac{1}{2}-\tilde{b}_{j_i}}{m+n+\frac{1}{2}-j_i-\tilde{b}_{j_i}} \le i$$

which follows from $m+n+\frac{1}{2}-j-\tilde{b}_j \ge 1$.

Therefore,

$$E_s \le \sum_{j_1=1}^{m}\left(\left(m+n+\frac{1}{2}-j_1\right)^2 - \tilde{b}_{j_1}^2\right)(3m+2n)^{s-1}s!$$
$$\times \sum_{j_1<\cdots<j_s\le m}\prod_{i=2}^{s}\left(m+n+\frac{1}{2}-j_i-\tilde{b}_{j_i}\right)$$
$$\le s\left[(3m+2n)\sum_{j=1}^{m}\left(m+n+\frac{1}{2}-j-\tilde{b}_j\right)\right]^{s-1}\cdot E_1,$$

since

$$\sum_{j=1}^{m}\left(m+n+\frac{1}{2}-j-\tilde{b}_j\right)=\sum_{i=1}^{n}\left(\tilde{a}_i-i+\frac{1}{2}\right)=\sum a_i.$$

This proves the second bound.

To prove the final claim, substitute the bounds above into (5) and use

$$\frac{\sigma^2}{n^2}(3m+2n)\sum a_i = O\left(\frac{C}{\log n}\right). \qquad \square$$

PROOF OF PROPOSITION 3.1 FOR SMALL REPRESENTATIONS. When $\sum a_i \le \frac{n}{\sigma \log n}$, the above lemmas reduce the proof of Proposition 3.1 to the estimate (now independent of θ)

$$\frac{E_1(\mathbf{a})}{2n} > \frac{\log d_{\mathbf{a}}}{\log n}. \tag{13}$$

This is most convenient to verify in the shift notation $\mathbf{s} \leftrightarrow \mathbf{a}$, $a_i = \sum_{j\le i} s_j$. Given a shift index $\mathbf{s}$ write $\mathbf{s} = \sum_{i=1}^{n} s_i \mathbf{e}_i$ for its decomposition in standard basis vectors. It is immediate from the expression (11) for E_1 that

$$E_1(\mathbf{s}) \ge \sum_{i=1}^{n} s_i E_1(\mathbf{e}_i).$$

Since Lemma A.2 guarantees that $\log d_{\mathbf{s}} \le \sum_{i=1}^{n} s_i \log d_{\mathbf{e}_i}$, we have reduced to checking the claim for $\mathbf{s}$ of form $\mathbf{e}_i$.

Now

$$E_1(\mathbf{e}_i) = \sum_{j=i}^{n} 2j = n^2 + n + i - i^2.$$

Lemma A.3 proves the estimate

$$\log d_{\mathbf{e}_i} \le m\left[1+\log 2+\log\frac{n}{m}+\log\frac{n}{i-\frac{1}{2}}\right]+2(n-i)[1+\log 2]+\log\frac{n}{i-\frac{1}{2}},$$

and thus we have reduced to checking

$$\frac{n^2+n+i-i^2}{2n} \geq \min(i,n-i)\left[\frac{\log\frac{n}{\min(i,n-i)}}{\log n} + \frac{\log\frac{n}{i}}{\log n}\right] + O\left(\frac{n-i}{\log n}\right).$$

For $i = \vartheta n$ with $0 < \vartheta \leq \frac{1}{2}$, this reduces to

$$\left(\frac{1-\vartheta^2}{2}\right)n + \frac{1+\vartheta}{2} \geq \frac{2n}{\log n}\vartheta \log\frac{1}{\vartheta} + O\left(\frac{n}{\log n}\right)$$

which holds for all n sufficiently large. For $n - i = \vartheta n$ with $0 < \vartheta \leq \frac{1}{2}$, the corresponding statement is

$$\vartheta\left(1-\frac{\vartheta}{2}\right)n + \left(1-\frac{\vartheta}{2}\right) \geq \vartheta n\frac{\log\frac{1}{\vartheta(1-\vartheta)}}{\log n} + O\left(\frac{\vartheta n}{\log n}\right),$$

which, again, holds uniformly in ϑ for all n sufficiently large. □

3.2. *Insights from the trivial character.* To gain a heuristic understanding of our saddle point arguments for the character ratio of medium and large representations, we consider initially the case of the trivial character $\chi_{\mathbf{0}}$. Of course, $r_{\mathbf{0}}(\theta) \equiv 1$; we analyze $r_{\mathbf{a}}(\theta)$ for other $\mathbf{a}$ by viewing the associated integral as a perturbation of the integral for $r_{\mathbf{0}}(\theta)$.

In the special case $\mathbf{a} = \mathbf{0}$, set

$$\omega_j = \frac{j-\frac{1}{2}}{n}$$

for α_j. We aim to choose $\alpha = \omega$ in Lemma 2.1 where ω is the location of a real saddle point in the integral. In this way, the dominant part of the character ratio is given by the part of the integral very near $z = \omega$. To this end, introduce

$$g_{\mathbf{0}}(z) = \theta z - \frac{1}{n}\sum_{j=1}^{n}\log\left(z^2+\omega_j^2\right) \tag{14}$$

so that (3) becomes[3]

$$r_{\mathbf{0}}(\theta) = \frac{(2n-1)!}{(2n\sin\frac{\theta}{2})^{2n-1}}\oint_{(\omega)} e^{ng_{\mathbf{0}}(z)}\,dz.$$

A saddle point in the integral occurs for ω solving

$$g_{\mathbf{0}}'(\omega) = \theta - \frac{1}{n}\sum_{j=1}^{n}\frac{2\omega}{\omega^2+\omega_j^2} = 0. \tag{15}$$

The information that we need regarding $g_{\mathbf{0}}$ and its first few derivatives near the saddle point is contained in the following lemma, whose proof we postpone to the end of this section.

[3]The notation $\int_{(\omega)}$ indicates a contour on the line $\Re(z) = \omega$.

LEMMA 3.4. *For $\frac{\log n}{\sqrt{n}} < \theta < \pi - \frac{(\log n)^2}{n}$, the saddle point ω of $g_{\mathbf{0}} = g_{\mathbf{0},\theta}$ is given by*

$$\omega = \big(1 + O\big(\log^{-2} n\big)\big) \cot \frac{\theta}{2}.$$

In particular,

$$1 + \omega^2 = \frac{1 + O(\frac{1}{\log n})}{\sigma^2}.$$

For fixed $\theta \in (0, \pi)$,

$$\omega = \big(1 + O_\theta\big(n^{-1}\big)\big) \cot \frac{\theta}{2}.$$

For $j \ge 2$, we have $g_{\mathbf{0},\theta}^{(j)} = g_{\mathbf{0}}^{(j)}$ is independent of θ. Uniformly, for $\omega' \in [\frac{(\log n)^2}{n}, \frac{\sqrt{n}}{\log n}]$ we have the asymptotics

$$g_{\mathbf{0}}^{(2)}\big(\omega'\big) = \frac{2}{1 + \omega'^2}\big(1 + O\big(n^{-1}\big(1 \wedge \omega'^{-2}\big)\big)\big), \qquad g_{\mathbf{0}}^{(3)}\big(\omega'\big) \sim \frac{2\omega'}{(1 + \omega'^2)^2},$$

and for real t satisfying $|t| < \omega' \vee \frac{1}{2}$, the bound

$$\big|g_{\mathbf{0}}^{(4)}\big(\omega' + it\big)\big| \le 40\big(1 \wedge \omega'^{-4}\big).$$

In particular, for ω the saddle point and for all n sufficiently large,

$$\Re\bigg(g_{\mathbf{0}}(\omega) - g_{\mathbf{0}}\bigg(\omega + \frac{i\sqrt{1 + \omega^2}}{3}\bigg)\bigg) \ge \frac{1}{40}.$$

Finally, for any $\omega' > 0$ and $t \ge 0$,

$$\Re g_{\mathbf{0}}\bigg(\omega' + i\bigg(t + \frac{1}{n}\bigg)\bigg) < \Re g_{\mathbf{0}}\big(\omega' + it\big). \tag{16}$$

Assuming this lemma, it is a standard exercise in the saddle point method to write

$$r_{\mathbf{0}}(\theta) = \frac{(2n-1)! e^{n g_{\mathbf{0}}(\omega)}}{2\pi (2n \sin \frac{\theta}{2})^{2n-1}} \bigg\{ \int_{|t| < 10^4 \sqrt{\frac{1+\omega^2}{\log n}}} + \int_{10^4 \sqrt{\frac{1+\omega^2}{\log n}} < |t| < \frac{\sqrt{1+\omega^2}}{3}} + \int_{|t| > \frac{\sqrt{1+\omega^2}}{3}} \bigg\}$$
$$\times e^{n g_{\mathbf{0}}(\omega + it) - n g_{\mathbf{0}}(\omega)}\, dt.$$

In the first integral, we Taylor expand the exponent as

$$-n g_{\mathbf{0}}^{(2)}(\omega) \frac{t^2}{2} + i n g_{\mathbf{0}}^{(3)}(\omega) \frac{t^3}{6} + O\Big(n|t|^4 \sup_{|s| < 10^4 \sqrt{\frac{1+\omega^2}{\log n}}} \big|g_{\mathbf{0}}^{(4)}(\omega + is)\big|\Big).$$

Thus, the first integral is equal to

$$\left(1+O\left(\frac{1}{n}\right)\right)\sqrt{\frac{2\pi}{ng_{\mathbf{0}}^{(2)}(\omega)}}.$$

Near the boundary $t=\pm\frac{1\vee\omega}{\log n}$ the integrand is bounded in size by $e^{-n\log^{-2}n(1+o(1))}$; using this bound and the fact that the integrand decreases in every period of length $\frac{1}{n}$ [see (16) above], we may easily bound the second integral. For the third, use this and note that for $|t|>\omega\vee 1$, the integrand decreases by factors of e^{-cn} when t doubles. Thus, we may express

$$1=r_{\mathbf{0}}(\theta)=\frac{(2n-1)!}{(2n\sin\frac{\theta}{2})^{2n-1}}\frac{e^{ng_{\mathbf{0}}(\omega)}}{\sqrt{2\pi ng_{\mathbf{0}}^{(2)}(\omega)}}\left(1+O\left(\frac{1}{n}\right)\right). \tag{17}$$

Notice that the same reasoning as above allows us to replace the integrand with its absolute value.

LEMMA 3.5. *We have*

$$1+O\left(\frac{1}{n}\right)=\frac{(2n-1)!}{(2n\sin\frac{\theta}{2})^{2n-1}}\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|e^{ng_{\mathbf{0}}(\omega+it)}\right|dt. \tag{18}$$

In view of this lemma, we may bound the character ratios $|r_{\mathbf{a}}(\theta)|$ by bounding the real difference

$$\Re\bigl(g_{\mathbf{a}}(\omega+it)-g_{\mathbf{0}}(\omega+it)\bigr).$$

PROOF OF LEMMA 3.4. Write

$$g_{\mathbf{0}}'(z)=\theta-\frac{1}{n}\sum_{j=1}^{n}\left[\frac{1}{z+i\frac{j-\frac{1}{2}}{n}}+\frac{1}{z-i\frac{j-\frac{1}{2}}{n}}\right]=\theta+i\sum_{j=0}^{2n-1}\frac{1}{-inz-n+\frac{1}{2}+j}.$$

This sum may be expressed in terms of the digamma function $\psi(x)=\frac{\Gamma'}{\Gamma}(x)$. This function satisfies the following properties:

1. For integer $k\geq 1$,

$$\psi(x+k)-\psi(x)=\sum_{j=0}^{k-1}\frac{1}{x+j}.$$

2. We have $\psi(x)=\psi(1-x)-\pi\cot(\pi x)$.
3. ψ has expansion uniformly in angular sectors about 0 that omit the negative real axis

$$\psi(x)=\ln(x)-\frac{1}{2x}-\frac{1}{12x^2}+O\bigl(x^{-3}\bigr).$$

Using the first two properties, we may write

$$\begin{aligned}g'_{\mathbf{0}}(z) &= \theta + i\Big(\psi\Big(n+\frac{1}{2}-inz\Big) - \psi\Big(-n+\frac{1}{2}-inz\Big)\Big)\\ &= \theta + i\Big(\psi\Big(n+\frac{1}{2}-inz\Big) - \psi\Big(n+\frac{1}{2}+inz\Big) + \pi\tan(i\pi nz)\Big).\end{aligned}$$

Suppose $z=\omega>0$ is real. Then

$$i\pi\tan(i\pi n\omega) = -\pi + O(e^{-\pi n\omega}),$$

while $\psi(x)=\log x - \frac{1}{2x} + O(x^{-2})$ gives

$$\begin{aligned}&i\Big(\psi\Big(n+\frac{1}{2}-in\omega\Big) - \psi\Big(n+\frac{1}{2}+in\omega\Big)\Big)\\ &\quad= 2\tan^{-1}\Big(\frac{\omega}{1+\frac{1}{2n}}\Big) + \frac{1}{n}\frac{\omega}{1+\omega^2} + O\Big(\frac{1\wedge\omega^{-2}}{n^2}\Big).\end{aligned} \tag{19}$$

Thus, the claim regarding location of the saddle point follows from

$$\frac{\theta}{2} = \cot^{-1}\Big(\frac{\omega}{1+\frac{1}{2n}}\Big) + O\Big(\frac{1\wedge\omega^{-1}}{n}\Big) + O(e^{-\pi n\omega}).$$

We may express higher derivatives of g_0 in terms of the polygammas $\psi_k(z) = \frac{d^k}{dz^k}\psi(z)$:

$$\begin{aligned}g_{\mathbf{0}}^{(2)}(z) &= n\Big(\psi_1\Big(n+\frac{1}{2}-inz\Big) + \psi_1\Big(n+\frac{1}{2}+inz\Big) - \pi^2\sec^2(i\pi nz)\Big),\\ g_{\mathbf{0}}^{(3)}(z) &= -in^2\Big(\psi_2\Big(n+\frac{1}{2}-inz\Big) - \psi_2\Big(n+\frac{1}{2}+inz\Big) - 2\pi^3\tan\cdot\sec^2(i\pi nz)\Big),\\ g_{\mathbf{0}}^{(4)}(z) &= -n^3\Big(\psi_3\Big(n+\frac{1}{2}-inz\Big) + \psi_3\Big(n+\frac{1}{2}+inz\Big)\\ &\qquad - 2\pi^4(\sec^4 - 2\tan^2\sec^2)(i\pi nz)\Big).\end{aligned}$$

On $\Re(z)=\omega$, the terms involving sec are exponentially small, so may be ignored. All of the claims follow from

$$\begin{aligned}&\psi_1(z) = z^{-1} + O(|z|^{-2}), \qquad \psi_2(z) = -z^{-2} + O(|z|^{-3}),\\ &\psi_3(z) = 2z^{-3} + O(|z|^{-4});\end{aligned}$$

we just check the explicit bound for $g_{\mathbf{0}}^{(4)}$:

$$\begin{aligned}g_{\mathbf{0}}^{(4)}(\omega'+it) &\sim 2\big[(1+t-i\omega')^{-3} + (1-t+i\omega')^{-3}\big]\\ &= 4\Big[\frac{1+3(t-i\omega')^2}{(1-(t-i\omega')^2)^3}\Big] = 4\Big[\frac{1+3t^2-3\omega'^2-6it\omega'}{(1-t^2+\omega'^2+2it\omega')^3}\Big].\end{aligned}$$

Now $|1-t^2+\omega'^2+2it\omega'|^2 = 1+2\omega'^2+\omega'^4+t^4+2\omega'^2t^2-2t^2$. When $\omega' \geq \frac{1}{2}$ and $t \leq \omega'$ we have

$$\left|1-t^2+\omega'^2+2it\omega'\right|^2 \geq \left(1+\omega'^4\right),$$

so that

$$\left|g_{\mathbf{0}}^{(4)}(\omega'+it)\right| \leq 4\frac{1+9\omega'^2}{(1+\omega'^4)^{\frac{3}{2}}} \leq 40\left(1 \wedge \omega'^{-4}\right).$$

When $\omega' < \frac{1}{2}$, $t \leq \frac{1}{2}$ and, therefore, $|1-t^2+\omega'^2+2it\omega'|^2 \geq \frac{1}{2}$ while $|t-i\omega'|^2 \leq \frac{1}{2}$. Thus, in this case $g_{\mathbf{0}}^{(4)} \leq 4 \times \frac{5}{2} \times 2^{\frac{3}{2}} < 40$.

We may estimate $\Re(g_{\mathbf{0}}(\omega) - g_{\mathbf{0}}(\omega + \frac{i\sqrt{1+\omega^2}}{3}))$ by Taylor expansion about ω.

Finally, to prove the last claim, note that

$$\Re\left(g_{\mathbf{0}}\left(\omega' + i\left(t+\frac{1}{n}\right)\right) - g_{\mathbf{0}}\left(\omega'+it\right)\right) = \log\left|\frac{\omega'+i(t-\frac{2n-1}{2n})}{\omega'+i(t+\frac{2n+1}{2n})}\right| < 0. \qquad \square$$

3.3. *Character ratios of moderate representations.* In this section, we extend the proof of Proposition 3.1 to representations $\rho_{\mathbf{a}}$ satisfying $a_n \leq \frac{2\cdot 10^6 n}{\sigma}$, $\sigma = \sin\frac{\theta}{2}$. Since we have treated the case $\sum a_j \leq \frac{n}{\sigma \log n}$ in the section on small representations, we may assume that this no longer holds. In this section, we make use of the assumption that $\theta > \frac{\log n}{\sqrt{n}}$ and we further assume that $\pi - \theta \geq \frac{\log^2 n}{n}$. The case in which θ is closer to π is treated in Section 3.5.

In Appendix A, we prove the following estimate regarding the dimension of moderate representations $\mathbf{a}$. For all sufficiently large n, we have

$$\exp\left(\frac{n(5-\log\sigma)}{4\cdot 10^6 \log n}\right) \leq d_{\mathbf{a}} \leq \exp\left(\frac{2\cdot 10^7 n^2}{\sigma}\right).$$

In analogy with (14) for the trivial character, introduce (recall $\alpha_j = \frac{\tilde{a}_j}{n}$)

$$g_{\mathbf{a}}(z) = \theta z - \frac{1}{n}\sum_{j=1}^{n} \log\left(z^2+\alpha_j^2\right), \tag{20}$$

so that, fixing the line of integration at the saddle point $\omega \sim \cot\frac{\theta}{2}$ for the trivial representation,

$$r_{\mathbf{a}}(\theta) = \frac{(2n-1)!}{(2n\sin\frac{\theta}{2})^{2n-1}} \oint_{(\omega)} e^{ng_{\mathbf{a}}(z)}\,dz. \tag{21}$$

Heuristically, since

$$\frac{(2n-1)!}{(2n\sin\frac{\theta}{2})^{2n-1}}\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|e^{ng_{\mathbf{0}}(\omega+it)}\right|dt = 1 + O\left(\frac{1}{n}\right)$$

we may bound

$$\left|r_{\mathbf{a}}(\theta)\right| \le \sup_t \left|\frac{e^{ng_{\mathbf{a}}(\omega+it)}}{e^{ng_{\mathbf{0}}(\omega+it)}}\right|.$$

In practice, we put in the sup bound only for small $|t| \ll \frac{1}{\sigma\sqrt{\log n}}$ and rely on the rapid decay of the integrand in t to take care of the rest of the integral.

We prove the following two estimates. In the main part of the integral, we prove the following.

