ON RICCI COEFFICIENTS OF NULL HYPERSURFACES WITH TIME FOLIATION IN EINSTEIN VACUUM SPACE-TIME: PART II

QIAN WANG

Abstract. This paper is the sequel of [9]. We prove several decomposition results which are crucial to the proof of the main result in [9].

1. Introduction

In [9] we considered the casual geometry of null cones with time foliation in a $(3+1)$-dimensional Einstein vacuum space-time and provided a series of estimates on the Ricci coefficients which are significant for proving the improved breakdown criterion in [8]. The derivation of those estimates in [9] is based on some decomposition results which were stated without proof. The purpose of this paper is to give the proofs of these important decomposition results.

We first review the setup in [9]. Let $(\mathbf{M}, \mathbf{g})$ be a $(3+1)$-dimensional globally hyperbolic Einstein vacuum space-time foliated by a time function t whose level sets are denoted by Σ_t. Let $\mathbf{D}$ and $\mathbf{R}$ denote respectively the covariant differentiation and the Riemannian curvature tensor of the space-time $(\mathbf{M}, \mathbf{g})$ and let $\mathbf{T}$ denote the future directed unit normal to Σ_t. We may introduce the lapse function n and the second fundamental form k by

$$n = (-\mathbf{g}(\mathbf{D}t, \mathbf{D}t))^{1/2} \qquad \text{and} \qquad k(X,Y) = -g(\mathbf{D}_X \mathbf{T}, Y),$$

where X, Y are vector field tangent to Σ_t. Let g be the induced Riemannian metric on Σ_t and let ∇ be the corresponding covariant differentiation. Under transport coordinates along the integral curves of $\mathbf{T}$, the metric $\mathbf{g}$ takes the form

$$\mathbf{g} = -n^2 dt^2 + g_{ij} dx^i dx^j.$$

We consider an outgoing null cone $\mathcal{H}$ contained in $(\mathbf{M}, \mathbf{g})$ with vertex p verifying $t(p) = 0$. Let S_t denote its intersections with Σ_t which are assumed to be diffeomorphic to the standard sphere $\mathbb{S}^2$, let N denote outward unit normal of S_t in Σ_t, and let γ be the induced metric on S_t whose covariant differentiation is denoted by $\nabla\!\!\!\!/$. We denote the Gauss curvature of S_t by K and define its radius by $r = \sqrt{(4\pi)^{-1}|S_t|}$, where $|S_t|$ denotes the area of S_t. Without loss of generality we may assume $\mathcal{H} = \cup_{0<t\le 1} S_t$. Along the null cone we introduce the null generator vector field L whose integral curves are null geodesics through p. We introduce on $\mathcal{H}$ the null lapse function

$$a^{-1} := -\mathbf{g}(L, \mathbf{T}).$$

By normalizing L we can assume $a(p) = 1$. Let s be the affine parameter of L which is chosen such that $s(p) = 0$ and $L(s) = 1$. We can define a conjugate null vector field $\underline{L}$ on $\mathcal{H}$ with $\mathbf{g}(L, \underline{L}) = -2$ and such that $\underline{L}$ is orthogonal to each S_t. In addition we can choose an orthonormal frame $(e_A)_{A=1,2}$ tangent to S_t such that $(e_A)_{A=1,2}$, $e_3 = \underline{L}$, $e_4 = L$ form a null frame. Relative to this null frame we define the null components of $\mathbf{R}$ as follows

$$\alpha_{AB} = \mathbf{R}(L, e_A, L, e_B), \qquad \beta_A = \frac{1}{2}\mathbf{R}(e_A, L, \underline{L}, L),$$
$$\rho = \frac{1}{4}\mathbf{R}(\underline{L}, L, \underline{L}, L), \qquad \sigma = \frac{1}{4}{}^\star\mathbf{R}(\underline{L}, L, \underline{L}, L),$$
$$\underline{\beta}_A = \frac{1}{2}\mathbf{R}(e_A, \underline{L}, \underline{L}, L), \qquad \underline{\alpha}_{AB} = \mathbf{R}(\underline{L}, e_A, \underline{L}, e_B).$$

Moreover, we can define the Ricci coefficients χ, $\underline{\chi}$, ζ, $\underline{\zeta}$, ϖ via the frame equations

$$\mathbf{D}_A L = \chi_{AB} e_B - \zeta_A L, \qquad \mathbf{D}_A \underline{L} = \underline{\chi}_{AB} e_B + \zeta_A \underline{L},$$
$$\mathbf{D}_L \underline{L} = 2\underline{\zeta}_A e_A, \qquad \mathbf{D}_{\underline{L}} L = 2\zeta_A e_A - 2\varpi L$$

and introduce the mass aspect functions μ by

$$\mu = -\frac{1}{2}\mathbf{D}_3 \mathrm{tr}\chi + \frac{a^2}{4}(\mathrm{tr}\chi)^2 - \varpi \mathrm{tr}\chi,$$

where $\mathrm{tr}\chi$ denotes the trace of χ, i.e. $\mathrm{tr}\chi = \gamma^{AB}\chi_{AB}$. We will also use $\hat{\chi}$ to denote the traceless part of χ. Similarly we can define $\mathrm{tr}\underline{\chi}$ and $\hat{\underline{\chi}}$.

