# Horospherical Density Over Recurrent Points in Laminations Associated to Rational Maps

Travis Waddington

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## 1 Introduction

## 2 Background

Central to the following discussion will be a generalized notion of *homoclinic points*. A homoclinic point usually refers to a point of transverse intersection of the stable and unstable manifolds of a fixed point. We will not define these terms here, as they are not germane to the dicussion at hand; the crux of the idea, as far as this paper is concerned, is that the homoclinic point converges to the fixed point in both forwards and backwards time (when restricted to appropriate subsequnces).

In our scenario, the role of the unstable manifolds will be played by the leaves of $A_f$ (because of leafwise expansion), while the role of the stable manifolds will be played by the fibers of $\pi_0$ (because of transversal contraction).

**Definition 2.1.** *Let $\hat{a}$ be a recurrent point in $A_f$ such that $\pi_0(\hat{a}) = a$ is not finitely postcritical. Denote by the homoclinic points of $\hat{a}$ the set $HC(\hat{a}) = \{\hat{y} \in L(\hat{a}) | \pi_0(\hat{y}) = a$ and $\pi_0'(\hat{y}) \neq 0\}$.*

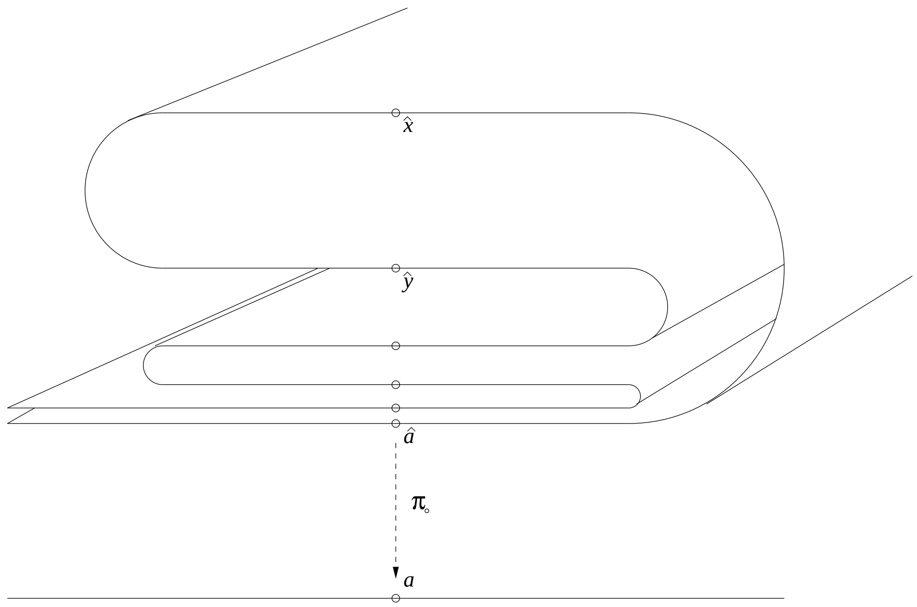


***Fig. 1.*** *$\hat{x}$ and $\hat{y}$ above $\hat{a}$ are elements of $HC(\hat{a})$ . In this figure, $A_f$ has been drawn as a 1-dimensional curve projecting down to a 1-dimensional $\hat{\mathbb{C}}$ ; the extra dimension going back into the page is the scaling bundle, $H_f$ .*

Because $\hat{a}$ is not fixed, the notions of stable and unstable manifolds do not really apply. Furthermore, since the stable direction is transversal, the "stable manifold" is not a manifold, but a Cantor set - the transversal structure of our lamination. However, the forward histories of $\hat{a}$ and $\hat{x}$ are identical, and by the shrinking lemma their past histories converge. Since $a$ is not postcritical, this means that $\hat{f}^n(\hat{x}) \to \hat{f}^n(\hat{a})$ as $n \to \pm\infty$. Finally, because $\hat{a}$ is recurrent, there are subsequences $\{n_j\} \to \pm\infty$ such that $\hat{f}^{n_j}(\hat{x}) \sim \hat{f}^{n_j}(\hat{a}) \to \hat{a}$. In this sense, the homoclinic points of $\hat{a}$ do limit to $\hat{a}$ both forwards and backwards, justifying the terminology.

Fix some horosphere over (i.e., a Euclidean metric on) some affine leaf $L \subset A_f$ . Let $\sigma$ be any metric on $\hat{\mathbb{C}}$ ; we will mostly take $\sigma$ to be the spherical metric. The pullback $\pi_0^*(\sigma)$ determines a section over (i.e., a metric on) $L$. For any point $\hat{x} \in L$, we can measure the difference in the heights of the horosphere and the $\sigma$-section over $\hat{x}$; denote this by $\beta_\sigma(\hat{x})$, noting that the value depends implicitly on our choice of horosphere. If we now take another point $\hat{y} \in L$, we can look at the difference $\Delta_\sigma(\hat{x}, \hat{y}) \ = \beta_\sigma(\hat{y}) - \beta_\sigma(\hat{x})$. Note that $\Delta_\sigma(\hat{x}, \hat{y})$ no longer depends on the choice of horosphere, since any two horosppheres are separated by a constant height, but does still depend on the choice of $\sigma$. This difference is the *basic cocycle* of $\sigma$. It should be noted at this time that, if $\pi_0(\hat{x}) = \pi_0(\hat{y})$, then we would lose our dependence on $\sigma$ as well. In particular, this is the case for all homoclinic points of a given recurrent point.

