

Research Statement

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I study discrete dynamical systems using topological and geometric techniques. My background is in holomorphic dynamics of a single variable, where in the past 30 years geometric and topological approaches have led to a rich flowering of powerful results. My goal as a researcher is to deepen this relationship in the field of single-variable holomorphic dynamics, and to extend it to, or else find alternative topological tools for, other areas of dynamics. I take an active interest in the fields of topology and geometry, especially those parts that pertain closely to dynamics such as hyperbolic geometry, 3-manifolds, Teichmüller and moduli spaces, laminations and foliations, and quasiconformal mappings. I am also interested in the application of thermodynamical formalism to dynamical systems.

My work thus far involves the classification of the dynamics of rational functions on the Riemann sphere and the description of the geometries of their Julia sets. In particular, I approach these problems via the investigation of the topological and measure-theoretic properties of the inverse limit spaces of iterated rational maps from the Riemann sphere to itself, $f : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$. My main results concern the minimality of certain dynamically meaningful laminations in these inverse limit spaces.

BACKGROUND

In the early 1980s, Dennis Sullivan introduced a noncanonical correspondence between the fields of Kleinian groups and rational dynamics. This correspondence, known as Sullivan's dictionary, proved to be a major breakthrough for the study of the dynamics of rational maps. In 1985, Sullivan [Sul85] established this correspondence and used it to prove the famous No Wandering Domains theorem, and as a corollary classified the Fatou components of rational maps. Since then, the ideas and techniques established by that dictionary have been mined to great effect; much of the astounding progress that has been made in the field in the last 25 years has relied on techniques adapted from Sullivan's dictionary.

However, even though it has been very useful, this dictionary is incomplete. It is not a functorial or rigorous correspondence, but rather an interconnected list of strong conceptual equivalences; as such, there is no easy way to fill in a missing line. Hyperbolic 3-manifolds play a very important role in the study of Kleinian groups, where they arise naturally as the result of quotienting \mathbb{H}^3 by the action of a given group. This direct correspondence allows the tools of topology and geometry to be brought to bear on the study of the dynamics of Kleinian groups. Unfortunately, because one cannot quotient the Riemann sphere by the action of a d-to-one rational map, there was no corresponding notion in the study of rational dynamics.

In 1997, Mikhail Lyubich and Yair Minsky [LM97] proposed a construction to fill this gap in the dictionary. First, building off the notion of solenoidal Riemann surfaces introduced by Sullivan in [Sul92], Lyubich and Minsky put a laminar topology on the space of infinite backward orbits of points under a rational map $f : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$. After pruning away a small pathological set, the remaining space is an affine lamination \mathcal{A}_f the leaves of which are copies of \mathbb{C} or of orbifolds based on \mathbb{C} , and which typically has the local transversal structure of a Cantor set. The map f easily extends to a bijection $\hat{f} : \mathcal{A}_f \rightarrow \mathcal{A}_f$ that takes leaves affinely to leaves.

\mathcal{A}_f is always affine but almost never Euclidean; that is, it is only for a small and well-understood family of exceptional rational functions (specifically Lattes examples, Tchebyshev maps, and maps conjugate to z^d) that the leaves of \mathcal{A}_f can be given a continuous family of Euclidean metrics.[LK01] Since this means that there is no way to fix a single scale for any given leaf, it is natural to consider the bundle of choices of scale over \mathcal{A}_f . This scaling bundle over \mathbb{C} has exactly the structure of the halfspace model for \mathbb{H}^3 , so taking the scaling bundle of \mathcal{A}_f yields a new lamination \mathcal{H}_f , with the same transversal structure but whose leaves now look like \mathbb{H}^3 . Because this is the scaling bundle, a section of this bundle in a given leaf of \mathcal{H}_f constitutes a choice of metric on the corresponding leaf of \mathcal{A}_f ; if this section were a horosphere, which is just a constant section of the scaling bundle, then the metric on the leaf of \mathcal{A}_f would be Euclidean. By pushing the choice of leafwise metric in \mathcal{A}_f forward by \hat{f} to get a metric on the image of the leaf, we can naturally extend the action of \hat{f} to \mathcal{H}_f .