LEMMA 3.6. *Let* $\mathbf{a}$ *satisfy* $a_n \le \frac{2\cdot 10^6 n}{\sigma}$ *and let* $z = \omega + it$ *with* $|t| \le 10^4\sqrt{\frac{1+\omega^2}{\log n}}$. *Then for sufficiently large* $C > 0$ *for* n *sufficiently large*,

$$\tag{22} \frac{1}{2}n^2(\log n + C + 1)\big(1+\omega^2\big)\cdot \Re\big(g_{\mathbf{a}}(z) - g_{\mathbf{0}}(z)\big) \le -\log d_{\mathbf{a}}.$$

In the tail of the integral, we prove the following.

LEMMA 3.7. *We have the bound*

$$\int_{|t|>10^4\sqrt{\frac{1+\omega^2}{\log n}}} \left|e^{ng_{\mathbf{a}}(\omega+it)}\right| dt \le \exp\left(ng_{\mathbf{0}}(\omega) - \frac{(10^8 + o(1))n}{\log n}\right).$$

We give the short deduction of Proposition 3.1 for moderate representations.

PROOF OF PROPOSITION 3.1.. Recall that (18) of Section 3.2 gives

$$1 + O\left(\frac{1}{n}\right) = \frac{(2n-1)!}{(2n\sin\frac{\theta}{2})^{2n-1}}\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|e^{ng_{\mathbf{0}}(\omega+it)}\right| dt.$$

Thus, we have the bound

$$\tag{23} \begin{aligned} &\left|r_{\mathbf{a}}(\theta)\right|\left(1 - O\left(\frac{1}{n}\right)\right) \\ &\quad\le \sup_{|t|\le 10^4\sqrt{\frac{1+\omega^2}{\log n}}} \exp\big(n\Re\big(g_{\mathbf{a}}(\omega+it) - g_{\mathbf{0}}(\omega+it)\big)\big) \\ &\qquad + \sqrt{2\pi g_0^{(2)}(\omega)n}\int_{|t|>10^4\sqrt{\frac{1+\omega^2}{\log n}}} \exp\big(n\Re\big(g_{\mathbf{a}}(\omega+it) - g_{\mathbf{0}}(\omega)\big)\big)\, dt. \end{aligned}$$

By Lemmas 3.6 and 3.7, the RHS is bounded by

$$\le \exp\left(-\frac{2\sigma^2\log d_{\mathbf{a}}}{n(\log n + O(1))}\right) + O\left(\exp\left(\frac{-10^8 n(1+o(1))}{\log n}\right)\right).$$

Since $d_{\mathbf{a}} \le \exp(\frac{2\cdot 10^7 n^2}{\sigma})$, the last expression is, for some $c > 0$,

$$\le \left(1 + O\left(\exp\left(\frac{-cn}{\log n}\right)\right)\right)\exp\left(-\frac{2\sigma^2 \log d_{\mathbf{a}}}{n(\log n + O(1))}\right).$$

We have thus shown that

$$\log\left|r_{\mathbf{a}}(\theta)\right| \le \frac{-2\sigma^2}{n\log n}\log d_{\mathbf{a}}\left(1 + O\left(\frac{1}{\log n}\right)\right) + O\left(\frac{1}{n}\right).$$

Since $d_{\mathbf{a}} \ge \exp(\frac{n(5-\log\sigma)}{4\cdot 10^6 \log n})$ and $\sigma \ge \frac{\log n}{\sqrt{n}}$, it follows that in fact

$$\log\left|r_{\mathbf{a}}(\theta)\right| \le \frac{-2\sigma^2}{n\log n}\log d_{\mathbf{a}}\left(1 + O\left(\frac{1}{\log n}\right)\right),$$

which proves Proposition 3.1 for moderate representations. $\square$

For our two integral estimates it is convenient to consider the length function

$$\begin{aligned}\ell(x; t, \omega)^2 &= |\omega + it - ix|^2|\omega + it + ix|^2 \\ &= \left(x^2 + \omega^2 - t^2\right)^2 + 4\omega^2 t^2.\end{aligned} \tag{24}$$

This function plays a prominent role in the next two sections because we may write

$$\begin{aligned}&\Re\left[g_{\mathbf{a}}(\omega + it) - g_{\mathbf{0}}(\omega + it)\right] \\ &\quad = \frac{1}{n}\sum_{j=1}^{n}\left(\log \ell(\omega_j; t, \omega) - \log \ell(\alpha_j; t, \omega)\right).\end{aligned}$$

We record several of its simple properties.

LEMMA 3.8. *Let t and $\omega > 0$ be fixed and consider ℓ as a function of x only. If $|t| < \omega$ then ℓ is minimized at $x = 0$ with minimum $t^2 + \omega^2$. If $|t| \ge \omega$, then ℓ is minimized at $|x| = \sqrt{t^2 - \omega^2}$ with minimum $2|t|\omega$. In the case $|t| > \omega$, for $0 < \delta < \sqrt{t^2 - \omega^2}$ we have*

$$\ell\left(\sqrt{t^2 - \omega^2} - \delta\right) < \ell\left(\sqrt{t^2 - \omega^2} + \delta\right).$$

3.3.1. *Estimates in the bulk.* As in the small representation section, our argument is based upon making shifts to the index, but whereas in that section we shifted the rightmost indices first. Here, we shift indices from left to right. The main estimate which we prove for the integrand is as follows.

LEMMA 3.9. *Let* $\mathbf{s} \in \mathbb{N}^j$ *and let* $(\mathbf{s}, \mathbf{0}_{n-j})$ *index a representation in shift notation. Recall that we set* $\mathbf{e}_j$ *for the* j*th standard unit vector. For* t *satisfying* $t \le 10^4\sqrt{\frac{1+\omega^2}{\log n}}$,

$$\begin{aligned}
&\Re\big[g_{(\mathbf{s},\mathbf{0})+\mathbf{e}_j}(\omega+it) - g_{(\mathbf{s},\mathbf{0})}(\omega+it)\big] \\
&\quad \le -\frac{1}{2n}\frac{(1-\frac{j-1}{n})(1+\frac{j+2|\mathbf{s}|}{n})((1+\frac{|\mathbf{s}|}{n})^2+(\frac{j+|\mathbf{s}|}{n})^2+2\omega^2)}{((1+\frac{|\mathbf{s}|}{n})^2+\omega^2)^2} \\
&\qquad \times\bigg(1-\frac{4t^2}{1+\omega^2}+O\bigg(\frac{1}{n}\bigg)\bigg).
\end{aligned} \tag{25}$$

REMARK 4. For reference, note that if $j \approx n$ and $|\mathbf{s}|$ is much smaller than $n(1+\omega^2)$ then the RHS of the above bound is roughly $-\frac{2(1-\frac{j}{n})}{n(1+\omega^2)}$.

PROOF OF LEMMA 3.9. Let $\mathbf{a}$ correspond to $(\mathbf{s}, 0)$. Then $\alpha_j = \frac{j+|\mathbf{s}|-\frac{1}{2}}{n}$ and $\alpha_n = 1+\frac{|\mathbf{s}|-\frac{1}{2}}{n}$. We have

$$\begin{aligned}
&\Re\big[g_{(\mathbf{s},\mathbf{0})+\mathbf{e}_j}(\omega+it) - g_{(\mathbf{s},\mathbf{0})}(\omega+it)\big] \\
&\quad = \frac{1}{n}\log\frac{\ell(\alpha_j)}{\ell(\alpha_n+\frac{1}{n})} \\
&\quad = \frac{1}{2n}\log\frac{(\alpha_j^2+\omega^2-t^2)^2+4\omega^2t^2}{((\alpha_n+\frac{1}{n})^2+\omega^2-t^2)^2+4\omega^2t^2} \\
&\quad = \frac{1}{2n}\log\bigg[1-\frac{((\alpha_n+\frac{1}{n})^2-\alpha_j^2)((\alpha_n+\frac{1}{n})^2+\alpha_j^2+2\omega^2-2t^2)}{((\alpha_n+\frac{1}{n})^2+\omega^2-t^2)^2+4\omega^2t^2}\bigg] \\
&\quad \le -\frac{1}{2n}\frac{((\alpha_n+\frac{1}{n})^2-\alpha_j^2)((\alpha_n+\frac{1}{n})^2+\alpha_j^2+2\omega^2-2t^2)}{((\alpha_n+\frac{1}{n})^2+\omega^2+t^2)^2}.
\end{aligned}$$

Since $\alpha_n+\frac{1}{n} \ge 1$, the result follows on substituting the values of α_j and α_n and factoring out expressions involving t [use $\frac{t^2}{1+\omega^2} = \delta$ and $\frac{1-2\delta}{(1+\delta)^2} > 1-4\delta$]. □

The remainder of the proof of Lemma 3.6 is concerned with bounding the growth of the dimension. This is the most sensitive part of the shifting argument, so we prove an initial estimate first which holds only for $a_n \ll \frac{n}{\log n}$ in full generality, although it covers all moderate representations if θ is not too close to π. For convenience, we quote the dimension bound which we use, proved in Lemma A.6 of Appendix A.

LEMMA. *Let* $1 \le j \le n$ *and* $\mathbf{s} \in \mathbb{N}^j$. *Let* $m = \min(j, n-j+1)$ *and let* $1 \le \eta \le m$ *be a parameter. Write*

$$|\mathbf{s}|_{\eta,\mathrm{loc}} = \sum_{j-\eta \le i \le j} s_i.$$

We have the bound

$$\log \frac{d_{(\mathbf{s},\mathbf{0})+\mathbf{e}_j}}{d_{(\mathbf{s},\mathbf{0})}} \le m\left[\log \frac{n+|\mathbf{s}|_{\eta,\mathrm{loc}}}{m+|\mathbf{s}|_{\eta,\mathrm{loc}}} + \log \frac{n+j}{m+j} + 2\right] + \eta \log(n-j+\eta) + 2(n-j+1) + \log \frac{n}{j} + O(1).$$

Our initial lemma is as follows.

LEMMA 3.10. *Let* $(\mathbf{s}, \mathbf{0})$, $\mathbf{s} \in \mathbb{N}^j$, *be the shift index of* $\mathbf{a}$, *and let* $z = \omega + it$ *with* $|t| \le 10^4 \sqrt{\frac{1+\omega^2}{\log n}}$:

(i) *If* $a_n = |\mathbf{s}| \le \frac{n\sqrt{1+\omega^2}}{\log n}$ *then for sufficiently large fixed* C,

$$\frac{1}{2} n^2 (\log n + C)(1+\omega^2) \cdot \Re\left[g_{(\mathbf{s},\mathbf{0})+\mathbf{e}_j}(z) - g_{(\mathbf{s},\mathbf{0})}(z)\right] \le -\log \frac{d_{(\mathbf{s},\mathbf{0})+\mathbf{e}_j}}{d_{(\mathbf{s},\mathbf{0})}}. \tag{26}$$

(ii) *In the range* $\frac{n\sqrt{1+\omega^2}}{\log n} \le |\mathbf{s}| \le \frac{2\cdot 10^6 n}{\sigma}$ *there is a fixed constant* C' *such that if* $\omega > C'$ *then the bound* (26) *continues to hold.*

(iii) *For* $\omega < C'$, *there is a third fixed constant* $C'' > 0$ *such that*

$$\frac{1}{2} n^2 (\log n + C)(1+\omega^2) \cdot \Re\left[g_{(\mathbf{s},\mathbf{0})+\mathbf{e}_j}(z) - g_{(\mathbf{s},\mathbf{0})}(z)\right] \le -C''(n-j+1)\log n.$$

Each of C, C' *and* C'' *is independent of* n, θ, *and* $\mathbf{s}$. *In particular*:

(iv) *The bound* (26) *holds unless* $n - j + 1 \le n^\kappa$ *for a fixed universal* $\kappa < 1$.

PROOF. (i) The restrictions on $|\mathbf{s}|$ and t allow us to write (25) as

$$\Re\left[g_{(\mathbf{s},\mathbf{0})+\mathbf{e}_j}(z) - g_{(\mathbf{s},\mathbf{0})}(z)\right] \le -\frac{(1-\frac{j-1}{n})(1+\frac{j}{n})(1+\frac{j^2}{n^2}+2\omega^2)}{2n(1+\omega^2)^2}\left(1 + O\left(\frac{1}{\log n}\right)\right).$$

The relative error term may evidently be ignored by choosing the constant C sufficiently large. Set $m = n - j + 1$. For all j, the LHS above is less than

$$-\frac{2}{(1+\omega^2)n^2 \log n} \cdot m \log n \cdot \max\left(\left(1 - O\left(\frac{m}{n}\right)\right), c\right)$$

for some fixed $c > 0$. Meanwhile, we have a bound of $\geq -m\log(\frac{n}{m}) + O(m)$ for the RHS of (26) by taking $\eta = 1$ in Lemma A.6. The first claim follows by choosing C sufficiently large to cover the case of small m.

(ii) For $|\mathbf{s}| \leq \frac{2\cdot 10^6 n}{\sigma}$ if ω is larger than a sufficiently large fixed constant then in fact the net effect of the $\frac{|\mathbf{s}|}{n}$ terms in the RHS of (25) is negative, so that the above argument goes through without further restriction on $|\mathbf{s}|$.

(iii) In any case, in the range $|\mathbf{s}| \leq \frac{2\cdot 10^6 n}{\sigma}$, inclusion of the factors of $\frac{|\mathbf{s}|}{n}$ changes the RHS of (25) by at most a constant factor, which proves the third claim.

(iv) The claim regarding κ now follows, since uniformly in $|\mathbf{s}| \leq \frac{2\cdot 10^6 n}{\sigma}$ we have that the LHS of (26) is less than $-c' m \log n$, for a fixed $c' > 0$, while the RHS is $\geq -m\log(\frac{n}{m}) + O(m)$, with $m = n - j + 1$ as before. □

Lemma 3.10 allows us to prove Proposition 3.1 for moderate representations that satisfy $a_n \ll \frac{n}{\sigma \log n}$. While the strict increment inequality (26) does not necessarily hold for every increment in the range $a_n \ll \frac{n}{\sigma}$, we are able to complete the proof of Lemma 3.6 by proving that this estimate holds on average when nearby shifts are moved together.

PROOF OF LEMMA 3.6. Let $\mathbf{s}$ correspond to $\mathbf{a}$ and set $|\mathbf{s}| = a_n = m$. Using standard basis vectors, we may write (in accordance with left-to-right shift)

$$\mathbf{s} = \sum_{j=1}^{m} \mathbf{e}_{i_j}, \qquad i_1 \leq i_2 \leq \cdots \leq i_m;$$

set also $\mathbf{s}^j = \sum_{k=1}^{j} \mathbf{e}_{i_k}$ with $\mathbf{s}^0 = \mathbf{0}$. Obviously,

$$\begin{aligned}&\frac{1}{2}n^2(\log n + C + 1)(1+\omega^2)\cdot \Re[g_{\mathbf{a}}(z) - g_{\mathbf{0}}(z)] + \log d_{\mathbf{a}}\\ &\quad = \sum_{j=1}^{m}\left[\frac{1}{2}n^2(\log n + C + 1)(1+\omega^2)\cdot \Re[g_{\mathbf{s}^j}(z) - g_{\mathbf{s}^{j-1}}(z)] + \log \frac{d_{\mathbf{s}^j}}{d_{\mathbf{s}^{j-1}}}\right].\end{aligned} \tag{27}$$

If either $m < \frac{n\sqrt{1+\omega^2}}{\log n}$ or $\omega > C'$, then we may apply either (i) or (ii) of the previous lemma to conclude that each term in the sum is negative so that we are done. So we may assume that m is large and that ω is bounded.

Call $k = \lfloor \frac{n\sqrt{1+\omega^2}}{\log n} \rfloor$ and let $m_0 > k$ denote the index of the first positive term in the sum. Note that $i_{m_0} \geq n - n^{\kappa}$ for some fixed $\kappa < 1$, by (iv) of Lemma 3.10.

We first argue that we may assume that $m - m_0$ is large by noting that the first k terms in the sum of (27) are substantially negative. Recall that ω is assumed to be bounded. Applying Lemma A.10 with $\eta = n - i_k + 1$, we deduce for n sufficiently large that

$$d_{\mathbf{s}^k} \geq \exp\left(\frac{k(n - i_k + 1)}{3}\right) \geq \exp\left(\frac{n(n - i_k + 1)}{3\log n}\right).$$

Since the sum up to k in (27) is negative, even when $C+1$ is replaced by C, it follows by comparing with $\frac{\log d_{\mathbf{s}^k}}{\log n}$ that

$$\begin{aligned}&\frac{1}{2}n^2(\log n + C + 1)\left(1+\omega^2\right)\cdot\Re\left[g_{\mathbf{s}^k}(z) - g_{\mathbf{0}}(z)\right] + \log d_{\mathbf{s}^k}\\ &\qquad \le \left(1+o(1)\right)\frac{-n(n-i_k+1)}{3(\log n)^2}.\end{aligned} \tag{28}$$

Now for $j > k$, Lemma A.6 gives that

$$\log \frac{d_{\mathbf{s}^j}}{d_{\mathbf{s}^{j-1}}} \le (n-i_j+1)\left(\log n + O(1)\right) \le (n-i_k+1)\left(\log n + O(1)\right), \tag{29}$$

and so we immediately obtain that (27) is negative unless $m - m_0 \ge \frac{n}{4(\log n)^3}$.

Let $\delta > 0$ be a small, fixed, positive constant and set

$$J = \left\lceil \frac{\log(n - i_{m_0} + 1)}{\log(1+\delta)} \right\rceil \ll \log n.$$

We partition $\{i_{m_0}, i_{m_0+1}, \dots, i_m\}$ into J sets by defining

$$S_j = \{i_\lambda : i_\lambda \in n - I_j\}, \qquad I_j = \left[(1+\delta)^{j-1}, (1+\delta)^j\right), \qquad j = 1, 2, \dots, J.$$

We perform a trimming on the sets S_j. Let $M = \frac{2n}{3J(\log n)^4} \gg \frac{n}{(\log n)^5}$. We discard all S_j with $|S_j| < M$. From each remaining set, we form S_j' by removing the smallest $\frac{M}{2}$ of the i_λ from S_j. Altogether we have discarded at most $\frac{3}{2}JM \le \frac{n}{(\log n)^4}$ of the i_λ. Now in view of (29), the total contribution to (27) of the discarded i_λ is bounded by

$$\left(1+o(1)\right)(n-i_k+1)\frac{n}{(\log n)^3},$$

which is negligible; see (28).

We now claim that if δ was chosen to be appropriately small, then for each remaining $i_\lambda \in S_j'$,

$$\frac{1}{2}n^2(\log n + C + 1)\left(1+\omega^2\right)\cdot\Re\left[g_{\mathbf{s}^\lambda}(z) - g_{\mathbf{s}^{\lambda-1}}(z)\right] \le -\log\frac{d_{\mathbf{s}^\lambda}}{d_{\mathbf{s}^{\lambda-1}}}.$$

Setting $\mathbf{s} = \mathbf{s}^{\lambda-1}$, $j = i_\lambda$ and $\eta = 3\delta(n - i_\lambda + 1)$ in Lemma A.6 we find that $|\mathbf{s}|_{\eta,\mathrm{loc}} \ge \frac{M}{2}$, (it includes all of the deleted points from the set containing i_λ)

$$\log\frac{d_{\mathbf{s}^\lambda}}{d_{\mathbf{s}^{\lambda-1}}} \le 3\delta(n-i_k+1)\log n + O\left((n-i_k+1)\log\log n\right).$$

Choosing 3δ sufficiently smaller than C'' from the previous lemma proves the claim and completes the proof. $\square$

3.3.2. *Integral tail estimate.* Lemma 3.6 completes our supremum bound for small t, so we now turn to bounding the tail of the integral in (21).