# 3 Main Theorems

**Theorem 3.1.** *Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational map which has no branch exceptional points. Let $\hat{a}$ be a recurrent point in $A_f$, and let $\pi_0(\hat{a}) = a \in \hat{\mathbb{C}}$ not be finitely postcritical.*

*Then a horosphere above $\hat{a}$ is dense in $H_f$ if and only if $\exists \hat{x}, \hat{y} \in HC(\hat{a})$ such that $\Delta_\sigma(\hat{a}, \hat{x}) < 0 < \Delta_\sigma(\hat{a}, \hat{y})$.*

In other words, there must be homoclinic points of $\hat{a}$ with both less and more expansive histories than $\hat{a}$ itself. In a later section, we will say something about how often this condition is met. Note that the recurrence of $\hat{a}$ immediately tells us that the corresponding affine leaf, which we shall denote $L(\hat{a})$, cannot be associated to a parabolic periodic point, as $\hat{f}$ acts on such leaves by translation. This is important to note, because such leaves exhibit nondense horospheres.

**Theorem 3.2.** *As above, let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational map which has no branch exceptional points, let $\hat{a}$ be a recurrent point in $A_f$, and let $\pi_0(\hat{a}) = a \in \hat{\mathbb{C}}$ not be finitely postcritical.*

*Then the quotient of a horosphere above $\hat{a}$ by $\hat{f}$ is dense in the quotient lamination, $M_f$.*

# 4 Definitions and Structures

Since we are interested in cocycles of points that are transversally related, it will be useful to introduce a vertical transition map.

For $\hat{y} \in HC(\hat{a})$, let $\gamma$ be any path in $L(\hat{a})$ connecting $\hat{a}$ to $\hat{y}$, avoiding critical points of $\pi_0$. Covering $\gamma$ by a finite number of flowboxes, we can find a small flowbox around $\hat{a}$ that can be dragged through our covering along $\gamma$ to line up with a small flowbox around $\hat{y}$. We can further restrict these two flowboxes so that they project to the same neighborhood of $a$.

Now, for $\hat{x}$ in this small flowbox of $\hat{a}$, let $\psi_{\hat{a}\hat{y}}(\hat{x})$ be the point obtained by dragging $\hat{x}$ along $\gamma$; that is, let $\psi_{\hat{a}\hat{y}}(\hat{x})$ be the unique point in the small flowbox of $\hat{y}$ given above so that $\pi_0(\psi_{\hat{a}\hat{y}}(\hat{x})) = \pi_0(\hat{x})$.

**Definition 4.1.** $\forall \hat{y} \in HC(\hat{a})$ *the above procedure defines a Vertical Transition*

*Map $\hat{x} \mapsto \psi_{\hat{a}\hat{y}}(\hat{x})$ , well-defined in some flowbox around $\hat{a}$ that depends on $\hat{a}$ and on $\hat{y}$.*

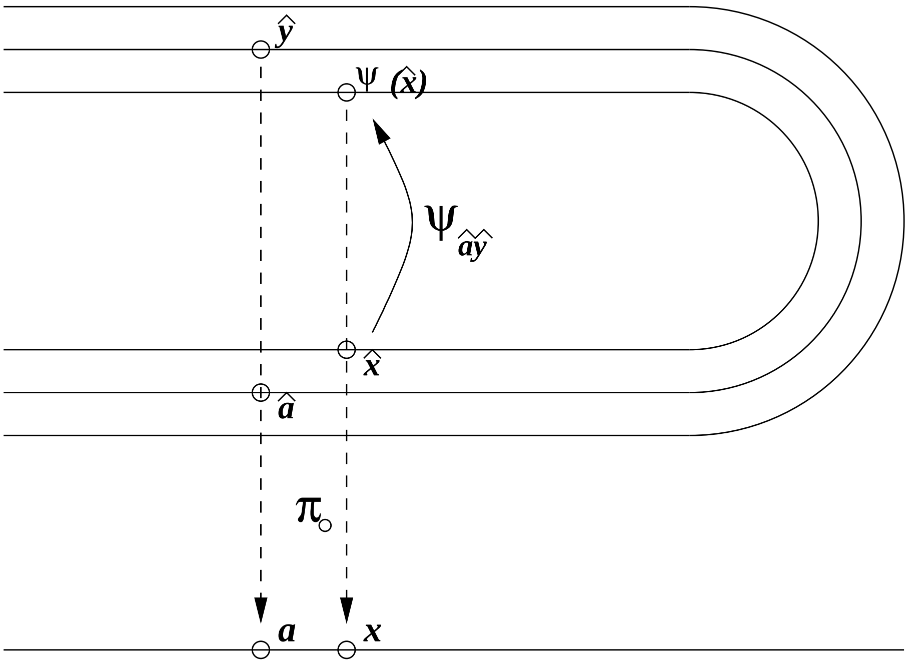


***Fig. 2.*** *The vertical transition map from a point $\hat{x}$ in a neighborhood of $\hat{a}$ to a neighborhood of $\hat{y} \in HC(\hat{a})$ .*

We immediately compose the basic cocycle with this transition map to obtain the function that will be the subject of the rest of this discussion.