The resulting map $\hat{f} : \mathcal{H}_f \rightarrow \mathcal{H}_f$ encodes the dynamical complexity of the d-to-one iterated system $f : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ into a single bijection. In essence, we have traded complicated d-to-one dynamics on a simple space for relatively simple bijective dynamics on a very complicated space. In broad strokes, this already seems to accomplish our goal - we have encoded the dynamics of the rational map into a space, and by using the tools of topology and geometry to understand this space, we can bring them to bear on the original dynamical problem. However, what we really have is an invertable dynamics acting on a space; a Kleinian group acts invertably on \mathbb{H}^3 . Thus, for a true analogue of the hyperbolic 3-manifold, we should quotient by the action of \hat{f} , which is precisely what Lyubich and Minsky do. The resulting quotient lamination, $\mathcal{M}_f = \mathcal{H}_f / \hat{f}$, is the correct rational dynamics translation of the hyperbolic 3-manifold in Sullivan's dictionary.

Now that we have this translation, the next objective is to study the geometry and topology of these laminations, and thereby to learn new things about the rational functions that generate them. This program was started by Lyubich and Minsky [LM97], who used the newly extended dictionary to translate a proof of Mostow's rigidity theorem for hyperbolic 3-manifolds line by line into the context of rational dynamics; the result is a strengthening and extension of Thurston's rigidity theorem for postcritically finite rational maps. There are other attempts currently underway to further understand these laminations, and thereby to prove new results in rational dynamics — the work of Kawahira [Kaw03] and Cabrera [Cab07] is particularly exciting.

THESIS RESULTS

My work focuses on certain dynamically interesting sublaminations of \mathcal{H}_f and \mathcal{M}_f . Since \mathbb{H}^3 is naturally foliated by horospheres, and since \mathcal{H}_f is by definition laminated by copies of \mathbb{H}^3 , \mathcal{H}_f is also laminated by horospheres. Because \hat{f} acts affinely, it preserves the horosphere lamination. Thus, when the lamination is quotiented by the action of \hat{f} to yield $\mathcal{M}_f = \mathcal{H}_f / \hat{f}$, the horosphere lamination descends to a lamination on \mathcal{M}_f . This lamination by horospheres arises naturally when you try to put a Euclidean metric on \mathcal{A}_f . However, it turns out to have an even nicer dynamical interpretation. For two different points in the same leaf of \mathcal{A}_f that both project to the same point in $\hat{\mathbb{C}}$, the difference between the heights of a given horosphere at those two points is a function of the relative expansion in those points' histories. In other words, recall that a point of \mathcal{A}_f is a backwards orbit of a point in $\hat{\mathbb{C}}$. Then we are looking at two alternate histories of the same point in $\hat{\mathbb{C}}$. The fact that they are in the same leaf means that, eventually, these histories run together — we say they have the same prehistory. But the recent histories can be quite different, and one history can have greater expansion than another. This difference is measured by the difference in the height of a given horosphere at the two points.

We know that only the exceptional Euclidean functions (Lattes, Tchebyshev, and z^d) have Euclidean laminations. Thus, for every other function, there will be horospheres that approach themselves transversally at a different height. A natural question to ask, then, is how widespread this behavior is amongst horospheres, and at what heights they approach themselves. A few years ago, Glutsyuk [Glu08] showed that, for f not an exceptional Euclidean function, any horosphere associated to the fixed orbit of a repelling periodic point is dense in the quotient lamination \mathcal{M}_f . Using this result, he was able to characterize the quotient horosphere laminations for critically non-recurrent rational maps. He also demonstrated a family of critically non-recurrent functions with fixed-orbit horospheres that were not dense prior to quotienting.

My goal in [Wad09] was to extend and generalize these results to as broad a class of maps as possible. The fundamental premise was that what was true for a periodic point should be roughly true for a roughly periodic (i.e., recurrent) point. My first step, then, was to generalize some of the tools used to understand periodic points to apply to recurrent points. A central idea in dynamics is the notion of homoclinic points of a given fixed point. Very roughly, these points converge to the fixed point in both forward and backward time, in some reasonably well-controlled fashion. This is realized geometrically as the intersection of stable and unstable manifolds of the fixed point. It turns out that this notion can be generalized usefully to the notion of homoclinic points of a recurrent point: if a is a recurrent point in \mathcal{A}_f , and b has the prehistory and the same current position as a , then we say that b is a homoclinic point of a . Because fixed points are

very rare and recurrent points are very common, this generalized notion, while slightly weaker, is much more broadly applicable.