LEMMA 3.11. *Introduce the function*

$$m_\omega(t) = \max_{\mathbf{a}: a_n \le \frac{2\cdot 10^6 n}{\sigma}} \Re\big(g_{\mathbf{a}}(\omega + it)\big). \tag{30}$$

This satisfies the following properties:

1. *The maximum in $m_\omega(t)$ is achieved at $\mathbf{a} = \mathbf{0}$ when $t^2 \le \omega^2 + \frac{1}{4}$.*
2. *The monotonicity property $m_\omega(t) \ge m_\omega(t + \frac{1}{n})$ holds for all $t \ge 0$.*
3. *For $t > 4 \cdot 10^6 \sqrt{1+\omega^2}$, there is a $c > 0$ such that $m_\omega(2t) \le m_\omega(t) - cn$.*

PROOF. Recall

$$\Re\big(g_{\mathbf{a}}(\omega + it)\big) = \omega\theta - \frac{1}{n}\sum_j \log\big(\ell(\alpha_j)\big),$$

with

$$\ell(x; \omega, t) = \big(x^2 + \omega^2 - t^2\big)^2 + 4\omega^2 t^2.$$

Set $c = \sqrt{\max(0, t^2 - \omega^2)}$. Since in $x > 0$ we have $\ell(x; \omega, t)$ is increasing in $|x - c|$, the optimal $\mathbf{a}$ has α_j forming a continuous block about c, that is, is of the form $(k)^n$ for some k. For (1), by the last claim of Lemma 3.8 for $0 \le \delta \le c$, $\ell(c - \delta) < \ell(c + \delta)$, so that the optimal choice is a continuous block including 0.

The second claim holds, since if $(k)^n$, $k \ge 1$ achieves the maximum in $m_\omega(t + \frac{1}{n})$ then

$$m_\omega(t) \ge \Re\big(g_{(k-1)^n}(\omega + it)\big) > \Re\left(g_{(k)^n}\left(\omega + i\left(t + \frac{1}{n}\right)\right)\right) = m_\omega\left(t + \frac{1}{n}\right),$$

while if $\mathbf{0}$ achieves the maximum in $m_\omega(t + \frac{1}{n})$ then

$$m_\omega(t) \ge \Re\big(g_{\mathbf{0}}(\omega + it)\big) > \Re\left(g_{\mathbf{0}}\left(\omega + i\left(t + \frac{1}{n}\right)\right)\right) = m_\omega\left(t + \frac{1}{n}\right),$$

by applying the last claim of Lemma 3.4.

Finally, notice that the restriction on a_n is equivalent to $\alpha_n \le 1 + (2 \cdot 10^6 + o(1))\sqrt{1+\omega^2}$. For all $|t| > 4 \cdot 10^6 \sqrt{1+\omega^2}$, the maximizing $\mathbf{a}$ is easily seen to be the block with a_n as large as possible. The last claim now follows from Euclidean geometry. $\square$

We now bound the tail of the integral for $r_{\mathbf{a}}(\theta)$.

PROOF OF LEMMA 3.7. Recall that this is the claim

$$\int_{|t| > 10^4 \sqrt{\frac{1+\omega^2}{\log n}}} \big|e^{n g_{\mathbf{a}}(\omega + it)}\big|\, dt \le \exp\left(n g_{\mathbf{0}}(\omega) - \frac{(10^8 + o(1))n}{\log n}\right).$$

In the integral, we may evidently replace $g_{\mathbf{a}}(\omega+it)$ with $m_\omega(t)$. For $|t| = 10^4\sqrt{\frac{1+\omega^2}{\log n}} + O(\frac{1}{n})$, for sufficiently large n, $m_\omega(t) = \Re(g_{\mathbf{0}}(\omega+it))$ and Taylor expansion of $g_{\mathbf{0}}$ around ω gives; see Lemma 3.4,

$$\Re g_{\mathbf{0}}(\omega+it) \le g_{\mathbf{0}}(\omega) - \frac{10^8+o(1)}{\log n}.$$

The bound now follows easily on applying the monotonicity of m_ω and rapid decay for $|t| > 4\cdot 10^6\sqrt{1+\omega^2}$. $\square$

3.4. *Large representations.* Among those large representations, for which $a_n > \frac{2\cdot 10^6 n}{\sigma}$, we distinguish further two kinds. Let $k = n - \lfloor\frac{\sigma n}{2}\rfloor$. If $a_k < \frac{2\cdot 10^6 k}{\sigma}$ we say that $\rho_{\mathbf{a}}$ has "controlled growth". Otherwise, we say that $\rho_{\mathbf{a}}$ is "giant".

3.4.1. *Controlled growth.* In the case that $\rho_{\mathbf{a}}$ has controlled growth, let $k < m < n$ be maximal such that $a_m < \frac{2\cdot 10^6 m}{\sigma}$. We are going to view $r_{\mathbf{a}}(\theta)$ as a perturbation of the character ratio $r_{\mathbf{a}(m)}(\theta)$ on $\mathrm{SO}(2m+1)$, where $\mathbf{a}(m)$ denotes $\mathbf{a}$ truncated at a_m.

Let ω' denote the saddle point for $g_{\mathbf{0}(m)}$, solving $g'_{\mathbf{0}(m)}(\omega') = 0$ for $\mathrm{SO}(2m+1)$, and set $\omega^* = \frac{m}{n}\omega'$.

We choose a contour in the integral formula for $r_{\mathbf{a}}(\theta)$ passing through ω^* and given by $\mathscr{C} = \mathscr{C}_1 \cup \mathscr{C}_2$ where

$$\mathscr{C}_1 = \left\{ z = \omega^* + it : |t| \le \frac{m}{n} 10^4 \sqrt{\frac{1+\omega'^2}{\log m}} \right\}$$

and where $\mathscr{C}_2$ depends upon ω^*,

$$\omega^* > \frac{1}{400}: \qquad \mathscr{C}_2 = \left\{ z = \omega^* + it : |t| > 10^4 \frac{m}{n}\sqrt{\frac{1+\omega'^2}{\log m}} \right\},$$

$$\begin{aligned} \omega^* \le \frac{1}{400}: \qquad \mathscr{C}_2 = &\left\{ \omega^* + it : 10^4\frac{m}{n}\sqrt{\frac{1+\omega'^2}{\log n}} \le |t| \le \frac{m}{n}\frac{\sqrt{1+\omega'^2}}{3} \right\} \\ &\cup \left\{ \alpha \pm \frac{m}{n}\frac{i\sqrt{1+\omega'^2}}{3} : \alpha \in \left[\omega^*, \frac{1}{400}\right] \right\} \\ &\cup \left\{ \frac{1}{400} + it : |t| > \frac{m}{n}\frac{\sqrt{1+\omega'^2}}{3} \right\}. \end{aligned}$$

For the integrand on $\mathscr{C}_1$, we prove the following estimate.

LEMMA 3.12. *For* $|t| < 10^4 \frac{m}{n}\sqrt{\frac{1+\omega^2}{\log m}}$ *and for a sufficiently large fixed constant* C,

$$\begin{aligned}&\Re\left(ng_{\mathbf{a}}(\omega^* + it) - mg_{\mathbf{0}(m)}\left(\omega' + \frac{n}{m}it\right)\right) - (n-m)\log(\sigma^2)\\ &\qquad \le -\frac{2\sigma^2 \log d_{\mathbf{a}}}{n(\log n + C)}.\end{aligned} \tag{31}$$

For the integral on $\mathscr{C}_2$, we prove the following.

LEMMA 3.13. *There is* $c > 0$ *such that*

$$\begin{aligned}&\frac{\sqrt{2\pi m g_{\mathbf{0}}^{(2)}(\omega')}}{(\sin\frac{\theta}{2})^{2(n-m)}} \int_{z\in\mathscr{C}_2} \exp\bigl(n\Re g_{\mathbf{a}}(z) - mg_{\mathbf{0}}(\omega')\bigr)\, d|z|\\ &\qquad \le \exp\left(\frac{-cn}{\log n} - \frac{2\sigma^2\log d_{\mathbf{a}}}{n(\log n + C)}\right).\end{aligned}$$

We give the short deduction of Proposition 3.1 for controlled-growth representations.

PROOF OF PROPOSITION 3.1. Note that, as we are comparing characters on $\mathrm{SO}(2n+1)$ and $\mathrm{SO}(2m+1)$, the leading factors in the integral representation have ratio

$$\frac{(2n-1)!}{(2m-1)!}\frac{(2m\sin\frac{\theta}{2})^{2m-1}}{(2n\sin\frac{\theta}{2})^{2n-1}} < \frac{1}{(\sin\frac{\theta}{2})^{2(n-m)}}.$$

Arguing as in the proof for moderate representations,

$$\begin{aligned}&\left(1 - O\left(\frac{1}{n}\right)\right)|r_{\mathbf{a}}(\theta)|\\ &\quad \le \frac{1}{(\sin\frac{\theta}{2})^{2(n-m)}} \sup_{|t|\le 10^4\frac{m}{n}\sqrt{\frac{1+\omega^2}{\log m}}} \exp\left(\Re\left(ng_{\mathbf{a}}(\omega^*+it) - mg_{\mathbf{0}}\left(\omega' + \frac{n}{m}it\right)\right)\right)\\ &\qquad + \frac{\sqrt{2\pi m g_{\mathbf{0}}^{(2)}(\omega')}}{(\sin\frac{\theta}{2})^{2(n-m)}} \int_{z\in\mathscr{C}_2} \exp\bigl(n\Re g_{\mathbf{a}}(z) - mg_{\mathbf{0}}(\omega')\bigr)\, d|z|.\end{aligned}$$

Putting in the bounds of Lemmas 3.12 and 3.13, we obtain

$$\log|r_{\mathbf{a}}(\theta)| \le O\left(\frac{1}{n}\right) - \frac{2\sigma^2\log d_{\mathbf{a}}}{n(\log n + C)}.$$

The error term of size $O(\frac{1}{n})$ is dealt with as for the moderate representations. $\square$

3.4.2. *Proof of estimates*. Throughout this section, we argue by keeping track of the incremental change to the integrand $e^{ng_{\mathbf{a}(m)}(z)}$ and to $d_{\mathbf{a}(m)}$ as we append successively each a_j, $j > m$. In Appendix A, we write the dimension formula incrementally as

$$d_{\mathbf{a}} = \prod_{k=1}^{n} d_{\mathbf{a}}(k), \qquad d_{\mathbf{a}}(k) = \frac{\tilde{a}_k}{k-\frac{1}{2}} \prod_{1\le j<k} \frac{\tilde{a}_k^2 - \tilde{a}_j^2}{(k-\frac{1}{2})^2 - (j-\frac{1}{2})^2},$$
$$d_{\mathbf{a}(m)} = \prod_{k=1}^{m} d_{\mathbf{a}}(k).$$

For those $k > m$, Lemma A.1 gives

$$\log d_{\mathbf{a}}(k) \le (2k-1)\log \alpha_k + O(n).$$

PROOF OF LEMMA 3.12. We may write the LHS of (31) as [recall $\sigma^2 = \frac{1+O((\log n)^{-2})}{1+\omega^2}$]

$$m\Re\left(g_{\mathbf{a}(m)}\left(\omega' + i\frac{nt}{m}\right) - g_{\mathbf{0}(m)}\left(\omega' + i\frac{nt}{m}\right)\right)$$
$$- \sum_{j=m+1}^{n} \Re \log \frac{\alpha_j^2 + (\omega^* + it)^2}{1+\omega^2} + O\left(\frac{n-m}{(\log n)^2}\right).$$

Since $\rho_{\mathbf{a}(m)}$ is either a small or a moderate representation of $\mathrm{SO}(2m+1)$,

$$m\Re\big(g_{\mathbf{a}(m)}(\omega' + it) - g_{\mathbf{0}(m)}(\omega' + it)\big) \le -\frac{2\sigma^2 \log d_{\mathbf{a}(m)}}{m(\log m + C)},$$

and, therefore, the proof is completed on observing that, for any fixed $c > 0$, for n sufficiently large we have

$$\sum_{j=m+1}^{n} \left[\frac{2\sigma^2 \log d_{\mathbf{a}}(j)}{n(\log n + C)} - \Re \log \frac{\alpha_j^2 + (\omega^* + it)^2}{1+\omega^2}\right] < \frac{-c(n-m)}{(\log n)^2}$$

in view of $t = o(1)$, the bound for $d_{\mathbf{a}}(j)$ above, and using $\alpha_j \ge 10^6\sqrt{1+\omega^2}$ for all $m < j \le n$. $\square$

We prove the following estimate before proving Lemma 3.13.

LEMMA 3.14. *Keep the definitions of m and $\mathscr{C}_2$ from above and let $m < j \le n$. We have*

$$\inf_{z\in\mathscr{C}_2} |z^2 + \alpha_j^2| \ge 2\left(\frac{1}{400} \vee \omega^*\right)\alpha_j.$$

PROOF. Observe that on any line $\Re(z) = \alpha$, the minimum of $|z^2 - (i\alpha_j)^2|$ is at least $2\alpha\alpha_j$. It only remains to check that for $\omega^* < \frac{1}{400}$, the minimum of $|(\omega^* + it)^2 - (i\alpha_j)^2|$ for $t < \frac{m}{n}\frac{\sqrt{1+\omega^{*2}}}{3}$ exceeds $\frac{\alpha_j}{200}$, but this is obvious geometrically. $\square$

We now bound the integral over the contour $\mathscr{C}_2$.

PROOF OF LEMMA 3.13. In view of the last lemma, the integral is bounded by

$$\exp\left(O\left(\frac{n-m}{(\log n)^2} + \log n\right)\right)$$
$$\times \prod_{j=m+1}^{n} \frac{1+\omega^2}{2\alpha_j(\frac{1}{400} \vee \omega^*)} \int_{z\in\mathscr{C}_2} \exp\left(\Re\left[m g_{\mathbf{a}(m)}\left(\frac{n}{m}z\right) - m g_{\mathbf{0}(m)}(\omega')\right]\right) d|z|.$$

We claim that the latter integral is bounded by (note that the first factor covers the leading error above)

$$\exp\left(\frac{-(10^8 + o(1))m}{\log m}\right) \le \exp\left(\frac{-cn}{\log n} - \frac{2\sigma^2 \log d_{\mathbf{a}(m)}}{m(\log m + C)}\right).$$

If $\omega^* > \frac{1}{400}$, then this follows from Lemma 3.7 of the previous section. If $\omega^* \le \frac{1}{400}$, Lemma 3.7 still bounds the vertical part of the integral that is nearest the real axis, so it remains to bound the horizontal part, and the vertical part that extends to $\pm i\infty$. By Lemma 3.4,

$$g_{\mathbf{0}(m)}\left(\omega' + \frac{i\sqrt{1+\omega'^2}}{3}\right) - g_{\mathbf{0}(m)}(\omega') \le \frac{-1}{40}.$$

Using $\theta < \pi$ and $n - m < m$, we find that throughout the horizontal part of $\mathscr{C}_2$, where $|t| = \frac{\sqrt{1+\omega^2}}{3}$, the integrand is bounded by

$$\exp\left(-\left(\frac{1}{40} - \frac{\pi}{200}\right)m + o(m)\right) < \exp\left(-\frac{m}{200}\right).$$

Therefore, the remainder of the integral contributes $\exp(-\frac{m}{200} + o(m))$ by mimicking the proof of Lemma 3.7.

Using $\frac{1}{2\max(a,b)} \le \frac{1}{\sqrt{a^2+b^2}}$, each term in the product is bounded by $\frac{20\sqrt{1+\omega^2}}{\alpha_j} \le (\frac{\sqrt{1+\omega^2}}{\alpha_j})^{\frac{1}{10}}$. Moreover, the bound for $\alpha_j > 1$,

$$\log d_{\mathbf{a}}(j) \le (2j-1)\log\alpha_j + O(n),$$

implies that

$$-\frac{2\sigma^2\sum_{j=m+1}^n \log d_{\mathbf{a}}(j)}{n(\log n + C)} \geq \frac{1}{10}\sum_{j=m+1}^n \log\left(\frac{\sqrt{1+\omega^2}}{\alpha_j}\right)$$

for all n sufficiently large, which suffices to complete the dimension increment. □

3.4.3. *Giant representations.* Once the representation is giant, trivial considerations suffice to bound the character ratio.

PROOF OF PROPOSITION 3.1. We take the integral contour on the line $\Re(z) = \omega \vee 1$ so as to avoid nearby poles of the integrand, and put in a sup bound for all but the first factor from the product, reserving the first factor to ensure convergence. This yields

$$\begin{aligned}\left|r_{\mathbf{a}}(\theta)\right| &\leq O\left(\sqrt{ng_0^{(2)}(\omega)}\right)\int_{\Re(z)=1\vee\omega} e^{n\Re(g_{\mathbf{a}}(z)-g_{\mathbf{0}}(\omega))}\,d|z| \\ &\ll \frac{\sqrt{n}e^{\theta(1\vee\omega-\omega)n}}{\sqrt{1+\omega^2}}\int\left|\frac{(\frac{1}{n})^2+\omega^2}{\alpha_1^2+((1\vee\omega)+it)^2}\right|dt \sup_{\Re(z)=1\vee\omega}\prod_{j=2}^n\left|\frac{(\frac{j-\frac{1}{2}}{n})^2+\omega^2}{\alpha_j^2+z^2}\right|.\end{aligned}$$

The integral is $O(\omega\wedge\omega^2)$. If $\alpha_j < 1\vee\omega$, then the denominator of the jth term in the product is minimized at $\Re(z) = 0$ with minimum value $\alpha_j^2 + (1\vee\omega)^2$. Thus, the jth term is bounded by 1. Otherwise, if $\alpha_j \geq 1\vee\omega$ the minimum value of the denominator is $2\alpha_j(1\vee\omega) > \alpha_j\sqrt{1+\omega^2}$. Therefore, we obtain the bound

$$\left|r_{\mathbf{a}}(\theta)\right| \leq O\left(n^{\frac{1}{2}}e^{((1\vee\omega)-\omega)\theta n}\right)\prod_{\substack{j:\alpha_j>1\vee\omega\\ j>1}}\frac{\sqrt{1+\omega^2}}{\alpha_j}.$$

Lemma A.1 gives the dimension bound

$$\log d_{\mathbf{a}} \leq O\left(n^2\right) + 2n\sum_{k:\alpha_k\geq 1}\log\alpha_k.$$

We deduce that for giant representations

$$\begin{aligned}\log\left|r_{\mathbf{a}}(\theta)\right| + \frac{2\sigma^2\log d_{\mathbf{a}}}{n\log n} &\leq -\sum_{\substack{j:\alpha_j>1\vee\omega\\ j>1}}\left(\left(1-\frac{2\sigma^2}{\log n}\right)\left(\log\alpha_j - \frac{1}{2}\log\left(1+\omega^2\right)\right)\right) \\ &\quad + \delta_{\omega<1}(1-\omega)\theta n + O\left(\frac{\sigma^2|\log\sigma|n}{\log n}\right).\end{aligned} \tag{32}$$

By virtue of being giant, for

$$k = n - \left\lfloor \frac{n}{2\sqrt{1+\omega^2}} \right\rfloor$$

we have

$$a_k \geq \frac{2 \cdot 10^6 k}{\sqrt{1+\omega^2}} \quad \Rightarrow \quad \alpha_k \geq \frac{10^6}{\sqrt{1+\omega^2}}.$$

First, consider $\omega < 1$. In this case, the sum over $j \geq k$ contributes

$$\lesssim -\frac{6 \ln 10}{2\sqrt{2}} n < -4.88n$$

while the sum over $j < k$ contributes

$$\lesssim n \frac{\ln 2}{2} < 0.35n.$$

Since $0.35 + \pi < 4.88$, (32) is asymptotically negative.