**Definition 4.2.** *Given $\hat{y} \in HC(\hat{a})$ and $\hat{x}$ in an appropriate flowbox (depending on $\hat{y}$) of $\hat{a}$, let $\Delta_{\hat{a}\hat{y}}(\hat{x}) = \Delta_\sigma(\hat{x}, \psi_{\hat{a}\hat{y}}(\hat{x}))$.*

Thus, $\Delta_{\hat{a}\hat{y}}(\hat{x})$ represents the difference in the height of $\sigma$ between a point $\hat{x}$ near $\hat{a}$ and the "corresponding" point near $\hat{y}$. Of course, because these points are in the same transversal, this value is actually independent of the choice of $\sigma$.

**Remark 4.3.** *$\Delta_{\hat{a}\hat{y}}(\hat{x})$ is real analytic on leaves, and is transversally continuous.*

Real analyticity on the leaves is almost immediate from the definitions. The complex structure on $A_f$ was defined so that the projection maps, including $\pi_0$, would be analytic; as such, when we pull $\sigma$ back to $H_f$ we get an analytic section over any given leaf. A horosphere is a constant section over that leaf, so the difference in heights between the two is an analytic real-valued function.

Transversal continuity is a little more difficult to see, but comes essentially from the fact that the height difference between any two given points on a given leaf comes from the "recent past"; this, in turn, comes from the uniform continuity of the basic cocycle with respect to the metric induced by a special section. Let $\hat{x}$ and $\hat{w}$ be two points in the domain of $\psi_{\hat{a}\hat{y}}$ that are also in the same transversal. By the Shrinking Lemma, for any positive $\varepsilon$ we can find some large $N$ so that $\hat{f}^{-N}(\psi_{\hat{a}\hat{y}}(\hat{x})\ )$ is very close to $\hat{f}^{-N}(\hat{x})$, close enough that, by the uniform continuity of the basic cocycle, $\Delta_\sigma(\hat{f}^{-N}(\psi_{\hat{a}\hat{y}}(\hat{x})\ ), \hat{f}^{-N}(\hat{x})) < \varepsilon/2$. Now, choose $\hat{w}$ so close to $\hat{x}$ that $\hat{f}^{-N}(\hat{w})$ is still very close to $\hat{f}^{-N}(\hat{x})$; in particular, they are still in the same transversal, so that the last $N$ terms of $\hat{w}$ and $\hat{x}$ agree. Then, by the same argument (because the continuity is uniform), $\Delta_\sigma(\hat{f}^{-N}(\psi_{\hat{a}\hat{y}}(\hat{w})\ ), \hat{f}^{-N}(\hat{w})) < \varepsilon/2$. That is, far enough in the past, any two points are arbitrarily close together at every step, so that the metrics above and below cannot disagree too much. It is only once these points diverge that their expansions can differ significantly. Thus, as one pair of points approximates the history of another pair of points farther and farther into the past, it also duplicates more and more of the expansion differences, so that the height differential of the first pair approximates the height differential of the second pair more and more closely.

# 5 Additivity and Close Terms

This section represents the bulk of the technical work in this paper. In it, we will analyze the values of $\Delta_\sigma(\hat{a}, \hat{y})$ attained by points in $\hat{y} \in HC(\hat{a})$ . We will show that these values have approximate additivity (additivity in the closure) and that there is no smallest gap between values.

**Lemma 5.1.** $\overline{\{\Delta_\sigma(\hat{a}, \hat{y})\ |\hat{y} \in HC(\hat{a})\ \}}$ *is an additive semigroup.*

*Proof.* Let $\hat{x}, \hat{y} \in HC(\hat{a})$ . We need to show that there exists a sequence $\{\hat{z_j}\} \subset HC(\hat{a})$ such that $\Delta_\sigma(\hat{a}, \hat{z_j}) \to \Delta_\sigma(\hat{a}, \hat{x}) + \Delta_\sigma(\hat{a}, \hat{y})$ .

This is straightforward. By recurrence, there is a sequence of times $\{-n_j\}$ such that $\hat{f}^{-n_j}(\hat{a}) \to \hat{a}$. $\hat{f}^{-1}$ contracts leafwise, so for any $\hat{z} \in HC(\hat{a})$ , $\hat{f}^{-n_j}(\hat{z}) \to \hat{a}$ as well.

Now, set $\hat{z_j} = \hat{f}^{n_j} \circ \psi_{\hat{a}\hat{y}} \circ \hat{f}^{-n_j}(\hat{x})$.

It is easy to check that $\hat{z_j} \in HC(\hat{a})$ .