Notice that the conditions for b to be a homoclinic point of a also imply that we can measure the height difference of a horosphere at those two points. This sets the stage for the basic engine of my results:

Proposition 1. *Let a be a recurrent point of \hat{f} in \mathcal{A}_f , let $HC(a)$ be the set of homoclinic points of a , and let $\Delta(y, a)$ denote the height difference of any horosphere between the points a and y for $y \in HC(a)$. Then the set $\{\Delta(y, a) | y \in HC(a)\}$ is an additive monoid. The monoid is nontrivial if f is not an exceptional Euclidean function.*

The idea is to show that, for f not in the list of exceptional functions, this set contains arbitrarily small positive and negative elements, in which case it must be all of \mathbb{R} . By a series of dynamical and geometric arguments, it is possible to show that the set must contain arbitrarily close elements, but these close elements may be very large, and because the set is a monoid we cannot subtract. However, a single element of the opposite sign is enough to ratchet our way back down towards 0. Alternatively, we can quotient by the dynamics of \hat{f} . This yields the two main results:

Theorem 2. *Let $f : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ be a rational map that is not an exceptional Euclidean map. Let a be a recurrent point in \mathcal{A}_f whose prehistory is not that of a parabolic fixed orbit. Then any horosphere above a is dense in \mathcal{M}_f .*

In other words, for all non-exceptional maps, almost every horosphere is dense after quotienting. Recall that all we need to progress to close terms to density is a single element of the opposite sign. Thus, all we need is a single element of each sign, and we are guaranteed to have whichever sign we need for ratcheting. In other words, our recurrent point must be *non-extremal*: there must be alternate histories with both more and less expansion.

Theorem 3. *Same conditions as above.*

Then any horosphere above a is dense in \mathcal{H}_f if and only if a is non-extremal

Thus, it turns out that for most maps, most horospheres are dense even before quotienting, which was surprising - it turns out that Glutsyuk's family of counterexamples was very special, and did not represent typical behavior at all.

The question of when this non-extremal condition is met turns out to be quite interesting. The answer to this question is tied to Lyapunov exponents. Recall that the Lyapunov exponent of a measure μ with respect to a map f is the average exponent of expansion with respect to μ :

$$\Lambda_\mu = \int \log|f'| d\mu$$

By convexity of the set of invariant measures, we know that the set of Lyapunov exponents of invariant measures supported on the Julia set \mathcal{J} is just an interval. Let us denote by Λ_{min} and Λ_{max} the infimum and supremum of these Lyapunov exponents, so that this interval is just $[\Lambda_{min}, \Lambda_{max}]$. For f geometrically finite (or probably even Collet-Eckmann — in progress!), the following result characterizes the non-extremal condition in terms of Lyapunov exponents:

Proposition 4. *Let f be geometrically finite, and let a be a typical recurrent point with respect to some f -invariant measure μ with Lyapunov exponent Λ_μ . Then a is extremal if and only if Λ_μ is an endpoint of $[\Lambda_{min}, \Lambda_{max}]$.*

This result naturally gives rise to two topologically exceptional f -invariant subsets of the Julia set: \mathcal{J}_{min} , the support of measures of minimal Lyapunov exponent, and \mathcal{J}_{max} , the support of the measures of maximal Lyapunov exponent. These sets are *topologically distinct* from the rest of \mathcal{J}_f because it is exactly their recurrent points that give rise to nondense horospheres in \mathcal{H}_f . It turns out that they are dynamically distinct as well. Typical points of most Lyapunov exponents exhibit arbitrarily large deviations from the average rate of expansion. The expansion of typical points of these extremal sets, on the other hand, is sharply bound to the expected expansion, as though the point were part of a periodic cycle. Thus these

sets, topologically distinguished by the horospheres of the Lyubich-Minsky lamination, are dynamically distinguished in the original Julia set. This is a particularly gratifying result, as the original overarching goal of this line of inquiry was to learn new dynamical results by studying the topology and geometry of these laminations.