For $\omega > 1$, the sum over $j \geq k$ contributes

$$\lesssim -\frac{3 \ln 10}{\sqrt{1+\omega^2}} n < \frac{-6.9n}{\sqrt{1+\omega^2}}$$

while the sum over $j < k$ contributes

$$\lesssim n \ln\left(\sqrt{1 + \frac{1}{\omega^2}}\right) \leq \frac{n}{2\omega^2} < \frac{n}{\sqrt{1+\omega^2}}.$$

Thus, again, (32) is asymptotically negative. □

3.5. *The case of θ close to π.* Our treatment of moderate and large representations in the last two sections completes the proof Theorem 1.2 for θ varying with n in the range $\frac{\log n}{\sqrt{n}} \leq \theta \leq \pi - \frac{(\log n)^2}{n}$. The case $\theta \approx \pi$ is difficult because the location of the saddle point is harder to determine. We now show that a modification of Rosenthal's argument in the case $\theta = \pi$ suffices to cover the range $\theta \geq \pi - \frac{(\log n)^2}{n}$.

For this range of θ, $\sin \frac{\theta}{2} = 1 - O(n^{-2+\varepsilon})$, and so we seek the estimate

$$\log\left|r_{\mathbf{a}}(\theta)\right| \leq -\frac{2 \log d_{\mathbf{a}}}{n(\log n + O(1))}.$$

The section on small representations gives this estimate already for any $\mathbf{a}$ satisfying $\sum_j a_j \leq \frac{n}{\log n}$, so we may assume that this does not hold. Applying Lemma A.9, we deduce that

$$\log d_{\mathbf{a}} \gg \frac{n}{\log n}.$$

Introduce

$$\overline{r}_{\mathbf{a}}(\pi) = \frac{(2n-1)!}{2^{2n-1}} \sum_{j=1}^{n} \left| \frac{\sin(\tilde{a}_j \pi)}{\tilde{a}_j \prod_{r \neq j} (\tilde{a}_r^2 - \tilde{a}_s^2)} \right|.$$

Note that

$$r_{\mathbf{a}}(\theta) \leq \left(\sin\frac{\theta}{2}\right)^{-2n+1} \overline{r}_{\mathbf{a}}(\pi) \leq \left(1 + O\left(\frac{(\log n)^4}{n}\right)\right) \overline{r}_{\mathbf{a}}(\pi).$$

Rosenthal's upper bound in the case $\theta = \pi$ is proven by showing that $\log \overline{r}_{\mathbf{a}}(\pi) \leq \frac{-2\log d_{\mathbf{a}}}{n(\log n + C)}$. Therefore, bounding the remaining factor in $r_{\mathbf{a}}(\theta)$,

$$\log\left|r_{\mathbf{a}}(\theta)\right| \leq O\left(\frac{(\log n)^4}{n}\right) - \frac{2\log d_{\mathbf{a}}}{n(\log n + C)} \leq -\frac{2\log d_{\mathbf{a}}}{n(\log n + C)}\left(1 + O\left(n^{-1+\varepsilon}\right)\right).$$

Thus, the error may be absorbed into the constant.

4. Mixture of rotations. The remainder of the paper concerns random walks wherein at each step the angle of rotation is chosen from a fixed distribution. If ξ is this probability distribution on $\mathbb{T}_0$, the mixture walk P_ξ has Fourier coefficients

$$\xi(r_{\mathbf{a}}) = \int_{\mathbb{T}_0} r_{\mathbf{a}}(\theta)\, d\xi(\theta).$$

Generically, we expect that the mixing time in total variation is controlled by the eigenvalue at the lowest dimensional nontrivial representation. In this case, this is the natural representation of dimension $2n+1$, with eigenvalue

$$\xi(r_{(\mathbf{0},1)}) = \int_{\mathbb{T}_0} \left(1 - \frac{4(\sin\frac{\theta}{2})^2}{2n+1}\right) d\xi(\theta) = 1 - \frac{4\xi(\sigma^2)}{2n+1},$$

leading to a predicted total variation mixing time of

$$\frac{\log d_{(\mathbf{0},1)}}{-\log \xi(r_{(\mathbf{0},1)})} \sim \frac{n \log n}{2\xi(\sigma^2)}. \tag{33}$$

In the case of the mixture walk, the situation is less clear because the quantity $\frac{\log d_{\mathbf{a}}}{-\log|\xi(r_{\mathbf{a}})|}$ is not necessarily maximized at the natural representation. Heuristically, this is suggested by our Proposition 3.1, which proves the bound

$$\left|\xi(r_{\mathbf{a}})\right| \leq \int_{\mathbb{T}_0} d_{\mathbf{a}}^{\frac{-2\sigma^2}{n(\log n + C)}}\, d\xi(\theta).$$

This is only an upper bound, but for large representations $\rho_{\mathbf{a}}$ this bound suggests that $\xi(r_{\mathbf{a}})$ is largely controlled by the part of ξ nearest 0 (the issue is that the integration is not in the exponent).

In the proof that follows, we confirm the natural representation prediction (33) for the mixing time in total variation, proving Theorem 1.1, but as the above discussion suggests, we do not follow the customary path of bounding the L^1 norm with the L^2 norm, instead using a truncation argument to bypass the larger dimensional representations. In the following section, we prove a cut-off for the L^2 norm at a point which depends on the smallest point in the support of measure ξ, thus confirming Theorem 1.3.

4.1. *Random θ in total variation*: *Proof of Theorem* 1.1. The lower bound follows from a standard application of the second moment method (see [7]) applied to the function $\chi_{(\mathbf{0},1)}$. The necessary estimates appear in Example 1:

$$d_{(\mathbf{0},1)} = 2n+1, \qquad \xi(r_{(\mathbf{0},1)}) = 1 - \frac{2\xi(\sigma^2)}{n} + O(n^{-2}),$$

$$d_{(\mathbf{0},1,1)} = n(2n+1), \qquad \xi(r_{(\mathbf{0},1,1)}) = 1 - \frac{4\xi(\sigma^2)}{n} + O(n^{-2}),$$

$$d_{(\mathbf{0},2)} = n(2n+3), \qquad \xi(r_{(\mathbf{0},2)}) = 1 - \frac{4\xi(\sigma^2)}{n} + O(n^{-2})$$

together with the decomposition

$$\chi_{(\mathbf{0},1)}^2 = \chi_{\mathbf{0}} + \chi_{(\mathbf{0},1,1)} + \chi_{(\mathbf{0},2)}.$$

We refer the reader to Rosenthal's proof of the lower bound in the case of deterministic θ, [18] Theorem 2.1, where the details are the same.

For the upper bound, recall that we set $\mu_\theta = \delta_{\mathrm{Id}} \cdot P_\theta$ for the generating measure of the fixed-θ walk. Proposition 3.1 guarantees that there exists a $C > 0$ such for all n sufficiently large, and for all nontrivial representations $\mathbf{a}$,

$$\log\left|\hat{\mu}_\theta(\chi_{\mathbf{a}})\right| = \log\left|r_{\mathbf{a}}(\theta)\right| \le -\frac{2\sigma^2(\theta)}{n(\log n + C)} \log d_{\mathbf{a}}.$$

Let $c > 0$ and let $t = \frac{n(\log n + 2C + 2c)}{2\xi(\sigma)^2}$. Conditioning on the choices of θ at each step of the walk, and applying the triangle inequality, we have

$$\left\|\delta_{\mathrm{Id}} \cdot P_\xi^t - \nu\right\|_{\mathrm{TV}} \le \iiint_{\boldsymbol{\theta} \in \mathbb{T}_0^t} \|\mu_{\theta_1} * \cdots * \mu_{\theta_t} - \nu\|_{\mathrm{TV}}\, d\xi(\theta_1) \cdots d\xi(\theta_t).$$

Since the total variation distance is bounded by 1, for any measurable set $E \subset \mathbb{T}_0^t$ we obtain the bound

$$\left\|\delta_{\mathrm{Id}} \cdot P_\xi^t - \nu\right\|_{\mathrm{TV}} \le \xi^{\otimes t}(E) + \iiint_{\boldsymbol{\theta} \in E^c} \left(\frac{1}{4} \sum_{\mathbf{a} \ne \mathbf{0}} d_{\mathbf{a}}^2 \prod_{j=1}^t \left|\hat{\mu}_{\theta_j}(\chi_{\mathbf{a}})\right|^2\right)^{\frac{1}{2}} d\xi(\theta_1) \cdots d\xi(\theta_t),$$

by applying the upper bound lemma on E^c. We now define

$$E = \left\{\boldsymbol{\theta} \in (\mathbb{T}_0)^t : \sum_{j=1}^{t} 2\sigma^2(\theta_j) \le n(\log n + C + c)\right\}.$$

Standard tail estimates give that there exists a constant $K > 0$ such that

$$\xi^{\otimes t}(E) \ll \exp\left(-\frac{(C+c)^2 n}{K \log n}\right).$$

Meanwhile, applying Proposition 3.2, the square of the integrand over E^c is bounded by

$$\begin{aligned}\sup_{\boldsymbol{\theta}\in E^c} \frac{1}{4}\sum_{\mathbf{a}\neq \mathbf{0}} \exp\left(2\log d_{\mathbf{a}}\left[1 - \frac{1}{n(\log n + C)}\sum_{j=1}^{t} 2\sigma^2(\theta_j)\right]\right) &\ll \sum_{\mathbf{a}\neq\mathbf{0}} d_{\mathbf{a}}^{\frac{-c}{\log n}} \\ &= O\big(e^{-\frac{c}{8}}\big),\end{aligned}$$

completing the proof.

4.2. *Random θ walk in L^2: Theorem* 1.3. Our proof of Theorem 1.3 exhibits a competition between character ratio and dimension growth for small and moderate representations. The large representations are inconsequential. We first prove the lower bound, which is illustrative. Then we discuss how to modify the argument from the fixed θ setting to obtain the upper bound.

In this section, it is convenient to use as reference points block representations of the form $\mathbf{a}(s,t) = (\mathbf{0}_{n-n^t}, (n^s)_{n^t})$, $0 < s, t < 1$, where n^s, n^t are assumed to be integers. We write $\mathbf{a}(s) = \mathbf{a}(s,s)$. In Lemma A.7 of Appendix A, we give the estimate

$$\log d_{\mathbf{a}(s,t)} = (1 - s \vee t)n^{s+t}\big(\log n + O(1)\big). \tag{34}$$

The estimate regarding character ratios which we need for the lower bound in Theorem 1.3 is the following one.

PROPOSITION 4.1. *Uniformly for θ in compact subsets of $(0,\pi)$ and for $\frac{1}{2} \le u \le \frac{3}{4}$, we have*

$$r_{\mathbf{a}(u)}(\theta) = \big(1 + O\big(n^{4u-3}\big)\big)\exp\left(-2\left(\sin\frac{\theta}{2}\right)^2 n^{2u-1}\right).$$

PROOF OF LOWER BOUND IN THEOREM 1.3. We give only the proof for the sharper lower bound assuming a positive one-sided derivative of ξ at q, the lower bound for general ξ, which requires only the leading order term, being easier. For this bound, it suffices to prove the following two results separately:

1. For $t = \frac{1}{2\xi(\sigma^2)} n(\log n - c)$,

$$\|\delta_{\mathrm{Id}} \cdot P_\xi^t - \nu\|_2 \to \infty, \qquad n \to \infty, c \to \infty.$$

2. For $t = \frac{1}{4\sigma^2(q)} n(\log n - 3\log\log n - c)$,

$$\|\delta_{\mathrm{Id}} \cdot P_\xi^t - \nu\|_2 \to \infty, \qquad n \to \infty, c \to \infty.$$

We prove both estimates by dropping all but one term in the sum

$$\|\delta_{\mathrm{Id}} \cdot P_\xi^t - \nu\|_2^2 = \sum_{\mathbf{a} \neq \mathbf{0}} d_{\mathbf{a}}^2 |\xi(r_{\mathbf{a}})|^{2t}.$$

In the first case, take the natural representation $\mathbf{a} = (\mathbf{0}, 1)$, with dimension $d_{(\mathbf{0},1)} = 2n+1$ and character ratio $\xi(r_{(\mathbf{0},1)}) = 1 - \frac{2\xi(\sigma^2)}{n} + O(\frac{1}{n^2})$, which plainly suffices.

Notice that the second case is contained in the first unless $\sigma^2(q) < \frac{\xi(\sigma^2)}{2} \le \frac{1}{2}$, so we assume $\sigma^2(q) < \frac{1}{2}$ from now on. In this case, we consider a representation of the form $\mathbf{a}(u)$, where u is slightly larger than $\frac{1}{2}$. Introduce parameters Δ and β characterized by

$$e^\beta = \frac{1}{2}\Delta \log \Delta = \frac{\log n}{2(\sin \frac{q}{2})^2}$$

and set $u = \frac{1}{2} + \frac{\beta}{2\log n}$. Note that our dimension evaluation in (34) gives

$$\left(1 + O\left(\frac{1}{\log n}\right)\right) \log d_{\mathbf{a}(u)} = \left(\frac{1}{2} - \frac{\log\log n}{2\log n}\right) e^\beta n \log n.$$

Since ξ has positive one-sided derivative at q, we have

$$\xi([q, q + \Delta^{-1}]) \ge \frac{\delta}{\Delta}$$

for some $\delta > 0$. Proposition 4.1 gives an asymptotic evaluation of $r_{\mathbf{a}(u)}(\theta)$ for $\theta \in [q, \pi - \varepsilon]$, so in evaluating the contribution of $\theta \in [\pi - \varepsilon, \pi]$ to $\xi(r_{\mathbf{a}(u)})$ we simply put in the bound of Proposition 3.1 for the fixed θ walk,

$$\log |r_{\mathbf{a}(u)}(\theta)| \lesssim -\frac{(2 - \frac{\varepsilon^2}{2}) \log d_{\mathbf{a}(u)}}{n(\log n + O(1))}.$$

Putting in the asymptotic of Proposition 4.1 for $r_{\mathbf{a}(u)}(\theta)$ for θ on the small interval $[q, q+\Delta^{-1}]$ and using only that the character ratio is nonnegative on the remaining bulk, $\theta \in [q + \Delta^{-1}, \pi - \varepsilon]$, we deduce the lower bound

$$\begin{aligned}\xi(r_{\mathbf{a}(u)}) &\geq (1 + O(n^{3u-2})) \int_{\theta \in [q, q+\Delta^{-1}]} \exp\left(-2\left(\sin\frac{\theta}{2}\right)^2 e^{\beta}\right) d\xi(\theta) \\ &\quad - \int_{|\theta-\pi|<\varepsilon} \exp(-(1+O(\varepsilon^2))e^{\beta})\, d\xi(\theta) \\ &\geq \delta\Delta^{-1} \exp\left(-2\left(\sin\frac{q}{2}\right)^2 e^{\beta}\right) \exp(-2\Delta^{-1}e^{\beta}) - \exp(-(1+O(\varepsilon^2))e^{\beta}).\end{aligned}$$

Since $\sigma^2(q) < \frac{1}{2}$, this furnishes an asymptotic of shape

$$\xi(r_{\mathbf{a}(u)}) \gtrsim \frac{\delta}{\Delta^2} \exp\left(-2\left(\sin\frac{q}{2}\right)^2 e^{\beta}\right)$$

if ε is chosen sufficiently small [use $\exp(2\Delta^{-1}e^{\beta}) = \Delta \asymp \frac{\log n}{\log\log n}$ while $e^{\beta} \asymp \log n$]. Therefore,

$$\log \xi(r_{\mathbf{a}(u)}) \geq -2\left(\sin\frac{q}{2}\right)^2 e^{\beta} - 2\log\log n - O(1) = -\log n - 2\log\log n + O(1).$$

Recall that

$$\log d_{\mathbf{a}(u)} = \frac{1}{2(\sin\frac{q}{2})^2}\left(\frac{1}{2} - \frac{\log\log n}{\log n} + O\left(\frac{1}{\log n}\right)\right) n(\log n)^2.$$

We deduce that

$$\frac{n(\log n - 3\log\log n)}{4(\sin\frac{q}{2})^2} \log \xi(r_{\mathbf{a}(u)}) + \log d_{\mathbf{a}(u)} \geq O(\log d_{\mathbf{a}(u)} (\log n)^{-1}).$$

The result follows since the error can be absorbed in the constant c. $\square$

The proof of Proposition 4.1 is by a direct saddle point evaluation. We first prove a preliminary lemma.

LEMMA 4.2. *Let D be a bounded rectangle contained in the right half-plane $\{z \in \mathbb{C} : \Re(z) > 0\}$. Let $\frac{1}{2} \leq u < 1 - \delta$ for some $\delta > 0$, and assume n^u is an integer. Uniformly for $z \in D$ we have for all $m \geq 1$*

$$g_{\mathbf{a}(u)}^{(m)}(z) = g_{\mathbf{0}}^{(m)}(z) - n^{2(u-1)} g_{\mathbf{0}}^{(m+2)}(z) + O_{m,D}(n^{4(u-1)}).$$

In particular, suppose $\theta \in (\varepsilon, \pi - \varepsilon)$ for some $\varepsilon > 0$. Let $\omega' > 0$ solve the saddle point equation $g'_{\mathbf{a}(u)}(\omega') = 0$ and let ω be the usual saddle point $g'_{\mathbf{0}}(\omega) = 0$. We have

$$\omega' = \omega + n^{2(u-1)} \frac{g_{\mathbf{0}}^{(3)}(\omega)}{g_{\mathbf{0}}^{(2)}(\omega)} + O_{\varepsilon}(n^{4(u-1)}).$$

Also,

$$g_{\mathbf{a}(u)}(\omega') = g_{\mathbf{0}}(\omega) - n^{2(u-1)} g_{\mathbf{0}}^{(2)}(\omega) + O_\varepsilon(n^{4(u-1)}),$$
$$g_{\mathbf{a}(u)}^{(2)}(\omega') = g_{\mathbf{0}}^{(2)}(\omega) + O_\varepsilon(n^{2(u-1)}).$$

For all $t > 0$, *and all* n *sufficiently large we have*

$$\Re g_{\mathbf{a}(u)}\left(\omega' + it + \frac{i}{n}\right) \le \Re g_{\mathbf{a}(u)}(\omega' + it).$$

Finally, there are constants $c_1, c_2 > 0$ *such that if* $t > c_1$ *then*

$$\Re\big(g_{\mathbf{a}(u)}(\omega + 2it) - g_{\mathbf{a}(u)}(\omega + it)\big) < -c_2.$$

PROOF. Set $\phi(z) = i[\psi(n + \frac{1}{2} - inz) - \psi(-n + \frac{1}{2} - inz)]$, and recall that $g_{\mathbf{0}}'(z) = \theta + \phi(z)$. We have

$$g_{\mathbf{a}(u)}'(z) = g_{\mathbf{0}}'(z) + \big(\phi(z + in^{u-1}) - 2\phi(z) + \phi(z - in^{u-1})\big).$$

The term in parentheses is $-n^{2(u-1)} g_{\mathbf{0}}^{(3)}(z) + O(n^{4(u-1)})$ by Taylor expansion. The claims for the other derivatives follow similarly.

The facts regarding ω' and $g_{\mathbf{a}(u)}^{(m)}(\omega')$ may be deduced by standard calculus. The monotonicity property for $g_{\mathbf{a}(u)}$ on $\Re(z) = \omega'$ is proven in the same way as the related claim for $g_{\mathbf{0}}$, in Lemma 3.4, that is,

$$\begin{aligned}
&\Re\left(g_{\mathbf{a}(u)}\left(\omega + i\left(t + \frac{1}{n}\right)\right) - g_{\mathbf{a}(u)}(\omega + it)\right) \\
&\quad = \log\left|\frac{\omega + i(t - 1 + \frac{1}{2(n-n^u)})}{\omega + i(t + 1 + \frac{1}{2(n-n^u)})}\right|\left|\frac{\omega + i(t + 1 + \frac{1}{2n})}{\omega + i(t - 1 + \frac{1}{2n})}\right|\left|\frac{\omega + i(t - 1 + \frac{1}{2(n+n^u)})}{\omega + i(t + 1 + \frac{1}{2(n+n^u)})}\right| \\
&\quad = O(n^{-3+2u}) + \log\left|\frac{\omega + i(t - 1 + \frac{1}{2n})}{\omega + i(t + 1 + \frac{1}{2n})}\right| < 0.
\end{aligned}$$

The last statement may be checked geometrically. □

Using the last lemma, we can now evaluate $\log|r_{\mathbf{a}(u)}(\theta)|$.