$\Delta_\sigma(\hat{a}, \hat{z_j})$

$= \Delta_\sigma(\hat{a}, \hat{x}) \ + \Delta_\sigma(\hat{x}, \hat{z_j})$

$= \Delta_\sigma(\hat{a}, \hat{x}) \ + \Delta_\sigma(\hat{f}^{-n_j}(\hat{x}), \psi_{\hat{a}\hat{y}} \circ \hat{f}^{-n_j}(\hat{x}))$

$= \Delta_\sigma(\hat{a}, \hat{x}) \ + \Delta_{\hat{a}\hat{y}}(\hat{f}^{-n_j}(\hat{x}))$

$\to \Delta_\sigma(\hat{a}, \hat{x}) \ + \Delta_{\hat{a}\hat{y}}(a)$

$= \Delta_\sigma(\hat{a}, \hat{x}) \ + \Delta_\sigma(\hat{a}, \hat{y})$

□

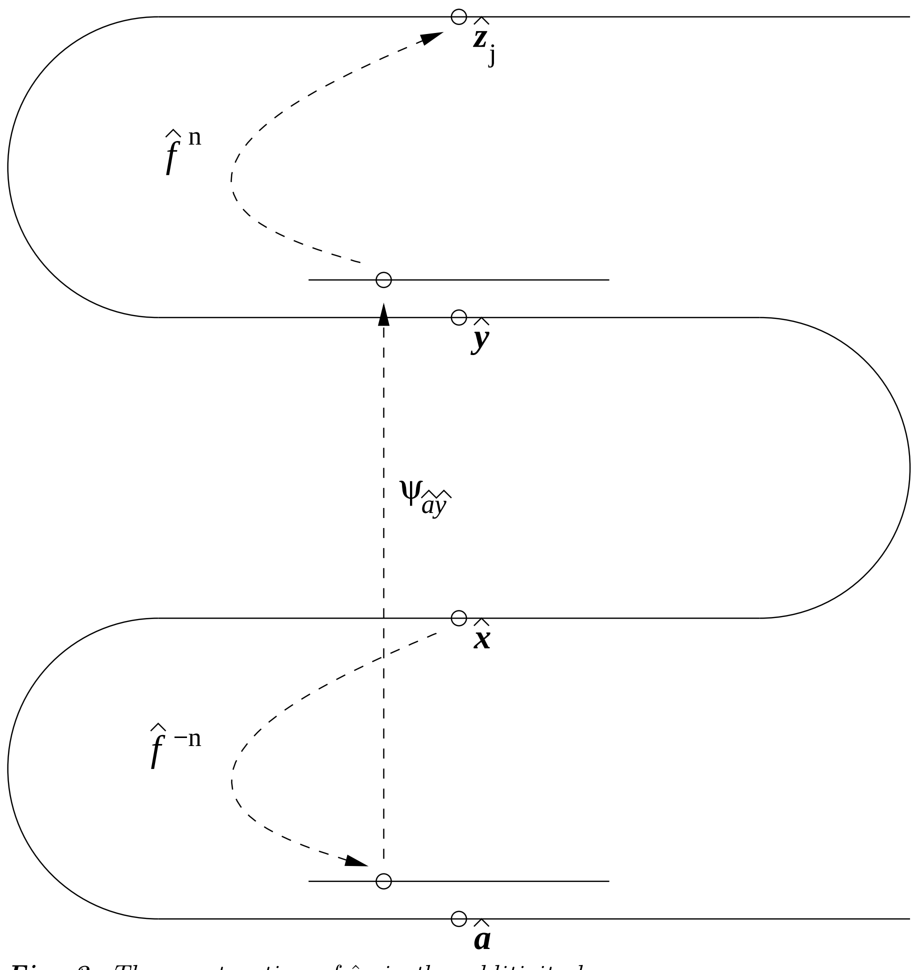


***Fig. 3.*** *The construction of $\hat{z}_j$ in the additivity lemma.*

Now, if we can show that $\overline{\{\Delta_\sigma(\hat{a},\hat{y})\ |\hat{y} \in HC(\hat{a})\ \}}$ has arbitrarily close terms, then we can use those terms plus additivity to construct arbitrarily long nets of arbitrarily close terms. We will concentrate these $\varepsilon$-nets over $\hat{a}$ to demonstrate horospherical density there. The hard part is showing that $\overline{\{\Delta_\sigma(\hat{a},\hat{y})\ |\hat{y} \in HC(\hat{a})\ \}}$ does indeed have arbitrarily close terms, so we will concentrate on that first.

The basic strategy for showing close terms comes from the proof of the Additivity Lemma, above. As $\hat{f}^{-n_j}(\hat{b}) \to \hat{a}$, $\Delta_{\hat{a}\hat{y}}(\hat{f}^{-n_j}(\hat{x}))$ converges to the single value $\Delta_\sigma(\hat{a},\hat{y})$ , so these values must be getting very close to each other. The problem is, they could be TOO close. That is, they could all be equal (for large n), in which case we do not get arbitrarily close terms.

Our strategy is to show that this cannot always be the case. We want to show that, for some choice of $\hat{x}$, $\hat{y}$, and $\{-n_j\}$, the values $\Delta_{\hat{a}\hat{y}}(\hat{f}^{-n_j}(\hat{x}))$ are not equal to $\Delta_\sigma(\hat{a},\hat{y})$ . This non-trivial convergence will give us the close terms that we need.

Actually, a slightly weaker condition will suffice. It is enough to show that, for some $\hat{y} \in HC(\hat{a})$ , there is no flowbox $B$ of $\hat{a}$ so that whenever $\hat{f}^{-n}(\hat{z}) \in B$ (for any $\hat{z} \in HC(\hat{a})$ and any $n > 0$), $\Delta_{\hat{a}\hat{y}}(\hat{f}^{-n_j}(\hat{z})) = \Delta_\sigma(\hat{a},\hat{y})$ .

If no such flowbox exists, then there are unequal terms arbitrarily close to $\Delta_\sigma(\hat{a},\hat{y})$ , and this will be enough; they don't need to come from the same tail.