FUTURE PLANS

The invariant sets \mathcal{J}_{min} and \mathcal{J}_{max} are potentially very important to the overall structure of the Julia set, and have not been studied before. It would be very interesting to understand what happens to these sets as the character of the map changes — as the map approaches an exceptional Euclidean map, for example, or as it undergoes parabolic bifurcation. Because these sets are topologically distinguished by the horosphere lamination, I conjecture that a good understanding of how these sets transform under such perturbations of f will come from a good understanding of the perturbations' effects on \mathcal{H}_f and on the horosphere lamination. The groundwork for this investigation has been laid - Kawahira [Kaw03] has described the structure of \mathcal{H}_f at certain parabolic parameters on the main cardioid of the Mandelbrot set, and Cabrera [Cab07] has provided a good description of \mathcal{H}_f for superattracting parameters of the Mandelbrot set. But the perturbative picture, and in particular the structure of the horosphere lamination under perturbation, are not yet understood at all. It should be mentioned that constructing numerical pictures of these sets for different parameters would be an excellent project for a young graduate student.

When f is not Collet-Eckmann, it is less clear precisely when a recurrent point is non-extremal. The real question in these cases has to do with the behavior of the extremal measures, and thus with the dynamics on the two invariant subsets. My immediate goal is to clear up these questions for as broad a class of maps as possible. This is important because the non-Collet-Eckmann maps are the maps that most need new tools for investigation. I weakly conjecture that the same correspondence between extremal Lyapunov exponents and non-dense horospheres will continue to hold, but I would not be shocked to discover a counterexample. Either result would be interesting. If the rule held, it would be a useful universal result, leading to a better understanding of the structure and dynamics of the Julia set. If a counterexample were found, it would be fascinating to understand how the non-extremal condition decouples from Lyapunov exponents, and these pathological cases might provide counterexamples to other conjectures. In either case, a more complete understanding of this relationship between horospherical density and Lyapunov exponent extremality would be essential to understanding the perturbative picture.

Just recently, Alexey Glutsyuk [Glu09] established that the quotiented horosphere lamination in \mathcal{M}_f for well-behaved (e.g. hyperbolic) maps is not merely minimal but is in fact uniquely ergodic. Again, though, this result provides a tool for studying only maps that are already well-understood. Another of my immediate goals is to extend this recent result to apply to less well-understood maps, again following the general yoga that what holds for a periodic point should also hold, in some modified, measurable sense, for a recurrent point.

I would also like to contrast the behavior of these extremal Lyapunov exponents and their supporting sets with similar structures in other systems. For instance, Contreras, Lopes, and Thieullen have some interesting results about Lyapunov-minimizing measures for $C^{1+\alpha}$ expanding maps of the circle ([CLT01]), and there has been some interesting work done on extremal Lyapunov measures for interval exchange maps. I want to compare those results with my results for rational maps, to see if there can be any cross-pollination of ideas, and whether any more general picture can be established.

These are all relatively immediate plans, following up on the results I have already proven. My long-term research goals are considerably more general. I am interested in all sorts of discrete dynamical systems, and I am looking forward to investigating new problems and new systems outside of functions of one complex variable. The rich structures that have been unearthed in holomorphic dynamics serve as a model for what it means to understand a family of chaotic dynamical systems, and I am intrigued by the challenge of exploring similar structures and arguments in fields that lack the powerful tools of 1-dimensional complex analysis. I am also interested in the application of dynamics to the sciences, and in the mathematical inspiration that can arise from real-world problems. Much of my initial interest in the field of dynamics was inspired by classes in Physics and Artificial Life that I took as an undergraduate at Caltech, and now that I understand the field of

dynamics as a mathematician rather than as a mathematically inclined undergraduate, I would like to revisit those promising connections. I am also intrigued by the fields of neurobiology and evolutionary biology; these fields are undergoing massive growth in terms of technical research, understanding, and available data, and dynamics is the correct mathematics with which to organize the results of this research into meaningful theory. Above all, though, I want to continue to explore the interplay between geometry, topology, and dynamics, and to find ways to comprehend and analyze dynamical systems not just as dynamics in a space, but as spaces themselves.

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