PROOF OF PROPOSITION 4.1. The assumption on θ ensures that $g_{\mathbf{0}}^{(2)}(\omega) = O(1)$. Standard application of the saddle point method gives

$$r_{\mathbf{a}(u)}(\theta) = \left(1 + O\left(\frac{1}{n}\right)\right)\frac{(2n-1)!}{(2n\sin\frac{\theta}{2})^{2n-1}}\frac{e^{ng_{\mathbf{a}(u)}(\omega')}}{\sqrt{2\pi n g_{\mathbf{a}(u)}^{(2)}(\omega')}}.$$

Comparing this to the integral for the trivial representation at its saddle point ω yields

$$\begin{aligned} r_{\mathbf{a}(u)}(\theta) &= \exp\bigl(n\bigl(g_{\mathbf{a}(u)}(\omega') - g_{\mathbf{0}}(\omega)\bigr)\bigr)\bigl(1 + O\bigl(n^{2(u-1)}\bigr)\bigr) \\ &= \exp\bigl(-n^{2u-1} g_{\mathbf{0}}^{(2)}(\omega) + O\bigl(n^{4u-3}\bigr)\bigr) \\ &= \exp\biggl(-2\biggl(\sin\frac{\theta}{2}\biggr)^2 n^{2u-1} + O\bigl(n^{4u-3}\bigr)\biggr). \end{aligned}$$

Here, we use $g_{\mathbf{0}}^{(2)}(\omega) = 2(\sin\frac{\theta}{2})^2(1 + O(\frac{1}{n}))$, with an error absorbed in the $O(n^{4u-3})$ term. $\square$

4.2.1. *Upper bound for L^2 mixture walk.* We now turn to the upper bound in Theorem 1.3. In this section, we prove the following variant of the character ratio bound in Proposition 3.1 for mixtures of character ratios.

PROPOSITION 4.3. *Let ξ be a probability measure supported in $(0, \pi)$ and let $q > 0$ and $q' < \pi$ be the smallest and largest points in its support. As usual, let $\sigma(\theta) = \sin\frac{\theta}{2}$. Let $\rho_{\mathbf{a}}$ be a nontrivial irreducible representation. There exists a constant $C > 0$ for which the following bound holds*:

$$\log \xi\bigl(|r_{\mathbf{a}}|\bigr) \le -\min\biggl(\frac{2\xi(\sigma^2)}{n(\log n + C)}, \frac{4\sigma^2(q)}{n(\log n + 2\log\log n + C)}\biggr) \log d_{\mathbf{a}}.$$

From this proposition, the deduction of the upper bound in Theorem 1.3 is the same as for Theorem 1.2.

As in the proof of Proposition 3.1 for fixed angle, the proof of Proposition 4.3 splits according as the representation is small, moderate or large. When the representation is small, in the range $\sum a_j \le \frac{n}{\sigma(q)\log n}$, one may apply Lemma 3.3 directly to deduce that, uniformly in θ,

$$r_{\mathbf{a}}(\theta) = 1 - \frac{E_1 \sigma^2(\theta)}{n^2}\biggl(1 + O\biggl(\frac{1}{\log n}\biggr)\biggr),$$

and that, in this range, $E_1 = O(\frac{n^2}{\log^2 n})$. Thus, for small representations,

$$\xi\bigl(|r_{\mathbf{a}}|\bigr) = \xi(r_{\mathbf{a}}) = 1 - \frac{E_1 \xi(\sigma^2)}{n^2}\biggl(1 + O\biggl(\frac{1}{\log n}\biggr)\biggr)$$

and

$$\log \xi\bigl(|r_{\mathbf{a}}|\bigr) = -\frac{E_1 \xi(\sigma^2)}{n^2}\biggl(1 + O\biggl(\frac{1}{\log n}\biggr)\biggr).$$

The proof that

$$\log \xi\left(|r_{\mathbf{a}}|\right) \le -\frac{2\xi(\sigma^2)\log d_{\mathbf{a}}}{n(\log n + C)}$$

now goes through as before.

Similarly, when the representation is large, in the regime where $a_n \ge \frac{2\cdot 10^6}{\sigma(q)}n$, arguments similar to our previous ones reduce the problem to the case of small and moderate dimensions. Recall that previously when the representation is large, we either build the character ratio out of a ratio on a smaller group by appending weights, or bound the integral trivially. We leave it to the reader to check that in our bounds from Section 3.4 on large representations, the incremental contributions to the log of the dimension are dominated by the contributions to the character ratio by a factor of at least $\log n$, so that large representations may be ignored.

We now turn to the main case of moderate representations, where there are some new ideas. One idea is that the greatest trade-off between character ratio and dimension growth occurs for a_n of size about $n^{\frac{1}{2}}$. For smaller a_n, the character ratio is sufficiently near 1 that there is no loss in integrating it directly as opposed to its logarithm. For larger a_n, the character ratio beats the dimension by a larger amount. Another idea, already illustrated in our proof of the L^2 lower bound, is that the dimension is most difficult to control when the indices a_j are clumped. We handle this case in Lemma A.8 below.

Before starting out, we make several simplifying reductions. As before, we bound the character ratio $r_{\mathbf{a}}(\theta)$ for $\theta \in [q, q']$ by bounding the associated integral,[4]

$$\left|r_{\mathbf{a}}(\theta)\right| \le \frac{(2n-1)!}{(2n\sin\frac{\theta}{2})^{2n-1}}\frac{1}{2\pi}\int_{|t|\ll_\xi \frac{1}{\sqrt{\log n}}} e^{n\Re(g_{\mathbf{a}}(\omega+it))}\,dt.$$

This we write as

$$\frac{(2n-1)!}{(2n\sin\frac{\theta}{2})^{2n-1}}\frac{1}{2\pi}\int_{|t|\ll_\xi \frac{1}{\sqrt{\log n}}} e^{L_{\mathbf{a}}(\theta,t)}e^{n\Re(g_{\mathbf{0}}(\omega+it))}\,dt, \tag{35}$$

where we introduce

$$L_{\mathbf{a}}(\theta,t) = n\Re\big(g_{\mathbf{a}}\big(\omega(\theta)+it\big) - g_{\mathbf{0}}\big(\omega(\theta)+it\big)\big) = \log\left|\prod_{j=1}^{n}\frac{(\omega(\theta)+it)^2+\omega_j^2}{(\omega(\theta)+it)^2+\alpha_j^2}\right|$$

and also $L_{\mathbf{a}}(\theta) = L_{\mathbf{a}}(\theta,0)$.

We now record several lemmas regarding $L_{\mathbf{a}}(\theta,t)$.

[4]Throughout this section, we work with integrals truncated at $|t| \ll_\xi \frac{1}{\sqrt{\log n}}$, thus neglecting the tail. The necessary argument to control the error from the tail is the same as in proof of Proposition 3.1.

LEMMA 4.4. *In the range* $\omega = \Theta(1)$, $\alpha_j = O(1)$ *and* $|t| = o(1)$,

$$\left|\frac{\omega_j^2 + (\omega + it)^2}{\alpha_j^2 + (\omega + it)^2}\right| = 1 - \frac{\alpha_j^2 - \omega_j^2}{\alpha_j^2 + \omega^2}[1 + O(t^2)].$$

In particular, if $\rho_{\mathbf{a}}$ *is a moderate representation and* $\theta \in [q, q']$, $|t| \ll \frac{1}{\log n}$ *then*

$$L_{\mathbf{a}}(\theta, t) = L_{\mathbf{a}}(\theta)\bigl(1 + O(t^2)\bigr).$$

PROOF. Write

$$\frac{\omega_j^2 + (\omega + it)^2}{\alpha_j^2 + (\omega + it)^2} = 1 - \frac{\alpha_j^2 - \omega_j^2}{\alpha_j^2 + \omega^2}\left[1 + \left(\frac{\alpha_j^2 + \omega^2}{\alpha_j^2 + (\omega + it)^2} - 1\right)\right].$$

Since $1 - \frac{\alpha_j^2 - \omega_j^2}{\alpha_j^2 + \omega^2} = \Omega(1)$, the proof of the first claim is completed by noting that $(\frac{\alpha_j^2 + \omega^2}{\alpha_j^2 + (\omega + it)^2} - 1)$ has imaginary part that is $O(|t|)$ and real part that is $O(t^2)$.

For the second statement, note that $\rho_{\mathbf{a}}$ moderate implies $\alpha_j = O(1)$, while $\theta \in [q, q']$ implies $\omega = \Theta(1)$. The first statement then applies, and shows that

$$\log\left|\frac{\omega_j^2 + (\omega + it)^2}{\alpha_j^2 + (\omega + it)^2}\right| = \bigl(1 + O(t^2)\bigr) \log \frac{\omega^2 + \omega_j^2}{\omega^2 + \alpha_j^2}.$$

The second claim follows on summing over j. □

LEMMA 4.5. *Let* $\rho_{\mathbf{a}}$ *be a moderate representation and let* $\theta \in [q, q']$. *Then*:

1. $L_{\mathbf{a}}(\theta) \asymp L_{\mathbf{a}}(q)$;
2. $\frac{1}{\log n} \ll -L_{\mathbf{a}}(q)$ *for all moderate* $\mathbf{a}$.

PROOF. Since $\omega = \Theta(1)$ and $\alpha_j = O(1)$ for all j the first statement follows immediately from the definition,

$$L_{\mathbf{a}}(\theta) = \sum \log\left(1 - \frac{\alpha_j^2 - \omega_j^2}{\omega^2 + \alpha_j^2}\right).$$

For the second statement, recall that $\mathbf{a}$ moderate guarantees $\sum_j a_j \gg \frac{n}{\log n}$ and $a_n \ll n$. We bound

$$-L_{\mathbf{a}}(q) = -\sum_j \log\left(1 - \frac{\alpha_j^2 - \omega_j^2}{\omega^2 + \alpha_j^2}\right) \gg \sum_j (\alpha_j - \omega_j) \gg \frac{1}{\log n}$$

since $1 \ll \omega_j \le \alpha_j \ll 1$. □

LEMMA 4.6. *There is a constant $c > 0$ such that for all moderate representations $\rho_{\mathbf{a}}$ we have*

$$\log \xi\big(|r_{\mathbf{a}}|\big) \le c L_{\mathbf{a}}(q).$$

Furthermore, there is a $c'(\xi) > 0$ such that if $-L_{\mathbf{a}}(q) < c'n$ then

$$\log \xi\big(|r_{\mathbf{a}}|\big) \le \left(1 + O\left(\frac{\log n}{n}\right)\right) \log \xi\big(e^{L_{\mathbf{a}}(\theta)}\big).$$

PROOF. By modeling the proof of Proposition 3.1 for moderate representations [see, e.g., (23)], we have

$$\xi\big(|r_{\mathbf{a}}|\big) \le \left(1 + O\left(\frac{1}{n}\right)\right) \sup_{\theta \in [q, q']} \sup_{|t| \ll_\xi \frac{1}{\sqrt{\log n}}} e^{-L_{\mathbf{a}}(\theta, t)}.$$

The first statement follows since $L_{\mathbf{a}}(\theta, t) \sim L_{\mathbf{a}}(\theta) \asymp L_{\mathbf{a}}(q)$ [the error term of size $O(\frac{1}{n})$ is negligible].

To prove the second statement, use (35) to write

$$\begin{aligned} \big|r_{\mathbf{a}}(\theta)\big| &\le e^{L_{\mathbf{a}}(\theta)} \frac{(2n-1)! e^{n g_{\mathbf{0}}(\omega)}}{(2n \sin \frac{\theta}{2})^{2n-1}} \\ &\quad \times \frac{1}{2\pi} \int_{|t| \ll_\xi \frac{1}{\sqrt{\log n}}} \exp\left(-\frac{n g_{\mathbf{0}}^{(2)}(\omega) t^2}{2} + O\big(t^2 L_{\mathbf{a}}(\theta)\big) + O\big(n t^4\big)\right) dt. \end{aligned}$$

For $L_{\mathbf{a}}(\theta) < c''n$ for a sufficiently small constant c'', the entire expression evaluates to [see (18)]

$$e^{L_{\mathbf{a}}(\theta)} \left(1 + O\left(\frac{1}{n}\right) + O\left(\frac{L_{\mathbf{a}}(\theta)}{n}\right)\right).$$

Thus,

$$\xi\big(|r_{\mathbf{a}}|\big) \le \xi\big(e^{L_{\mathbf{a}}}\big) \left(1 + O\left(\frac{1}{n}\right) + O\left(\frac{L_{\mathbf{a}}(q)}{n}\right)\right).$$

The claim follows since $\log \xi(e^{L_{\mathbf{a}}}) \asymp L_{\mathbf{a}}(q)$ and $-L_{\mathbf{a}}(q) \gg \frac{1}{\log n}$. $\square$

The above lemmas effectively reduce our problem to that of comparing $L_{\mathbf{a}}(\theta)$ to the dimension $\log d_{\mathbf{a}}$. Recall that earlier we introduced the dimension increment

$$d_{\mathbf{a}}(k) = \frac{\tilde{a}_k}{2k-1} \prod_{j<k} \frac{\tilde{a}_k^2 - \tilde{a}_j^2}{(k - \frac{1}{2})^2 - (j - \frac{1}{2})^2}$$

and that in Lemma A.1 we proved the bound

$$\log d_{\mathbf{a}}(k) \le a_k \log\left(1 + \frac{2k-1}{a_k}\right) + (2k-1) \log\left(1 + \frac{a_k}{2k-1}\right) + O(1). \tag{36}$$

Similarly, now we introduce the character ratio increment (recall $\alpha_j = \frac{a_j+j-\frac{1}{2}}{n}$)

$$L_{\mathbf{a}}(k,\theta) = \log\left(1 - \frac{\alpha_j^2 - \omega_j^2}{\omega(\theta)^2 + \alpha_j^2}\right).$$

We now prove several lemmas about these increments.

LEMMA 4.7. *Let $\rho_{\mathbf{a}}$ be a moderate representation. For all $j \gg n$ and θ in the support of ξ, we have*

$$-L_{\mathbf{a}}(j,\theta) \gg_\xi \frac{a_j}{n}.$$

If $j > n - \frac{Cn}{\log n}$ for some constant C and $a_j < \frac{n}{\log n}$, then

$$-L_{\mathbf{a}}(j,\theta) = \frac{2\sigma^2(\theta)a_j}{n}\left(1 + O\left(\frac{1}{\log n}\right)\right). \tag{37}$$

PROOF. For both claims, write $\alpha_j^2 - \omega_j^2 = \frac{a_j}{n} \cdot \frac{a_j+2j-1}{n}$. The first claim is straightforward. For the second, observe that $\frac{a_j+2j-1}{n} = 2 + O(\frac{1}{\log n})$ and $\frac{1}{\omega^2+\alpha_j^2} = \sigma^2(1 + O(\frac{1}{\log n}))$. □

The next lemma allows us to prove the type of estimate that we want for Proposition 4.3 for a collection of increments whose sum is not too large.

LEMMA 4.8. *Let C be a constant and let $G \subset \mathbb{Z} \cap [n - \frac{Cn}{\log n}, n]$ be a subset of indices such that $\sum_{j\in G} a_j < \frac{n}{\log n}$. Then for a sufficiently large constant C',*

$$\log \int_q^{q'} \exp\left(\sum_{j\in G} L_{\mathbf{a}}(j,\theta)\right) d\xi(\theta) \leq -\frac{2\xi(\sigma^2)\sum_{j\in G}\log d_{\mathbf{a}}(j)}{n(\log n + C')}.$$

PROOF. Since $a_j < \frac{n}{\log n}$ for all $j \in G$, we may substitute the asymptotic (37) into the LHS and integrate. Since the argument of the exponential is $O(\frac{1}{\log n})$, the LHS becomes

$$\left(1 + O\left(\frac{1}{\log n}\right)\right)\frac{-2\xi(\sigma^2)}{n}\sum_{j\in G} a_j.$$

The result now follows on noting that, in this range, $\log d_{\mathbf{a}}(j) \leq a_j(\log n + O(1))$. □

The following dimension lemma is proved as Lemma A.8 of Appendix A.

LEMMA. *Let B be the collection of indices*

$$B = \left\{ j \in \left[n - \frac{Cn}{\log n}, n \right] : a_j \in \big((1-\delta)S, S\big] \right\},$$

where $\delta = \frac{1}{\log n}$, *$C$ is a constant, and* $1 \le S \le \frac{n^{\frac{1}{2}}}{1+\log n}$. *Suppose that* $\frac{n^{\frac{1}{2}}}{\log n} \le |B| \le \frac{Cn}{\log n}$. *Then*

$$\log d_{\mathbf{a}}(B) \stackrel{def}{=} \sum_{j \in B} \log d_{\mathbf{a}}(j) \le \left(1 - \frac{\log |B|}{\log n}\right)|B| S\big(\log n + O(1)\big).$$

In particular, for each $\theta \in [q, q']$ we have

$$\sum_{j \in B} L_{\mathbf{a}}(j, \theta) \le \frac{-4\sigma^2(q) \log d_{\mathbf{a}}(B)}{n(\log n + 2 \log\log n + O(1))}. \tag{38}$$

Indeed, for each $j \in B$, Lemma 4.7 gives

$$-L_{\mathbf{a}}(j, \theta) = \frac{2\sigma^2(\theta) S}{n}\left(1 + O\left(\frac{1}{\log n}\right)\right)$$

so that

$$\sum_{j \in B} L_{\mathbf{a}}(j, \theta) \le -\frac{2\sigma^2(q)|B| S}{n}\left(1 + O\left(\frac{1}{\log n}\right)\right),$$

while

$$d_{\mathbf{a}}(B) \le \left(\frac{1}{2} + \frac{\log\log n}{\log n}\right)|B| S\big(\log n + O(1)\big).$$

LEMMA 4.9. *Let $1 \le k \le n$ and let $\rho_{\mathbf{a}}$ be a moderate representation with $a_j = 0$ for $j < k$. Let $\mathbf{a}'$ denote the string such that $a'(i) = a(i)$ for $i < k$ and $a'(i) = a(i) + 1$ for $i \ge k$. For some constants $C, C' > 0$*

$$\log d_{\mathbf{a}'} \le \log d_{\mathbf{a}} + Cn$$

while

$$L_{\mathbf{a}'} \le L_{\mathbf{a}} - C'\left(\frac{n-k}{n}\right).$$

PROOF. The dimension claim follows from Lemmas A.2 and A.3. The character ratio claim follows on checking that for each $i \ge k$ we have $\log(1 - \frac{\alpha_i^2 - \omega_i^2}{\omega^2 + \alpha_i^2})$ decreases by $\Omega(\frac{1}{n})$ after the shift. $\square$

PROOF OF THEOREM 1.3 UPPER BOUND, MODERATE REPRESENTATIONS. We begin with a few observations. We assume that $\rho_{\mathbf{a}}$ is moderate, with $a_n \le$

$\frac{2\cdot 10^6}{\sigma(q)}n$. Therefore, by Lemma A.5 of Section 3.3, $\log d_{\mathbf{a}} \ll n^2$. Therefore, we may restrict attention to character ratios for which $-\log \xi(|r_{\mathbf{a}}|) \ll \frac{n}{\log n}$. By Lemmas 4.5 and 4.6 of this section, this means that we may assume $-L_{\mathbf{a}}(q) \ll \frac{n}{\log n}$, so that, to within negligible error, we may replace

$$\xi(|r_{\mathbf{a}}|) \quad \leftrightarrow \quad \xi(e^{L_{\mathbf{a}}(\theta)}).$$

Next, observe that for a sufficiently large constant c, we may assume $a_i = 0$ for all $i < n - \frac{cn}{\log n}$. This is because any string with nonzero entries in this region may be obtained from one with all zeros by making shifts of the type described in Lemma 4.9, and if c is sufficiently large, these shifts improve the bound in Proposition 4.3.