We will assume such a flowbox exists, and derive a contradiction. But first, we need to come up with an appropriate choice of $\hat{y}$. This point is given by the following lemma.

**Lemma 5.2.** *$\exists \hat{y} \in HC(\hat{a})$ such that $\Delta_{\hat{a}\hat{y}}(\hat{z})$ is nonconstant as a function of $\hat{z}$ on a flowbox of $\hat{a}$.*

*Proof.* We will actually prove a slightly stronger statement, which is that $\Delta_{\hat{a}\hat{y}}(\hat{z})$ is nonconstant on the local leaf of $\hat{a}$. First, we will show that if $\Delta_{\hat{a}\hat{y}}(\hat{z})$ is constant on $L_{loc}(\hat{a})$ then $\Delta_{\hat{a}\hat{y}}(\hat{z}) = 0$ on that leaf. Then we will show that if $\Delta_{\hat{a}\hat{y}}(\hat{y}) = 0 \forall \hat{y} \in HC(\hat{a})$ then our original function $f$ must have branch exceptional points, which we specifically disallowed.

Consider $\psi_{\hat{a}\hat{y}} : L_{loc}(\hat{a}, U) \to L_{loc}(\hat{y}, U)$. Pick some Euclidean metric $g$ on $L(\hat{a})$ (in other words, a representative horosphere). Let $m_{\sigma,\hat{z}}$ denote the metric obtained at $\hat{z}$ by pulling back the spherical metric $\sigma$ through the projection $\pi_0 : L(\hat{a}) \to \hat{\mathbb{C}}$ .

$$\Delta_{\hat{a}\hat{y}}(\hat{z}) \; = ln\left|\frac{m_{\sigma,\psi_{\hat{a}\hat{y}}(\hat{z})}}{g}\right| - ln\left|\frac{m_{\sigma,\hat{z}}}{g}\right| = ln\left|\frac{m_{\sigma,\psi_{\hat{a}\hat{y}}(\hat{z})}}{m_{\sigma,\hat{z}}}\right|$$

$$|\psi'_{\hat{a}\hat{y}}(\hat{z})| = ((\pi_o^{-1})_{L_{loc}(\hat{y})} \circ (\pi_0)_{L_{loc}(\hat{a})})'(\hat{z}) = \left|\frac{g}{m_{\sigma,\psi_{\hat{a}\hat{y}}(\hat{z})}} \cdot \frac{m_{\sigma,\hat{z}}}{g}\right| = \left|\frac{m_{\sigma,\hat{z}}}{m_{\sigma,\psi_{\hat{a}\hat{y}}(\hat{z})}}\right|$$

So if $\Delta_{\hat{a}\hat{y}}(\hat{z})$ is constant on a leafwise neighborhood of a, that means that $\psi_{\hat{a}\hat{y}}$ is affine on that neighborhood, which means in turn that $\psi_{\hat{a}\hat{y}}$ extends to an affine self-map of $L(\hat{a})$.

If $|\psi_{\hat{a}\hat{y}}\,'| \neq 1$, then $\psi_{\hat{a}\hat{y}}$ must have some repelling or attracting fixed point. By construction, $\pi_0$ is $\psi_{\hat{a}\hat{y}}$ invariant, so $\pi_0$ must be constant near that fixed point. But $\pi_0$ is never constant. Thus, if $\Delta_{\hat{a}\hat{y}}(\hat{z})$ is constant in some leafwise neighborhood of $\hat{a}$, it must be identically zero in that leafwise neighborhood.

As an immediate consequence, if $\Delta_{\hat{a}\hat{y}}(\hat{z})$ is constant in some flowbox about $\hat{a}$, that constant must be zero.

Now, suppose that $\Delta_{\hat{a}\hat{y}}(\hat{z})$ is flowbox-constant near $\hat{a}$ for every $\hat{y} \in HC(\hat{a})$ . Then these constants are all zero. In particular, $\Delta_\sigma(\hat{a},\hat{y})$ is always zero, which means (by the cocycle property) that $\Delta_\sigma(\hat{y_1},\hat{y_2}) \; = 0$ for any pair $\hat{y}_1,\hat{y}_2 \in HC(\hat{a})$ .

Pick any simply connected neighborhood $W$ satisfying $W \cap J \neq \varnothing$. Lift W to two distinct univalent neighborhoods $W_1, W_2 \subset L(\hat{a})$. Now, pick some Euclidean metric (horosphere) $g$ on $L(\hat{a})$. We will show that the two projections of this metric onto $W$ from $W_1$ and $W_2$ are equal, and we will use this to show that $A_f$ is Euclidean. This can only happen if $f$ has a parabolic Thurston orbifold, which can in turn only happen if $f$ is Lattes, Tchebyshev, or $z^{\pm d}$. All of these cases have branch exceptional points, and are thus disallowed by our hypotheses.

Suppose that the two metrics are not the same. Equality on $W \cap J$ would, by analyticity, imply equality on all of $W$. Thus, we can restrict $W$ in such way that $W$ is still simply connected and still satisfies $W \cap J \neq \varnothing$ while further ensuring that the two metrics are unequal everywhere on $W$. Because $f^n(W)$

eventually covers almost all of $\hat{\mathbb{C}}$ , $\exists x \in W, n \in \mathbb{N} s.t. f^n(x) = a$. Now we can restrict W to some arbitrarily small neighborhood of this point $x$. Lift $W$ to a univalent neighborhood $U_1 \subset \hat{f}^{-n}(L(\hat{a}))$ transversally close to $W_1$. Lift $x$ to points $\hat{x}_1 \in W_1, \hat{x}_2 \in W_2, \hat{v}_1 \in U_1$. We can choose $U_1$ so that $\psi_{\hat{x}_1\hat{x}_2}$ is defined on $U_1$. Then let $\hat{v}_2 = \psi_{\hat{x}_1\hat{x}_2}(\hat{v}_1)$. Now, apply $\hat{f}^n$ to this whole picture. Now we have $\hat{f}^n(\hat{x}_1)$ transversally very close to $\hat{f}^n(\hat{v}_1)$ and $\hat{f}^n(\hat{v}_2) = \psi_{\hat{f}^n(\hat{x}_1)\hat{f}^n(\hat{x}_2)}(\hat{f}^n(\hat{v}_1))$. Furthermore, $\hat{f}^n(\hat{v}_{1,2}) \in HC(\hat{a})$ . And, since we picked $U_1$ to be transversally close to $W_1$, there is some $\hat{z}$ very close to $\hat{a}$ so that $\psi_{a\hat{f}^n(\hat{v}_1)}(\hat{z}) = \hat{f}^n(\hat{x}_1)$. Thus, $\Delta_{a\hat{f}^n(\hat{v}_1)}(\hat{z}) = 0$ and $\Delta_{a\hat{f}^n(\hat{v}_2)}(\hat{z}) = 0$, so $\Delta_{\hat{f}^n(\hat{v}_1)\hat{f}^n(\hat{v}_2)}(\hat{f}^n(\hat{x}_1)) = 0$. Finally, since by construction these points all have the same history for the last n steps, we have $\Delta_{\hat{v}_1\hat{v}_2}(\hat{x}_1) = 0$, which is the same as saying $\Delta_\sigma(\hat{x}_1, \hat{x}_2) = 0$. But this is just another way of saying that that the two pushdowns of a horospherical metric to $x$ agree, which contradicts our choice of $x$.

This shows that any horosphere on $L(\hat{a})$ projects coherently down to a metric on $\hat{\mathbb{C}}$ . We can immediately pull that metric back up to get onto all of $A_f$ . This new metric on $A_f$ must agree with the original Euclidean metric on $L(\hat{a})$. Since the metric is Euclidean on $L(\hat{a})$ which is dense in $A_f$ , it is Euclidean on all of $A_f$ . But the only functions with Euclidean $A_f$ are Lattes, Tchebyshev, or $z^d$. Thus, only disallowed functions could possibly have all their $\Delta_{\hat{a}\hat{y}}(\hat{z})$ functions constant. So for our function $f$, there must be some $\hat{y}$ in $HC(\hat{a})$ so that $\Delta_{\hat{a}\hat{y}}(\hat{z})$ is nonconstant in a flowbox around $\hat{a}$

□

Now, sticking with this choice of $\hat{y}$ as the target of our vertical transition function, we need to show that there is no "restrictive flowbox", that is,

**Lemma 5.3.** *For the conditions as stated, there is no flowbox $B$ of $\hat{a}$ so that when $\hat{f}^{-n}(\hat{z}) \in B$ (for any $\hat{z} \in HC(\hat{a})$ and any $n > 0$), $\Delta_{\hat{a}\hat{y}}(\hat{f}^{-n_j}(\hat{z})) = \Delta_\sigma(\hat{a}, \hat{y})$ .*

*Proof.* Suppose there is such a flowbox $B$. On $B \cap L_{loc}(\hat{a}$, the level set of

$\Delta_{\hat{a}\hat{y}}(\hat{z})$ is a finite union of smooth curves intersecting at $\hat{a}$. By continuity and analyticity, this is also true for nearby local leaves, so the level set of $\Delta_{\hat{a}\hat{y}}(\hat{z})$ is a continuous family of unions of real-analytic curves $\Gamma$.

Now, our hypothesis tells us that, for any $\hat{z} \in HC(\hat{a})$ and any $n$, if $\hat{f}^{-n}(\hat{z}) \in B$ then $\hat{f}^{-n}(\hat{z}) \in \Gamma$. Let $q \in \hat{\mathbb{C}}$ be any point in the Julia set of $f$ near $a$. By mixing, we know that infinitely many inverses branches of $a$ go arbitrarily close to q. Pick some $\hat{z_k} \in HC(\hat{a})$ whose first $n_k$ terms take it arbitrarily close to q, and whose next m terms follow some $\hat{q} \in B$, so that $\hat{f}^{-n_k}(\hat{z_k}) \in B$. Indeed, we can pick a sequence so that $\hat{f}^{-n_k}(\hat{z_k}) \to \hat{q}$. But those points are all in our continuous family of curves $\Gamma$, so we must have $\hat{q} \in \Gamma$. This means that the lifted Julia set is contained in $\Gamma$. Projection from a given leaf is analytic, so at $a \in \hat{\mathbb{C}}$ the Julia set is locally contained in a finite union of real-analytic curves. But again, this is only the case when $f$ is Tchebyshev, Lattes, or $z * d$

□

At this point, we have established that there is no "restrictive flowbox". In other words, there exists some $\hat{y} \in HC(\hat{a})$ and a sequence of points $\hat{f}^{-n_k}(\hat{z}_k) \to \hat{a}$ where $\hat{z}_k \in HC(\hat{a})$ such that $\Delta_{\hat{a}\hat{y}}(\hat{f}^{-n_k}(\hat{z}_k)) \neq \Delta_{\hat{a}\hat{y}}(a)$ even though $\Delta_{\hat{a}\hat{y}}(\hat{f}^{-n_k}(\hat{z}_k)) \to \Delta_{\hat{a}\hat{y}}(a)$ .