We next dispatch with any indices for which $a_j > \frac{n^{\frac{1}{2}}}{\log n}$. Recall that $a_j \ll n$ so that (36) gives

$$\log d_{\mathbf{a}}(j) \le a_j\big(\log n - \log a_j + O(1)\big).$$

If $a_j \ge \frac{n}{(\log n)^2}$, then the first part of Lemma 4.7 gives $L_{\mathbf{a}}(j,\theta) \gg \frac{a_j}{n}$. Thus,

$$-\frac{n\log n}{4\sigma^2(q)}L_{\mathbf{a}}(j,\theta) \gg \frac{\log n}{\log\log n}\log d_{\mathbf{a}}(j).$$

If $a_j \le \frac{n}{(\log n)^2}$, then the second part of Lemma 4.7 guarantees that, for a sufficiently large constant C,

$$\frac{n(\log n + 2\log\log n + C)}{4\sigma^2(q)}L_{\mathbf{a}}(j,\theta) + \log d_{\mathbf{a}}(j) \le 0,$$

but now the factor of $\log\log n$ is needed.

Now we treat the indices for which $a_j < \frac{n^{\frac{1}{2}}}{\log n}$ by splitting them into bins. Let $\delta = \frac{1}{\log n}$ and let

$$I_k = \left[\frac{n^{\frac{1}{2}}}{\log n}(1-\delta)^k, \frac{n^{\frac{1}{2}}}{\log n}(1-\delta)^{k-1}\right],$$

$$1 \le k \le K = \frac{\log \frac{n^{\frac{1}{2}}}{\log n}}{-\log(1-\delta)} \sim \frac{1}{2}(\log n)^2.$$

Say an interval I_k is "big" if there are at least $\frac{n^{\frac{1}{2}}}{\log n}$ indices j for which $a_j \in I_k$. By (38), if I_k is big then we have, for a sufficiently large C, for all $\theta \in [q, q']$,

$$\sum_{j:a_j\in I_k} L_{\mathbf{a}}(j,\theta) \le -\frac{4\sigma^2(q)}{n(\log n + 2\log\log n + C)}\sum_{j:a_j\in I_k}\log d_{\mathbf{a}}(j).$$

Let $\mathscr{B}$ be the collection of all indices j belonging to a big interval, together with all j for which $a_j > \frac{n^{\frac{1}{2}}}{\log n}$. Let $m \ge n - \frac{cn}{\log n}$ be the least index for which $a_m \ne 0$, and set $\mathscr{S} = [m, n] \setminus \mathscr{B}$. Observe that

$$L_{\mathbf{a}}(\theta) = \sum_{j\in\mathscr{B}} L_{\mathbf{a}}(j,\theta) + \sum_{j\in\mathscr{S}} L_{\mathbf{a}}(j,\theta) = L_{\mathbf{a},\mathscr{B}}(\theta) + L_{\mathbf{a},\mathscr{S}}(\theta),$$

and

$$\log d_{\mathbf{a}} = \sum_{j\in\mathscr{B}} d_{\mathbf{a}}(j) + \sum_{j\in\mathscr{S}} d_{\mathbf{a}}(j) = \log d_{\mathbf{a},\mathscr{B}} + \log d_{\mathbf{a},\mathscr{S}}.$$

Since each $j \in \mathscr{S}$ has a_j in an interval that contains at most $\frac{n^{\frac{1}{2}}}{\log n}$ other indices, we may bound

$$\sum_{j\in\mathscr{S}} a_j \le \frac{n^{\frac{1}{2}}}{\log n} \sum_{k=0}^{K-1} (1-\delta)^k \frac{n^{\frac{1}{2}}}{\log n} \le \frac{n}{\log n},$$

and, therefore, by Lemma 4.8,

$$\log \int_q^{q'} e^{L_{\mathbf{a},\mathscr{S}}(\theta)}\, d\xi(\theta) \le -\frac{2\xi(\sigma^2)\log d_{\mathbf{a},\mathscr{S}}}{n(\log n + C)}.$$

Now we bound

$$\begin{aligned}
\log \xi\bigl(|r_{\mathbf{a}}|\bigr) &\le \log \int_q^{q'} e^{L_{\mathbf{a}}(\theta)}\, d\xi(\theta) \\
&\le \sup_{\theta\in[q,q']} L_{\mathbf{a},\mathscr{B}}(\theta) + \log \int_q^{q'} e^{L_{\mathbf{a},\mathscr{S}}(\theta)}\, d\xi(\theta) \\
&\le -\frac{4\sigma^2(q)\log d_{\mathbf{a},\mathscr{B}}}{n(\log n + 2\log\log n + C)} - \frac{2\xi(\sigma^2)\log d_{\mathbf{a},\mathscr{S}}}{n(\log n + C)},
\end{aligned}$$

completing the proof. $\square$

5. L^∞ mixing time. In this section, we prove Corollary 1.2. First, we define, for two Borel probability measures on a compact metric space (Ω, ρ), the L^∞ *distance of* μ *and* ν *w.r.t.* ν, by

$$\|\mu - \nu\|_\infty := \|\mu - \nu\|_{\infty,\nu} := \sup_{f:\nu(f)\le 1} \bigl|\mu(f) - \nu(f)\bigr|.$$

If μ is not absolutely continuous with respect to ν then the L^∞ norm is ∞, since if $U = \{x \in \Omega : \frac{d\mu}{d\nu}(x) = \infty\}$ then for any Borel function f supported on U, $\nu(f + 1) = 1$. Conversely, if μ has a density with respect to ν, then we have the alternative definition:

$$\|\mu - \nu\|_\infty = \operatorname*{ess\,sup}_{x\in\Omega} \left|\frac{d\mu}{d\nu}(x) - 1\right|.$$

The equivalence of the two definitions is trivial in the discrete case. For compact metric space, use the fact that $\frac{d\mu}{d\nu}(x) = \lim_{r\to 0^+} \frac{\mu(B_r(x))}{\nu(B_r(x))}$, where $B_r(x)$ is metric ball of radius r around x.

Now specializing to the running distributions of a Markov chain, the first definition further implies that for any starting state x, the L^∞ distance $d_\infty(t) := \|P_x^t - \nu\|_\infty$ is monotone nonincreasing, since

$$\begin{aligned}\|P_x^{t+1} - \nu\|_\infty = \|P_x^t P - \nu P\|_\infty &= \sup_{f:\nu(f)\le 1} \left|P_x^t\big(P(f)\big) - \nu\big(P(f)\big)\right| \\ &\le \sup_{g:\nu(g)\le 1} \left|P_x^t(g) - \nu(g)\right| = \|P_x^t - \nu\|_\infty,\end{aligned}$$

using $\nu(f) \le 1$ implies $\nu(P(f)) \le 1$, by Markov property.

Now recall the Fourier inversion formula for compact Lie groups: for any $f \in L^2(\mathrm{SO}(2n+1))$,

$$f(x) = \sum_{\rho\in\widehat{\mathrm{SO}(2n+1)}} d_\rho \operatorname{Tr}\big[\hat{f}(\rho)\rho(x)^*\big].$$

Let $f_t = \frac{d\mu_t}{d\nu} - 1$, where $\mu_t = \delta_{\mathrm{Id}} \cdot P^t$. Since we are interested in the situation where $\operatorname{ess\,sup}_{x\in\Omega} \frac{d\mu_t}{d\nu}(x) < \infty$, certainly we may assume $f_t \in L^2(\mathrm{SO}(2n+1))$. Since $\mu_t = \mu^{*t}$ is a convolution product and invariant under conjugation, $\hat{f}_t(\rho_{\mathbf{a}}) = \xi(r_{\mathbf{a}}(\theta))^t I_{d_{\mathbf{a}}}$, that is, a constant times the identity matrix. In particular, $\hat{f}_t(\rho_{\mathbf{0}}) = 1$. Therefore,

$$f_t(x) = \sum_{\mathbf{a}\ne\mathbf{0}} d_{\mathbf{a}}^2 r_{\mathbf{a}}^t \chi_{\mathbf{a}}(x); \qquad \chi_{\mathbf{a}}(x) = \frac{1}{d_{\mathbf{a}}} \operatorname{Tr} \rho_{\mathbf{a}}(x).$$

Thus, when $t = 2s$ is even, the maximum of $|f_t(x)|$ occurs at $x \in \{\pm\mathrm{Id}\}$ [$r_{\mathbf{a}} := \frac{1}{d_{\mathbf{a}}} \operatorname{Tr} \hat{\mu}(\rho)$ can be either positive or negative]. But at $x = \mathrm{Id}$ say, we have $f_t(\mathrm{Id}) = \sum_{\mathbf{a}}^* d_{\mathbf{a}}^2 r_{\mathbf{a}}^t$, which coincides with $\|P_{\mathrm{Id}}^s - \nu_n\|_2$. Thus, $\|P_{\mathrm{Id}}^t - \nu_n\|_\infty = \|P_{\mathrm{Id}}^s - \nu_n\|_2$ and the corollary follows from monotonicity of the L^∞ distance.

6. Open problems.

We have seen the power of the contour representation formula (3) in proving sharp estimates for the size of the character ratio, which is instrumental in studying the convergence rate of the uniform plane Kac walk under various norms. Several issues still remain:

- What is the true cut-off window of L^2 and L^∞ convergence? We show that it is of order $O(n \log\log n)$ for measures of a specific type. For a sum of point masses, we may expect that $O(n)$ is the true order. Does a cut-off exist regardless of the measure?
- Theorem 1.3 artificially assumes that the generating measure μ be supported away from $\pi \in \mathbb{T}_0$. Probably this is an artifact of our proof and can be removed with more effort.

- For a fixed n, it would be nice to determine the first step at which the L^2 distance from uniform becomes finite. Carefully tracing our treatment of large representations shows that this happens in time $O(n)$, while a time $\geq n$ is necessary, because prior to this point the measure is supported on a lower dimensional submanifold.
- In Appendix B, we prove a generalization of the contour formula (3) to the case where the generating measure of the random walk is supported on bigger conjugacy classes of $\mathrm{SO}(2n+1)$, such as $\mathrm{SO}(2k) \subset \mathrm{SO}(2n+1)$. It remains to be seen whether multidimensional analogues of our arguments yield the corresponding mixing time analysis for the related random walks. A reasonable lower bound and candidate mixing time is available in this case from the second moment method applied to the natural representation.
- In [3], it is proved that under very general conditions on a finite state space Markov chains, ℓ^p-cut-off implies ℓ^q-cut-off for $q > p$. Does the same hold for our continuous state space walk?

APPENDIX A: BOUNDS FOR DIMENSIONS

This section collects together the basic estimates for dimensions of representations that we need throughout the mixing time upper bound argument. We conclude by proving Proposition 3.2.

Since $d_{\mathbf{0}} = 1$, the dimension equation (1) may be written as[5]

$$d_{\mathbf{a}} = \prod_{k=1}^{n} \frac{\tilde{a}_k}{k-\frac{1}{2}} \prod_{1\leq j<k\leq n} \frac{\tilde{a}_k^2 - \tilde{a}_j^2}{(k-\frac{1}{2})^2 - (j-\frac{1}{2})^2}.$$

We may treat this product incrementally as

$$d_{\mathbf{a}} = \prod_{k=1}^{n} d_{\mathbf{a}}(k), \qquad d_{\mathbf{a}}(k) = \frac{\tilde{a}_k}{k-\frac{1}{2}} \prod_{1\leq j<k} \frac{\tilde{a}_k^2 - \tilde{a}_j^2}{(k-\frac{1}{2})^2 - (j-\frac{1}{2})^2}.$$

If we replace $\tilde{a}_j$ with $j - \frac{1}{2} \leq \tilde{a}_j$ in the product above, then we find

$$d_{\mathbf{a}}(k) \leq \frac{a_k + k - \frac{1}{2}}{k-\frac{1}{2}} \frac{(a_k + 2k - 2)!}{a_k!(2k-2)!}. \tag{39}$$

In particular, this gives the following weak estimate for the dimension.

LEMMA A.1. *Recall the notation $\alpha_k = \frac{\tilde{a}_k}{n}$. We have, for each k,*

$$\log d_{\mathbf{a}}(k) \leq a_k \log\left(1 + \frac{2k-1}{a_k}\right) + (2k-1)\log\left(1 + \frac{a_k}{2k-1}\right) + O(1).$$

[5] Recall $\tilde{a}_j = a_j + j - \frac{1}{2}$.

In particular,

$$\log d_{\mathbf{a}} \le O(n^2) + 2n \sum_{k:\alpha_k \ge 1} \log \alpha_k.$$

PROOF. The first line follows from Stirling's approximation, and the second follows on summing. □

To get more refined estimates, we frequently make index-shift arguments, and so it is most convenient to attach to representation $\rho_{\mathbf{a}}$ the shift-index $\mathbf{s}$ given by

$$\mathbf{s} = (s_1, \dots, s_n), \qquad s_1 = a_1, \qquad s_i = a_i - a_{i-1}, \qquad i = 2, \dots, n.$$

Note that as $\mathbf{a}$ runs over nonzero weakly increasing strings of length n, $\mathbf{s}$ runs over all nonzero strings in $\mathbb{N}^n$, and $\mathbf{a}$ is recovered from $\mathbf{s}$ as $a_j = \sum_{i \le j} s_i$. We abuse notation by writing $d_{\mathbf{s}} = d_{\mathbf{a}}$ for $\mathbf{s}$ associated to $\mathbf{a}$.

Treating the product $d_{\mathbf{s}}$ now as a whole, we may split it as $d_{\mathbf{s}} = d_{\mathbf{s}}^- d_{\mathbf{s}}^+ d_{\mathbf{s}}^0$, where

$$d_{\mathbf{s}}^- = \prod_{1 \le j < k \le n} \frac{[\sum_{j < i \le k} s_i] + k - j}{k - j},$$

$$d_{\mathbf{s}}^+ = \prod_{1 \le j < k \le n} \frac{[\sum_{i \le j} s_i + \sum_{i \le k} s_i] + j + k - 1}{j + k - 1},$$

$$d_{\mathbf{s}}^0 = \prod_k \frac{[\sum_{i \le k} s_i] + k - \frac{1}{2}}{k - \frac{1}{2}}.$$

Note that $d_{\mathbf{s}}^* \ge 1$.

Let $\mathbf{e}_i$ denote the ith standard basis vector in $\mathbb{N}^n$. Our first set of bounds concern dimensions of shifts supported at the standard basis vectors.

LEMMA A.2. *For any* $\mathbf{s}_0 \in \mathbb{N}^{n-i}$, *we have*

$$\frac{d_{(\mathbf{0}_i, \mathbf{s}_0) + \mathbf{e}_{i+1}}}{d_{(\mathbf{0}_i, \mathbf{s}_0)}} \le d_{\mathbf{e}_{i+1}}.$$

PROOF. This is immediate from comparing separately each factor $d_{\mathbf{s}}^-$, $d_{\mathbf{s}}^+$ and $d_{\mathbf{s}}^0$. □

We can easily bound $d_{\mathbf{e}_i}$.

LEMMA A.3. *Let* $m = \min(i - 1, n - i + 1)$. *We have the bound*

$$\log d_{\mathbf{e}_i} \le m\left[1 + \log 2 + \log \frac{n}{m} + \log \frac{n}{i - \frac{1}{2}}\right] + 2(n - i)[1 + \log 2] + \log \frac{n + \frac{1}{2}}{i - \frac{1}{2}}.$$

In particular, $\log d_{\mathbf{e}_i} = O(n)$.

PROOF. Splitting $d_{\mathbf{e}_i} = d^-_{\mathbf{e}_i} d^+_{\mathbf{e}_i} d^0_{\mathbf{e}_i}$ as above, we bound each in turn.

$$d^-_{\mathbf{e}_i} = \prod_{1 \le j < i} \prod_{i \le k \le n} \frac{1+k-j}{k-j} \le \exp\left(\sum_{1 \le j < i} \sum_{i \le k \le n} \frac{1}{k-j}\right),$$

so dividing according to $k - j \le m$ and $k - j > m$ below, we have

$$\log d^-_{\mathbf{e}_i} \le \sum_{1 \le j < i} \sum_{i \le k \le n} \frac{1}{k-j} \le m + m\left[\frac{1}{m+1} + \cdots + \frac{1}{n-1}\right] \le m\left[1 + \log \frac{n}{m}\right].$$

Similarly,

$$d^+_{\mathbf{e}_i} = \prod_{1 \le j < i} \prod_{i \le k \le n} \frac{j+k}{j+k-1} \cdot \prod_{i \le j < k \le n} \frac{j+k+1}{j+k-1}$$
$$\le \exp\left(\sum_{1 \le j < i} \sum_{i \le k \le n} \frac{1}{j+k-1} + 2 \sum_{i \le j < k \le n} \frac{1}{j+k-1}\right)$$

so that

$$\begin{aligned}
\log d^+_{\mathbf{e}_i} &\le \sum_{1 \le j < i} \sum_{i \le k \le n} \frac{1}{j+k-1} + 2 \sum_{i \le j < k \le n} \frac{1}{j+k-1} \\
&\le m\left[\frac{1}{i} + \frac{1}{i+1} + \cdots + \frac{1}{2n-1}\right] + 2\left[\frac{1}{2i} + \frac{2}{2i+1} + \cdots + \frac{n-i}{n+i-1}\right] \\
&\quad + 2(n-i)\left[\frac{1}{n+i} + \frac{1}{n+i+1} + \cdots + \frac{1}{2n-1}\right] \\
&\le m \log \frac{2n}{i - \frac{1}{2}} + 2(n-i)[1 + \log 2].
\end{aligned}$$

Finally,

$$d^0_{\mathbf{e}_i} = \prod_{k \ge i} \frac{k + \frac{1}{2}}{k - \frac{1}{2}} = \frac{n + \frac{1}{2}}{i - \frac{1}{2}}.$$

□

As a consequence of this lemma, we get a crude bound for the dimension of any representation.

COROLLARY A.4. *For all sufficiently large n, each representation $\rho_{\mathbf{a}}$ has dimension bounded by*

$$d_{\mathbf{a}} \le e^{10na_n}.$$

If $a_i = 0$ for all $i \le \frac{n}{2}$, then

$$d_{\mathbf{a}} \le \exp\big(|\mathbf{a}|(\log n + 7)\big).$$

PROOF. Note that in shift notation, $a_n = |\mathbf{s}| = \sum_j s_j$. By Lemma A.2,

$$d_{\mathbf{s}} \le \prod_{i=1}^{n} d_{s_i \mathbf{e}_i}.$$

The maximum over i of the upper bound for $s^{-1} \log d_{s\mathbf{e}_i}$ in Lemma A.3 is less than $10n$ for all n sufficiently large. Combining these observations, the first claim of the corollary follows.

To prove the second, observe that $|\mathbf{a}| = \sum_i (n - i + 1)s_i$. For all $i > \frac{n}{2}$, Lemma A.3 gives $\log d_{s_i \mathbf{e}_i} \le (n - i + 1)s_i(\log n + 7)$, which suffices. □

We obtain the following bounds for medium size representations.