Now we will use these points to construct close terms in $HC(\hat{a})$ .

**Lemma 5.4.** $\forall \varepsilon > 0, \exists \hat{b}, \hat{c} \in HC(\hat{a})$ *such that* $\varepsilon > \left| \Delta_\sigma(\hat{a}, \hat{c}) - \Delta_\sigma(\hat{a}, \hat{b}) \right| > 0.$

*Proof.* From the sequence described above, select $\hat{z}_k$ so that $\Delta_{\hat{a}\hat{y}}(\hat{f}^{-n_k}(\hat{z}_k))$ is within $\varepsilon/2$ of, but not equal to, $\Delta_{\hat{a}\hat{y}}(a)$ . By convergence to $\hat{a}$ and continuity of $\Delta_{\hat{a}\hat{y}}(z)$ , there exists some $N > n_k$ so that $\Delta_{\hat{a}\hat{y}}(\hat{f}^{-N}(\hat{z}_k))$ is even closer to $\Delta_{\hat{a}\hat{y}}(a)$ .

Let $\hat{b} = \hat{f}^{n_k} \circ \psi_{\hat{a}\hat{y}} \circ \hat{f}^{-n_k}(\hat{z}_k)$, and let $\hat{c} = \hat{f}^N \circ \psi_{\hat{a}\hat{y}} \circ \hat{f}^{-N}(\hat{z}_k)$.

Then, since these two points both start out the same, and since their backward and forward motions cancel each other out, the only difference in their heights comes from the holonomy $\psi_{\hat{a}\hat{y}}$. In other words, $\Delta_\sigma(\hat{a}, \hat{b}) = \Delta_\sigma(\hat{a}, \hat{z_k}) + \Delta_{\hat{a}\hat{y}}(\hat{f}^{-n_k}(\hat{z}_k))$ , while $\Delta_\sigma(\hat{a}, \hat{c}) = \Delta_\sigma(\hat{a}, \hat{z_k}) + \Delta_{\hat{a}\hat{y}}(\hat{f}^{-N}(\hat{z}_k))$ .

Thus, $\Delta_\sigma(\hat{a}, \hat{b})$ and $\Delta_\sigma(\hat{a}, \hat{c})$ are within $\varepsilon$ of each other, but not equal.

□

Constructing these close terms represents the bulk of the technical achievement of this paper. The rest, including the two theorems, follows directly from the existence of these close terms. In particular, Theorem 1.1 is immediate.

*Proof.* Proof of Theorem 1.1

By the previous lemma, for any $\varepsilon$, $HC(\hat{a})$ contains points $b$ and $c$ whose cocycle values over $a$ are within $\varepsilon$ but not equal. If these values had opposite signs, then by additivity the cocycle values of points in $HC(\hat{a})$ would be $\varepsilon$-dense, proving our theorem, so let us assume they have the same sign. By our hypothesis, there is some $d$ in $HC(\hat{a})$ whose cocycle value against $a$ would have the opposite sign. Select a positive integer $l$ so that $l\varepsilon > \left|\Delta_\sigma(\hat{a}, \hat{d})\right|$. Consider the sequence $\{(l-i)\Delta_\sigma(\hat{a}, \hat{b}) + i\Delta_\sigma(\hat{a}, \hat{c})\}$ as $i$ ranges from 0 to $l$. This sequence forms an $\varepsilon$-net longer than $\Delta_\sigma(\hat{a}, \hat{d})$ , so for some value of i there must be an integer $m$ so that $(l-i)\Delta_\sigma(\hat{a}, \hat{b}) + i\Delta_\sigma(\hat{a}, \hat{c}) + m\Delta_\sigma(\hat{a}, \hat{d})$ and $(l-i-1)\Delta_\sigma(\hat{a}, \hat{b}) + (i+1)\Delta_\sigma(\hat{a}, \hat{c}) + m\Delta_\sigma(\hat{a}, \hat{d})$ have opposite signs. Thus, again, additivity gives us $\varepsilon$-density of cocycle values of homoclinic points.

Now we need to accumulate this density over $\hat{a}$. Pick any height $h$ and any $\varepsilon > 0$. By density, there exists some $\hat{y} \in HC(\hat{a})$ so that $\Delta_\sigma(\hat{a}, \hat{y})$ is within $\varepsilon$ of $h$. Select $\{n_j\}$ so that $\hat{f}^{-n_j}(\hat{a}) \to \hat{a}$. Let $\hat{z}_j = \hat{f}^{n_j} \circ \psi_{\hat{a}\hat{y}} \circ \hat{f}^{-n_j}(\hat{a})$. Note that as $n_j \to \infty$, $\hat{z}_j \to \hat{a}$. Meanwhile, the height of our horosphere $S_{\hat{a},0}$ over $\hat{z}_j$ is just $\Delta_{\hat{a}\hat{y}}(\hat{f}^{-n_j}(\hat{a}))$ , which in converges to $\Delta_{\hat{a}\hat{y}}(\hat{a}) = \Delta_\sigma(\hat{a}, \hat{y})$ .