LEMMA A.5. *Let* $\mathbf{a}$ *satisfy* $a_n \le \frac{2 \cdot 10^6 n}{\sigma}$ *and* $\sum a_j \ge \frac{n}{\sigma \log n}$. *Then for all sufficiently large* n, *we have the bounds*

$$\exp\left(\frac{n(5 - \log \sigma)}{4 \cdot 10^6 \log n}\right) \le d_{\mathbf{a}} \le \exp\left(\frac{2 \cdot 10^7 n^2}{\sigma}\right).$$

PROOF. The upper bound is immediate from Corollary A.4.

For the lower bound, write

$$d_{\mathbf{a}} \ge d_{\mathbf{a}}^{+} = \prod_{j>i} \frac{a_j + a_i + j + i - 1}{j + i - 1} \ge \prod_{j>i} \left(1 + \frac{a_i + a_j}{2n}\right).$$

The conditions guarantee that for $C := \frac{2 \cdot 10^6}{\sigma}$, $\frac{a_i + a_j}{2n} < C$, so that $1 + \frac{a_i + a_j}{2n} \ge \exp(\frac{(a_i + a_j)\log C}{2Cn})$, as may be checked by convexity. Thus,

$$\begin{aligned} d_{\mathbf{a}} &\ge \prod_{j>i} \exp\left(\frac{(a_i + a_j)\log C}{2Cn}\right) \ge \exp\left(\frac{(n-1)\log C}{Cn} \sum a_i\right) \\ &\ge \exp\left(\frac{n(5 - \log \sigma)}{4 \cdot 10^6 \log n}\right). \end{aligned}$$

□

The preceding bounds are useful when we perform the right-hand side shifts before the left ones. Sometimes we wish to perform left-hand side shifts first. The following lemma gives bounds in this case.

LEMMA A.6. *Let* $\mathbf{s} \in \mathbb{N}^j$. *Let* $m = \min(j, n - j + 1)$ *and let* $1 \le \eta \le m$ *be a parameter. Write*

$$|\mathbf{s}|_{\eta,\mathrm{loc}} = \sum_{j-\eta \le i \le j} s_i.$$

Then we have the bound

$$\log \frac{d_{(\mathbf{s},\mathbf{0})+\mathbf{e}_j}}{d_{(\mathbf{s},\mathbf{0})}} \le m\left[\log \frac{n+|\mathbf{s}|_{\eta,\mathrm{loc}}}{m+|\mathbf{s}|_{\eta,\mathrm{loc}}} + \log \frac{n+j}{m+j} + 2\right] + \eta \log(n-j+\eta) + 2(n-j+1) + \log \frac{n}{j} + O(1).$$

PROOF. With error at most $O(1)$, we have

$$\begin{aligned}\log \frac{d^-_{(\mathbf{s},\mathbf{0})+\mathbf{e}_j}}{d^-_{(\mathbf{s},\mathbf{0})}} &= \log\left[\prod_{1\le i<j}\prod_{j\le k\le n}\left(1+\frac{1}{k-i+\sum_{i<\ell\le j} s_\ell}\right)\right] \\ &\le \sum_{j-\eta\le i<j}\sum_{j\le k\le n}\frac{1}{k-i} + \sum_{i\le j-\eta}\sum_{j\le k\le n}\frac{1}{k-i+|\mathbf{s}|_{\eta,\mathrm{loc}}} \\ &\le \eta\log(n-j+\eta) + m\left(1+\log\frac{n+|\mathbf{s}|_{\eta,\mathrm{loc}}}{m+|\mathbf{s}|_{\eta,\mathrm{loc}}}\right)\end{aligned}$$

[see (40) below].

Meanwhile,

$$\begin{aligned}\log \frac{d^+_{(\mathbf{s},\mathbf{0})+\mathbf{e}_j}}{d^+_{(\mathbf{s},\mathbf{0})}} &\le \log \frac{d^+_{(\mathbf{0},\mathbf{0})+e_j}}{d^+_{(\mathbf{0},\mathbf{0})}} \\ &\le \log\left[\prod_{1\le i<j}\prod_{j\le k\le n}\left(1+\frac{1}{i+k-1}\right)\cdot\prod_{j\le i<k\le n}\left(1+\frac{1}{i+k-1}\right)\right].\end{aligned}$$

The first of these terms is bounded by

$$\begin{aligned}&\frac{1}{j}+\frac{2}{j+1}+\cdots+\frac{m}{j+m-1}+m\left[\frac{1}{j+m}+\cdots+\frac{1}{n+j-1}\right] \\ &\quad\le m\left[1+\log\frac{n+j}{m+j}\right].\end{aligned} \tag{40}$$

The second is bounded by

$$\begin{aligned}&\frac{1}{2j}+\frac{2}{2j+1}+\cdots+\frac{n-j+1}{n+j}+(n-j+1)\left[\frac{1}{n+j}+\cdots+\frac{1}{2n+2j-1}\right] \\ &\quad\le 2(n-j+1).\end{aligned}$$

Finally,

$$\log \frac{d^0_{(\mathbf{s},\mathbf{0})+\mathbf{e}_j}}{d^0_{(\mathbf{s},\mathbf{0})}} = \log\left[\prod_{k\ge j}\left(1+\frac{1}{k-\frac{1}{2}+|\mathbf{s}|}\right)\right] \le \log\frac{n+\frac{1}{2}}{j-\frac{1}{2}}. \qquad \square$$

The following estimate is used in our discussion of the L^2 mixing time.

LEMMA A.7. *Let* $\mathbf{a}(s,t) := (\mathbf{0}_{n-n^t}, (n^s)_{n^t})$. *Also let* $\mathbf{a}(u) = \mathbf{a}(u,u)$. *Then uniformly for* s, t *in any compact subset of* $(0,1)$,

$$\log d_{\mathbf{a}(s,t)} = (1 - s \vee t) n^{s+t} \big(\log n + O(1)\big).$$

PROOF. As usual, we write $\log d_{\mathbf{a}(s,t)} = \log d^0_{\mathbf{a}(s,t)} + \log d^+_{\mathbf{a}(s,t)} + \log d^-_{\mathbf{a}(s,t)}$. The first two terms are of lower order:

$$\log d^0_{\mathbf{a}(s,t)} = \sum_{n-n^t < j \le n} \log \frac{n^s + j - \frac{1}{2}}{j - \frac{1}{2}} = O\big(n^{s+t-1}\big),$$

$$\log d^+_{\mathbf{a}(s,t)} = \sum_{i \le n-n^t} \sum_{n-n^t < j \le n} \log \frac{n^s + j + i - 1}{j + i - 1} = O\big(n^{s+t}\big),$$

since the logarithm is $O(n^{s-1})$.

Meanwhile,

$$\begin{aligned}\log d^-_{\mathbf{a}(s,t)} &= \sum_{i \le n-n^t} \sum_{n-n^t < j \le n} \log \frac{n^s + j - i}{j - i} \\ &= \sum_{i \le n-n^t} \sum_{j \le n^t} \log \frac{n^s + j + i - 1}{j + i - 1} \\ &= \sum_{k=1}^{n^t} k \log \frac{k + n^s}{k} + n^t \sum_{n^t < k \le n-n^t} \log \frac{k + n^s}{k} \\ &\quad + \sum_{n-n^t < k \le n} (n-k) \log \frac{k + n^s}{k}.\end{aligned}$$

The last of these sums is $O(n^{s+t})$ since the logarithm is $O(n^{s-1})$. The first is also $O(n^{s+t})$ since it is bounded by [use $\log(1+x) \le x$]

$$\sum_{k \le n^t} k \log\left(1 + \frac{n^s}{k}\right) \le \sum_{k \le n^t} n^s = n^{s+t}.$$

In the middle sum, if $s > t$ then we split the sum further at n^s. The first part of the sum becomes

$$n^t \sum_{n^t < k \le n^s} \log \frac{k + n^s}{k} \le O\big(n^{s+t}\big) + n^t \sum_{k \le n^s} \log \frac{n^s}{k} = O\big(n^{s+t}\big).$$

We are left with the sum

$$n^t \sum_{n^{s \vee t} < k \le n-n^t} \log\left(1 + \frac{n^s}{k}\right) = n^t \sum_{n^{s \vee t} < k \le n-n^t} \left(\frac{n^s}{k} + O\left(\frac{n^{2s}}{k^2}\right)\right)$$

which evaluates to the main term, as desired. $\square$

We use the following estimate in the L^2 upper bound.

LEMMA A.8. *Let C be a constant, let $\delta = \frac{1}{\log n}$, let $1 \le S \le \frac{n^{\frac{1}{2}}}{1+\log n}$ and let B be the collection of indices*

$$B = \left\{ j \in \left[n - \frac{Cn}{\log n}, n \right] : a_j \in \bigl((1-\delta)S, S\bigr] \right\}.$$

Suppose that

$$\frac{n^{\frac{1}{2}}}{\log n} \le |B| \le \frac{Cn}{\log n}.$$

We have

$$\log d_{\mathbf{a}}(B) \stackrel{def}{=} \sum_{j \in B} \log d_{\mathbf{a}}(j) \le \left(1 - \frac{\log |B|}{\log n}\right) |B| S \bigl(\log n + O(1)\bigr).$$

PROOF. Let m be the last index in B, and, with an eye toward applying the block dimension bound in Lemma A.7, set $T = |B|$. Obviously, those indices greater than m do not affect $d_{\mathbf{a}}(B)$, so henceforth we write $\mathbf{a}$ in place of $\mathbf{a}(m)$, the weight truncated at m corresponding to a representation on $\mathrm{SO}(2m+1)$.

Factor $d_{\mathbf{a}}(B)$ as $d_{\mathbf{a}}(B) = E_{\mathbf{a}}(B) \cdot I_{\mathbf{a}}(B)$ where $E_{\mathbf{a}}(B)$ corresponds to factors pairing indices in B with those outside,

$$E_{\mathbf{a}}(B) = \prod_{j \in B} \frac{\tilde{a}_j}{j - \frac{1}{2}} \prod_{i \le m-T} \frac{\tilde{a}_j^2 - \tilde{a}_i^2}{(j-\frac{1}{2})^2 - (i-\frac{1}{2})^2},$$

and where $I_{\mathbf{a}}(B)$ pairs indices both inside B,

$$I_{\mathbf{a}}(B) = \prod_{j \in B} \prod_{m-T \le i < j} \frac{\tilde{a}_j^2 - \tilde{a}_i^2}{(j-\frac{1}{2})^2 - (i-\frac{1}{2})^2}.$$

Set $s = \frac{\log S}{\log m}$, $t = \frac{\log T}{\log m}$ and write as before $\mathbf{a}(S,T) = (\mathbf{0}_{m-m^t}, (m^s)_{m^t})$. Then we may bound $E_{\mathbf{a}}(B) \le d_{\mathbf{a}(s,t)}$ so that Lemma A.7 yields the bound

$$\log E_{\mathbf{a}}(B) \le (1 - s \vee t) ST \bigl(\log n + O(1)\bigr).$$

This is of the right size, so we now work to show that $\log I_{\mathbf{a}}(B)$ is of lower order by a factor of $\log n$.

Write $I_{\mathbf{a}}(B) = I_{\mathbf{a}}(B)^+ I_{\mathbf{a}}(B)^-$, where

$$I_{\mathbf{a}}(B)^+ = \prod_{j \in B} \prod_{m-T < i < j} \frac{a_i + a_j + i + j - 1}{i + j - 1},$$

$$I_{\mathbf{a}}(B)^- = \prod_{j \in B} \prod_{m-T < i < j} \frac{a_j - a_i + j - i}{j - i}.$$

Then we have the bound

$$\log I_{\mathbf{a}}(B)^+ \le \sum_{j\in B}\sum_{m-T<i<j} \log\left(1+\frac{S}{n(1-O(\frac{1}{\log n}))}\right)$$
$$\le \left(1+O\left(\frac{1}{\log n}\right)\right)\frac{ST^2}{2n},$$

which more than suffices, since $T \le \frac{Cn}{\log n}$. To bound $I_{\mathbf{a}}(B)^-$ consider the representation $\rho_{\mathbf{b}}$ on $\mathrm{SO}(2T+1)$ with $b_i = a_{m-T+i} - a_{m-T} \le \delta S$. Then $I_{\mathbf{a}}(B)^- = d_{\mathbf{b}}^- \le d_{\mathbf{b}}$. Since $S \le T$, the bound (36) gives [use $\log(1+x)\le x$]

$$\log d_{\mathbf{b}} \le \sum_{i=1}^{T} b_i\big(\log T + O(1)\big) \le \delta S T\big(\log n + O(1)\big).$$

Since $\delta = \frac{1}{\log n}$, this completes the proof. $\square$

It is also convenient to have some lower bounds for dimensions. The following bound is a useful reference point.

LEMMA A.9. *There is $c>0$ such that*

$$d_{\mathbf{a}} \ge \exp\left(c\min\left(n, \sum_j a_j\right)\right).$$

PROOF. Write $\mathbf{a} \leftrightarrow \mathbf{s}$ and write

$$d_{\mathbf{s}}^+ \ge \prod_{k=1}^{n}\prod_{1\le j\le \frac{k}{2}}\left(1+\frac{\sum_{i\le k} s_i}{j+k-1}\right) \ge \exp\left(c\sum_{k=1}^{n}\min(a_k,k)\right).$$

This suffices, since if $a_k \ge k$ for any k, then $a_\ell \ge k$ for all $\ell > k$, whence a lower bound of $\exp(c'(n-k)k)$. $\square$

The next lemma is useful for proving Proposition 3.2.

LEMMA A.10. *Let η, $1\le\eta\le n$ be a parameter, and set $\mathbb{N}^n = \mathbb{N}^{n-\eta}\oplus\mathbb{N}^{\eta}$. We have*

$$d_{(\mathbf{s}_1,\mathbf{s}_2)} \ge d_{(\mathbf{s}_1,\mathbf{0})}^+ d_{(\mathbf{0},\mathbf{s}_2)}^-.$$

Set $|\mathbf{s}_1| = \sum_{i\le n-\eta} s_i$. We have

$$d_{(\mathbf{s}_1,\mathbf{0})}^+ \ge \left(1+\frac{|\mathbf{s}_1|}{2n}\right)^{n\eta-\eta^2}.$$

In particular, for $|\mathbf{s}_1| < n$, *and for* $\eta < n^{1-\varepsilon}$ *and all sufficiently large* n,

$$d^+_{(\mathbf{s}_1,\mathbf{0})} \geq e^{\frac{|s_1|\eta}{3}}.$$

Define $|\mathbf{s}_2|_p = \sum_{i<\eta}((\eta - i + 1)s_i)$. *Then*

$$d^-_{(\mathbf{0},\mathbf{s}_2)} \geq \prod_{1\leq j<n-\eta} \prod_{1\leq k\leq\eta} \left(1 + \frac{\sum_{i\leq k} s_i}{\eta + j}\right) \geq \prod_{1\leq j<n-\eta} \left(1 + \frac{|\mathbf{s}_2|_p}{\eta + j}\right).$$

PROOF. The identity $d_{(\mathbf{s}_1,\mathbf{s}_2)} \geq d^+_{(\mathbf{s}_1,\mathbf{0})} d^-_{(\mathbf{0},\mathbf{s}_2)}$ is immediate from the definitions. Now we have

$$\begin{aligned} d^+_{(\mathbf{s}_1,\mathbf{0})} &= \prod_{1\leq j<k\leq n} \left(\frac{[\sum_{i\leq j} s_i + \sum_{i\leq k} s_i] + j + k - 1}{j + k - 1}\right) \\ &\geq \prod_{j<n-\eta} \prod_{n-\eta\leq k\leq n} \left(\frac{j + k - 1 + |\mathbf{s}_1|}{j + k - 1}\right) \geq \left(1 + \frac{|\mathbf{s}_1|}{2n}\right)^{n\eta-\eta^2}, \end{aligned}$$

proving the bound for $d^+_{(\mathbf{s}_1,\mathbf{0})}$ for sufficiently large n. Meanwhile, denoting s_i the components of $\mathbf{s}_2$,

$$\begin{aligned} d^-_{(\mathbf{0},\mathbf{s}_2)} &\geq \prod_{1\leq j<n-\eta} \prod_{1\leq k\leq\eta} \left(1 + \frac{\sum_{i\leq k} s_i}{k + n - \eta - j}\right) \\ &\geq \prod_{1\leq j<n-\eta} \prod_{1\leq k\leq\eta} \left(1 + \frac{\sum_{i\leq k} s_i}{\eta + j}\right) \qquad \text{(switching } j \leftrightarrow n - \eta - j\text{)} \\ &\geq \prod_{1\leq j<n-\eta} \left(1 + \frac{\sum_{1\leq k\leq\eta} \sum_{i\leq k} s_i}{\eta + j}\right) = \prod_{1\leq j<n-\eta} \left(1 + \frac{|\mathbf{s}_2|_p}{\eta + j}\right). \end{aligned}$$

□

We conclude the section by proving Proposition 3.2.

PROOF OF PROPOSITION 3.2. Set $\eta = \lfloor n^{3/4} \rfloor$. Then letting $\mathbb{N}^n = \mathbb{N}^{n-\eta} \oplus \mathbb{N}^\eta$ observe $d_{(\mathbf{s}_1,\mathbf{s}_2)} \geq d^+_{(\mathbf{s}_1,\mathbf{0})} d^-_{(\mathbf{0},\mathbf{s}_2)}$, $d_{\mathbf{0}} = 1$ and, therefore,

$$\begin{aligned} \sum_{\mathbf{0}\neq\mathbf{s}\in\mathbb{N}^n} d_{\mathbf{s}}^{\frac{-c}{\log n}} &= -1 + \sum_{\mathbf{s}} d_{\mathbf{s}}^{\frac{-c}{\log n}} \\ &\leq -1 + \left(\sum_{\mathbf{s}_1\in\mathbb{N}^{n-\eta}} (d^+_{(\mathbf{s}_1,\mathbf{0})})^{\frac{-c}{\log n}}\right) \cdot \left(\sum_{\mathbf{s}_2\in\mathbb{N}^\eta} (d^-_{(\mathbf{0},\mathbf{s}_2)})^{\frac{-c}{\log n}}\right), \end{aligned}$$

so it suffices to show that each sum on the right is $1 + O(e^{-c})$ as $n \to \infty$.

We first handle the sum over $\mathbf{s}_1$. Observe that $\#\{\mathbf{s} \in \mathbb{N}^n : |\mathbf{s}| = j\} \le n^j \wedge j^n$. Applying the bound of Lemma A.10,

$$\begin{aligned}
\sum_{\mathbf{s}_1 \in \mathbb{N}^\eta} \left(d^+_{(\mathbf{s}_1,\mathbf{0})}\right)^{\frac{-c}{\log n}} &\le 1 + \sum_{j=1}^{\infty} \#\{\mathbf{s} \in \mathbb{N}^{n-\eta} : |\mathbf{s}| = j\}\left(1 + \frac{j}{2n}\right)^{\frac{-cn^{\frac{7}{4}}}{\log n}} \\
&\le 1 + \sum_{j=1}^{4n} n^j e^{\frac{-c' jn^{\frac{3}{4}}}{\log n}} + \sum_{j=4n}^{\infty} j^n \left(\frac{2j}{2n}\right)^{\frac{-cn^{\frac{7}{4}}}{\log n}} \\
&= 1 + o(1), \qquad n \to \infty.
\end{aligned}$$

Now we consider the sum over $\mathbf{s}_2$. Again applying the bound of the lemma,

$$\sum_{\mathbf{s}_2 \in \mathbb{N}^\eta} \left(d^-_{(\mathbf{0},\mathbf{s}_2)}\right)^{\frac{-c}{\log n}} \le 1 + \sum_{q=1}^{\infty} \#\{\mathbf{s} \in \mathbb{N}^\eta : |\mathbf{s}|_p = q\}\left[\prod_{j<n-\eta}\left(1 + \frac{q}{\eta + j}\right)\right]^{\frac{-c}{\log n}}.$$

Evidently, $\#\{\mathbf{s} \in \mathbb{N}^\eta : |\mathbf{s}|_p = q\}$ is the number of partitions of q into parts of size at most η. This is bounded by the total number of partitions of q, which is $e^{O(\sqrt{q})}$, and by q^η. Thus, the above sum over q is bounded by

$$\begin{aligned}
&\sum_{1 \le q < \eta/2} \exp\left(O(\sqrt{q}) - \frac{cq(\log n - \log \eta)}{2 \log n}\right) \\
&\quad + \sum_{q=\eta/2}^{n^{5/4}} \exp\left(O(\sqrt{q}) - \frac{c\eta}{8}\right) \\
&\quad + \sum_{q=n^{5/4}}^{\infty} \exp\left(\eta \log q - \frac{c'(n-\eta)\log q}{\log n}\right), \qquad c' > 0
\end{aligned}$$

the first term is $O(e^{\frac{-c}{8}})$ and the remaining terms are $o(1)$ as $n \to \infty$. $\square$

APPENDIX B: CONTOUR FORMULA FOR CHARACTERS OF SO($2n+1$)

We now generalize our integral formula for the character ratio at a rotation to give a k-fold integral formula for the character ratios evaluated at a generic element of $\mathrm{SO}(2k) \subset \mathrm{SO}(2n+1)$.

THEOREM B.1. *Denote $\{L_i, 1 \le i \le n\}$ the roots of* $\mathrm{SO}(2n+1)$. *Let $\rho_{\mathbf{a}}$, $\mathbf{a} = a_1 \ge a_2 \ge \cdots \ge a_n \ge 0$, be an irreducible representation of* $\mathrm{SO}(2n+1)$ *corresponding to highest weight $\sum a_i L_i$, and for $1 \le i \le n$, write $\tilde{a}_i = a_i + i - \frac{1}{2}$. Let*

$\boldsymbol{\theta} = (\theta_1, \dots, \theta_k) \in (\mathbb{R}/2\pi\mathbb{Z})^k$, $\theta_i \neq 0$, $\theta_i \neq \theta_j$ *for* $i < j$ *be an element of a torus contained in* $\mathrm{SO}(2k) \subset \mathrm{SO}(2n+1)$. *We have the integral character ratio formula*

$$r_{\mathbf{a}}(\boldsymbol{\theta}) = \frac{r_{\mathbf{a}}(\boldsymbol{\theta})}{d_{\mathbf{a}}} = L(\boldsymbol{\theta}) \oint_{\mathscr{C}_1 \times \cdots \times \mathscr{C}_k} \cdots \oint M_{\mathbf{a};\boldsymbol{\theta}}(\mathbf{z})\, d\mathbf{z},$$

where the contours $\mathscr{C}_i$ *are chosen to have winding number* 1 *about each of the poles* $\pm\tilde{a}_i$ *of the integrand*

$$M_{\mathbf{a};\boldsymbol{\theta}}(\mathbf{z}) = \prod_{j=1}^{k} M_{\mathbf{a};\theta_j}(z_j) \times \prod_{1 \le j < \ell \le k} (z_\ell^2 - z_j^2); \qquad M_{\mathbf{a},\theta}(z) = \frac{\sin \theta z}{\prod_{j=1}^{n} (\tilde{a}_j^2 - z^2)}.$$

The leading term $L(\boldsymbol{\theta})$ *is independent of the representation* $\mathbf{a}$ *and given by*

$$L(\boldsymbol{\theta}) = \frac{(2n-1)!(2n-3)!\cdots(2n-2k+1)!}{2^{2nk-k^2} \prod_{j=1}^{k} \sin(\frac{\theta_j}{2})^{2n-2k+1} \prod_{1 \le i < j \le k} (\sin(\frac{\theta_i}{2})^2 - \sin(\frac{\theta_j}{2})^2)}.$$

PROOF. For $\boldsymbol{\theta} \in \mathbb{T}^n$ in general position on the maximal torus, the character $\chi_{\mathbf{a}}(\boldsymbol{\theta})$ is given by the following determinantal formula ([9], page 408, (24.28))[6]

$$\chi_{\mathbf{a}}(\boldsymbol{\theta}) = \begin{vmatrix} \mathrm{s}(\tilde{a}_1\theta_1) & \mathrm{s}(\tilde{a}_1\theta_2) & \cdot & \mathrm{s}(\tilde{a}_1\theta_n) \\ \mathrm{s}(\tilde{a}_2\theta_1) & \mathrm{s}(\tilde{a}_2\theta_2) & \cdot & \mathrm{s}(\tilde{a}_2\theta_n) \\ \cdot & \cdot & & \cdot \\ \mathrm{s}(\tilde{a}_n\theta_1) & \mathrm{s}(\tilde{a}_n\theta_2) & \cdot & \mathrm{s}(\tilde{a}_n\theta_n) \end{vmatrix} \Bigg/ \begin{vmatrix} \mathrm{s}(\tilde{0}_1\theta_1) & \mathrm{s}(\tilde{0}_1\theta_2) & \cdot & \mathrm{s}(\tilde{0}_1\theta_n) \\ \mathrm{s}(\tilde{0}_2\theta_1) & \mathrm{s}(\tilde{0}_2\theta_2) & \cdot & \mathrm{s}(\tilde{0}_2\theta_n) \\ \cdot & \cdot & & \cdot \\ \mathrm{s}(\tilde{0}_n\theta_1) & \mathrm{s}(\tilde{0}_n\theta_2) & \cdot & \mathrm{s}(\tilde{0}_n\theta_n) \end{vmatrix}.$$

Since the dimension is equal to the character value at $\boldsymbol{\theta} = \mathbf{0}$, we obtain the character ratio at $\boldsymbol{\theta} \in \mathbb{T}^k$,

$$r_{\mathbf{a}}(\boldsymbol{\theta}) = \frac{\begin{vmatrix} \mathrm{s}(\tilde{a}_1\theta_1) & \cdot & \mathrm{s}(\tilde{a}_1\theta_k) & \mathrm{s}(\tilde{a}_1\varepsilon_{k+1}) & \cdot & \mathrm{s}(\tilde{a}_1\varepsilon_n) \\ \mathrm{s}(\tilde{a}_2\theta_1) & \cdot & \mathrm{s}(\tilde{a}_2\theta_k) & \mathrm{s}(\tilde{a}_2\varepsilon_{k+1}) & \cdot & \mathrm{s}(\tilde{a}_2\varepsilon_n) \\ \cdot & & \cdot & \cdot & & \cdot \\ \mathrm{s}(\tilde{a}_n\theta_1) & \cdot & \mathrm{s}(\tilde{a}_n\theta_k) & \mathrm{s}(\tilde{a}_n\varepsilon_{k+1}) & \cdot & \mathrm{s}(\tilde{a}_n\varepsilon_n) \end{vmatrix}}{\begin{vmatrix} \mathrm{s}(\tilde{a}_1\varepsilon_1) & \mathrm{s}(\tilde{a}_1\varepsilon_2) & \cdot & \mathrm{s}(\tilde{a}_1\varepsilon_n) \\ \mathrm{s}(\tilde{a}_2\varepsilon_1) & \mathrm{s}(\tilde{a}_2\varepsilon_2) & \cdot & \mathrm{s}(\tilde{a}_2\varepsilon_n) \\ \cdot & \cdot & & \cdot \\ \mathrm{s}(\tilde{a}_n\varepsilon_1) & \mathrm{s}(\tilde{a}_n\varepsilon_2) & \cdot & \mathrm{s}(\tilde{a}_n\varepsilon_n) \end{vmatrix}}$$

[6] $\tilde{0}_i = i - \frac{1}{2}$. Henceforth, we write only s for sin.

$$\times \frac{\begin{vmatrix} s(\tilde{0}_1\varepsilon_1) & s(\tilde{0}_1\varepsilon_2) & \cdot & s(\tilde{0}_1\varepsilon_n) \\ s(\tilde{0}_2\varepsilon_1) & s(\tilde{0}_2\varepsilon_2) & \cdot & s(\tilde{0}_2\varepsilon_n) \\ \cdot & \cdot & & \cdot \\ s(\tilde{0}_n\varepsilon_1) & s(\tilde{0}_n\varepsilon_2) & \cdot & s(\tilde{0}_n\varepsilon_n) \end{vmatrix}}{\begin{vmatrix} s(\tilde{0}_1\theta_1) & \cdot & s(\tilde{0}_1\theta_k) & s(\tilde{0}_1\varepsilon_{k+1}) & \cdot & s(\tilde{0}_1\varepsilon_n) \\ s(\tilde{0}_2\theta_1) & \cdot & s(\tilde{0}_2\theta_k) & s(\tilde{0}_2\varepsilon_{k+1}) & \cdot & s(\tilde{0}_2\varepsilon_n) \\ \cdot & & \cdot & \cdot & & \cdot \\ s(\tilde{0}_n\theta_1) & \cdot & s(\tilde{0}_n\theta_k) & s(\tilde{0}_n\varepsilon_{k+1}) & \cdot & s(\tilde{0}_n\varepsilon_n) \end{vmatrix}},$$

where $\varepsilon_1, \dots, \varepsilon_n$ represent infinitesimals tending to 0.

Let $\varepsilon_1 \succ \varepsilon_2 \succ \cdots \succ \varepsilon_n$ where $a \succ b$ means b is eventually smaller than any fixed power of a as a and b tend to 0. The reason for this choice is that, when determinants involving $\boldsymbol{\varepsilon}$ are expanded as alternating sums below, we are able to neglect all but a single term.

In the terms containing epsilon arguments, we expand sin in its power series about 0 performing row reduction to eliminate lower order terms, then take the limit to eliminate higher powers, with the result that we may write $r_{\mathbf{a}} = \frac{R_{\mathbf{a}}}{R_{\mathbf{0}}}$ where

$$R_{\mathbf{a}} = \frac{\begin{vmatrix} \frac{s(\tilde{a}_1\theta_1)}{\tilde{a}_1} & \cdot & \frac{s(\tilde{a}_1\theta_k)}{\tilde{a}_1} & (-\tilde{a}_1^2)^{n-k-1} & (-\tilde{a}_1^2)^{n-k-2} & \cdot & 1 \\ \frac{s(\tilde{a}_2\theta_1)}{\tilde{a}_2} & \cdot & \frac{s(\tilde{a}_2\theta_k)}{\tilde{a}_2} & (-\tilde{a}_2^2)^{n-k-1} & (-\tilde{a}_2^2)^{n-k-2} & \cdot & 1 \\ \cdot & & \cdot & \cdot & \cdot & & \cdot \\ \frac{s(\tilde{a}_n\theta_1)}{\tilde{a}_n} & \cdot & \frac{s(\tilde{a}_n\theta_k)}{\tilde{a}_n} & (-\tilde{a}_n^2)^{n-k-1} & (-\tilde{a}_n^2)^{n-k-2} & \cdot & 1 \end{vmatrix}}{\begin{vmatrix} (-\tilde{a}_1^2)^{n-1} & (-\tilde{a}_1^2)^{n-2} & \cdot & 1 \\ (-\tilde{a}_2^2)^{n-1} & (-\tilde{a}_2^2)^{n-2} & \cdot & 1 \\ \cdot & \cdot & & \cdot \\ (-\tilde{a}_n^2)^{n-1} & (-\tilde{a}_n^2)^{n-2} & \cdot & 1 \end{vmatrix}}.$$

We observe that the denominator is the Vandermonde $\prod_{1 \le i < j \le n}(\tilde{a}_i^2 - \tilde{a}_j^2)$. Expanding the first k columns of the numerator, and dividing by the Vandermonde in the denominator, we obtain (note that the $\tilde{a}_i$ are increasing)

$$\begin{aligned} R_{\mathbf{a}}(\boldsymbol{\theta}) &= \sum_{\substack{1 \le i_1, \dots, i_k \le n \\ \text{distinct}}} \frac{(-1)^{\sum_{1 \le j < \ell \le k} \mathbf{1}(i_j < i_\ell)} \prod_{j=1}^k \sin(\theta_j \tilde{a}_{i_j}) \prod_{1 \le j < \ell \le k} |\tilde{a}_{i_j}^2 - \tilde{a}_{i_\ell}^2|}{\prod_{j=1}^k (\tilde{a}_{i_j} \prod_{\ell \ne i_j} (\tilde{a}_\ell^2 - \tilde{a}_{i_j}^2))} \\ &= \sum_{\substack{1 \le i_1, \dots, i_k \le n \\ \text{distinct}}} \frac{\prod_{j=1}^k \sin(\theta_j \tilde{a}_{i_j}) \prod_{1 \le j < \ell \le k} (\tilde{a}_{i_j}^2 - \tilde{a}_{i_\ell}^2)}{\prod_{j=1}^k (\tilde{a}_{i_j} \prod_{\ell \ne i_j} (\tilde{a}_\ell^2 - \tilde{a}_{i_j}^2))} \\ &= (-1)^{\binom{k}{2}} \oint \cdots \oint M_{\mathbf{a},\boldsymbol{\theta}}(\mathbf{z})\, d\mathbf{z}, \end{aligned}$$

since the factor $\prod_{1\le j<\ell\le k}(z_\ell^2 - z_j^2)$ in $M_{\mathbf{a},\boldsymbol{\theta}}(\mathbf{z})$ produces the factor

$$(-1)^{\binom{k}{2}} \prod_{1\le j<\ell\le k} (\tilde{a}_{i_j}^2 - \tilde{a}_{i_\ell}^2)$$

at each set of poles.

Backtracking a bit, we determine the constant term by

$$R_{\mathbf{0}} = \frac{\varepsilon_1^{2n-1}\varepsilon_2^{2n-3}\cdots\varepsilon_k^{2n-2k+1}}{(2n-1)!(2n-3)!\cdots(2n-2k+1)!}\frac{\det[\mathrm{s}(\tilde{0}_i\theta_j)|_{j\le k}\mathrm{s}(\tilde{0}_i\varepsilon_j)]}{\det[\mathrm{s}(\tilde{0}_i\varepsilon_j)]}.$$

Since $\mathrm{s}((j-\frac{1}{2})\theta)$ is a polynomial in $\mathrm{s}(\frac{\theta}{2})$ with highest order term $(-4)^{j-1}\times \mathrm{s}(\frac{\theta}{2})^{2j-1}$, performing row reductions we obtain

$$R_{\mathbf{0}} = \frac{\varepsilon_1^{2n-1}\varepsilon_2^{2n-3}\cdots\varepsilon_k^{2n-2k+1}}{(2n-1)!(2n-3)!\cdots(2n-2k+1)!}\frac{\det[\mathrm{s}(\frac{\theta_j}{2})^{2i-1}|_{j\le k}\mathrm{s}(\frac{\varepsilon_j}{2})^{2i-1}]}{\det[\mathrm{s}(\frac{\varepsilon_j}{2})^{2i-1}]}.$$

Keeping in mind the relative size of the infinitesimals and expanding the determinants alternating sums we see that, up to sign, the numerator contributes the determinant of its lower $k\times k$ minor times a product of infinitesimals, while the denominator contributes a single product of infinitesimals. This reduces the formula to

$$\begin{aligned} R_{\mathbf{0}} &= \frac{(-1)^{\binom{k}{2}}\varepsilon_1^{2n-1}\cdots\varepsilon_k^{2n-2k+1}}{s(\frac{\varepsilon_1}{2})^{2n-1}\cdots s(\frac{\varepsilon_k}{2})^{2n-2k+1}}\frac{\prod_{i=1}^k(s(\frac{\theta_i}{2})^{2n-2k+1})\det[s(\frac{\theta_j}{2})^{2(i-1)}]_{i,j=1}^k}{(2n-1)!(2n-3)!\cdots(2n-2k+1)!} \\ &= \frac{(-1)^{\binom{k}{2}}2^{2nk-k^2}\prod_{i=1}^k(s(\frac{\theta_i}{2})^{2n-2k+1})\prod_{1\le i<j\le k}(s(\frac{\theta_i}{2})^2 - s(\frac{\theta_j}{2})^2)}{(2n-1)!(2n-3)!\cdots(2n-2k+1)!}. \end{aligned}$$

□

REFERENCES

[1] ALDOUS, D. (1983). Random walks on finite groups and rapidly mixing Markov chains. In *Seminar on Probability, XVII. Lecture Notes in Math.* **986** 243–297. Springer, Berlin. MR0770418

[2] CARLEN, E. A., CARVALHO, M. C. and LOSS, M. (2003). Determination of the spectral gap for Kac's master equation and related stochastic evolution. *Acta Math.* **191** 1–54. MR2020418

[3] CHEN, G.-Y. (2006). The cut-off phenomenon for finite Markov chains. Ph.D. thesis, August 2006.

[4] DIACONIS, P. (1988). *Group Representations in Probability and Statistics*. IMS, Hayward, CA. MR0964069

[5] DIACONIS, P. (1996). The cutoff phenomenon in finite Markov chains. *Proc. Natl. Acad. Sci. USA* **93** 1659–1664. MR1374011

[6] DIACONIS, P. and SALOFF-COSTE, L. (2000). Bounds for Kac's master equation. *Comm. Math. Phys.* **209** 729–755. MR1743614

[7] DIACONIS, P. and SHAHSHAHANI, M. (1981). Generating a random permutation with random transpositions. *Z. Wahrsch. Verw. Gebiete* **57** 159–179. MR0626813

[8] EDWARDS, R. E. (1972). *Integration and Harmonic Analysis on Compact Groups*. Cambridge Univ. Press, London. MR0477598

[9] FULTON, W. and HARRIS, J. (1991). *Representation Theory*. *Graduate Texts in Mathematics* **129**. Springer, New York. MR1153249

[10] HEWITT, E. and ROSS, K. A. (1979). *Abstract Harmonic Analysis*. *Vol*. *i*, 2nd ed. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **115**. Springer, Berlin. MR0551496

[11] HOUGH, B. (2016). The random k cycle walk on the symmetric group. *Probab*. *Theory Related Fields* **165** 447–482. MR3500276

[12] JANVRESSE, E. (2001). Spectral gap for Kac's model of Boltzmann equation. *Ann*. *Probab*. **29** 288–304. MR1825150

[13] JIANG, Y. (2012). Total variation bound for Kac's random walk. *Ann*. *Appl*. *Probab*. **22** 1712–1727. MR2985175

[14] KAC, M. (1956). Foundations of kinetic theory. In *Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability*, 1954–1955, *Vol*. *III* 171–197. Univ. California Press, Berkeley. MR0084985

[15] MASLEN, D. K. (2003). The eigenvalues of Kac's master equation. *Math*. *Z*. **243** 291–331. MR1961868

[16] OLIVEIRA, R. I. (2009). On the convergence to equilibrium of Kac's random walk on matrices. *Ann*. *Appl*. *Probab*. **19** 1200–1231. MR2537204

[17] POROD, U. (1996). The cut-off phenomenon for random reflections. *Ann*. *Probab*. **24** 74–96. MR1387627

[18] ROSENTHAL, J. S. (1994). Random rotations: Characters and random walks on $\mathrm{SO}(N)$. *Ann*. *Probab*. **22** 398–423. MR1258882

[19] SERRE, J.-P. (1977). *Linear Representations of Finite Groups*. Springer, New York. MR0450380

[20] VON RENESSE, M.-K. and STURM, K.-T. (2005). Transport inequalities, gradient estimates, entropy, and Ricci curvature. *Comm*. *Pure Appl*. *Math*. **58** 923–940. MR2142879

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