Thus, the horosphere $S_{\hat{a},0}$ contains points $(\hat{z}_j, \Delta_{\hat{a}\hat{y}}(\hat{f}^{-n_j}(\hat{a})))$ which converge to a point above $\hat{a}$ with height within $\varepsilon$ of any height $h$, for any positive $\varepsilon$. This proves that the horosphere is dense above $\hat{a}$ and, by extension, above the leaf containing $\hat{a}$. Since $A_f$ is minimal, this establishes density of the horosphere in $H_f$ .

This proves the nontrivial direction of the theorem. The other direcction is obvious, as density would clearly imply cocycle values both positive and negative.

□

## 6 Conclusion: Proof of Theorem 1.2

Obviously, when the conditions for Theorem 1.1 hold, that implies Theorem 1.2. Furthermore, since all it requires is that $\hat{a}$ have neither the least nor the most expanding prehistory in $HC(\hat{a})$ , the conditions for Theorem 1.1 are frequently met. For instance, any time a periodic point has periodic expansion that is neither minimal nor maximal amongst periods, there are homoclinic points that spend time near more or less expansive periods; these can be made to have, correspondingly, either positive or negative values of the basic cocycle. However, there are certainly examples of recurrent points that do have maximal and minimal histories - A. Glutsyuk showed, for example, that the fixed lifts of the largest real fixed points of many real quadratic polynomials $z^2 + c$ have such extreme expansion-histories, minimal in the case $0 < c < 1/4$ and maximal in the case $c < 0$. But even in these cases, the horosphere becomes dense after quotienting $H_f$ by the action of $\hat{f}$. This is the content of Theorem 1.2.

*Proof.* Proof of Theorem 1.2

Let us pick a target height $h$ and a target accuracy $\varepsilon$; our goal will be to show that our horosphere $S_{\hat{a},0}$ contains a point $(\hat{z}, h')$ so that $\hat{z}$ is within $\varepsilon$ of $\hat{a}$ and $h'$ within $\varepsilon$ of $h$.

First, we know from the lemmas in the previous section that we can select two homoclinic points $\hat{b}, \hat{c} \in HC(\hat{a})$ so that $\varepsilon > \Delta_\sigma(\hat{a}, \hat{c}) - \Delta_\sigma(\hat{a}, \hat{b}) > 0$. As before, we will construct an $l$ element $\varepsilon$ net, but this time we choose $l$ so that the span of the net is longer than $\Delta_\sigma(\hat{a}, \hat{b})$ . This means that this $\varepsilon$-net overlaps the next net which, being even longer (it has one more element) overlaps the next net, and so on. Thus, stringing these nets together, we see that the cocycle values from $HC(\hat{a})$ are $\varepsilon$-dense beyond $l\Delta_\sigma(\hat{a},\hat{b})$ .

Now, select a bi-infinite sequence $\{n_j\}$ so that $\hat{f}^{n_j}(\hat{a}) \to \hat{a}$ as $j$ goes to either positive or negative infinity. Pick $j$ so that $ln\left|(\hat{f}^{n_j})'(\hat{a})\right|$ has sign opposite to, and is much larger than, $l\Delta_\sigma(\hat{a},\hat{b})$ , and so that $\hat{f}^{n_j}(\hat{a})$ is within $\varepsilon$ of $\hat{a}$. Then, by density, we can find a homoclinic points $y$ so that $\Delta_\sigma(\hat{a},\hat{y}) + ln\left|(\hat{f}^{n_j})'(\hat{a})\right| = h'$ is within $\varepsilon$ of $h$.

Now let $\hat{z}_k = \hat{f}^{n_k+n_j} \circ \psi_{\hat{a}\hat{y}} \circ \hat{f}^{-n_k}(\hat{a})$. As $k \to \infty$, $\hat{z}_k \to \hat{f}^{n_j}(\hat{a})$. Meanwhile, the back and forth by $\hat{f}^{n_k}$ has no effect on the height, so the height of the image of our horosphere over the final point is due entirely to $\psi_{\hat{a}\hat{y}}$ and to $\hat{f}^{n_j}$. Thus, this map takes $(\hat{a},0)$ to $(\hat{z}_k, \Delta_{\hat{a}\hat{y}}(\hat{f}^{-n_k}(\hat{a})) + ln\left|(\hat{f}^{n_j})'(\hat{a})\right|) \to (\hat{f}^{n_j}(\hat{a}), h')$. By the choice of the $\{n_j\}$, $(\hat{f}^{n_j}(\hat{a}), h') \to (\hat{a}, h')$, and $h'$ was chosen arbitrarily close to $h$.

This shows that the union of the images of our horosphere $S_{\hat{a},0}$ under $\hat{f}$ contains points arbitrarily close to $(\hat{a},h)$ for any height $h$, proving that the horosphere is dense above the leaf containing $\hat{a}$ in the quotient. Again, minimality of $A_f$ extends this density over all of the quotient $M_f$ .

□

# 7 References