Let $\mathrm{Tr}k = g^{ij}k_{ij}$ denote the trace of k. By setting $\lambda = -\mathrm{Tr}k/3$, the traceless part of k can be written as $\hat{k} := k + \lambda g$. Relative to the orthonormal frame $\{N, e_A, A = 1, 2\}$, we may decompose $\hat{k}$ along the null cone $\mathcal{H}$ by introducing the components

$$\eta_{AB} = \hat{k}_{AB}, \qquad \epsilon_A = \hat{k}_{AN}, \qquad \delta = \hat{k}_{NN}.$$

Let $\hat{\eta}_{AB}$ denote the traceless part of η. Since $\delta^{AB}\eta_{AB} = -\delta$, it is easy to see $\hat{\eta}_{AB} = \eta_{AB} + \frac{1}{2}\delta\gamma_{AB}$. By definition it is easy to check

$$\nu := -L(a) = -\nabla\!\!\!\!/\,_N \log n + \delta - \lambda,$$
$$\zeta_A = \nabla\!\!\!\!/\,_A \log a + \epsilon_A, \quad \underline{\zeta}_A = \nabla\!\!\!\!/\,_A \log n - \epsilon_A.$$

For ease of exposition, we will fix the following conventions

- $\pi\!\!\!/$ denotes the collection of $\hat{\eta}$, ϵ, δ, $\nabla_N \log n$, $\nabla\!\!\!\!/ \log n$, λ,
- $\iota := \mathrm{tr}\chi - \frac{2}{r}, V := \mathrm{tr}\chi - \frac{2}{s}, \quad \kappa := \mathrm{tr}\chi - (an)^{-1}\overline{an\mathrm{tr}\chi}$,
- A denotes the collection of $\hat{\chi}$, ζ, $\underline{\zeta}$, ν,
- $\underline{A}$ denotes the collection of A and $\hat{\underline{\chi}}$, $\nabla\!\!\!\!/ \log a$, $\pi\!\!\!/$,
- M denotes either $\nabla\!\!\!\!/ \mathrm{tr}\chi$ or the pair of quantities $(\mu, 0)$,
- R_0 denotes the collection of α, $\beta, \rho, \sigma, \underline{\beta}$,
- $S := S_t,\ \overset{\circ}{\gamma} := r^{-2}\gamma,\ \gamma^{(0)} := \gamma_{\mathbb{S}^2},\ \underline{K} := K - \frac{1}{r^2}$.

where, for a scalar function f, $\bar{f} := \frac{1}{|S_t|}\int_{S_t} f d\mu_\gamma$ denotes the average of f over S_t.

For any S_t-tangent tensor field F we introduce the norm

$$\mathcal{N}_1(F) = \|\nabla\!\!\!\!/\,_L F\|_{L^2(\mathcal{H})} + \|\nabla\!\!\!\!/ F\|_{L^2(\mathcal{H})} + \|r^{-1}F\|_{L^2(\mathcal{H})}.$$

We also introduce the curvature flux on $\mathcal{H}$ relative to t-foliation

$$\mathcal{R}(\mathcal{H})^2 = \int_0^1 \int_{S_t} an(|\alpha|^2 + |\beta|^2 + |\rho|^2 + |\sigma|^2 + |\underline{\beta}|^2) d\mu_\gamma dt.$$

The main result in [9], see [9, Theorem 1.1], provides a series of estimates on the Ricci coefficients under suitable assumptions on $\mathcal{R}(\mathcal{H})$ and $\mathcal{N}_1(\pi\!\!\!/)$ through a delicate bootstrap argument. The basic assumptions in [9] can be reformulated as follows.

Assumption 1.1. *(a) $C^{-1} < n < C$ on $\mathcal{H}$ for some positive constant C, and*

$$\mathcal{R}(\mathcal{H}) + \mathcal{N}_1(\pi\!\!\!/) \le \mathcal{R}_0, \ \text{ on } \mathcal{H} \tag{1.1}$$

with $\mathcal{R}_0$ sufficiently small.

(b) For some sufficiently small $0<\mathcal{R}_0<\Delta_0<1/2$ *there hold*

$$\|V\|_{L^\infty(\mathcal{H})}\le \Delta_0, \quad \|\hat{\chi},\nu,\zeta,\underline{\zeta}\|_{L_\omega^\infty L_t^2(\mathcal{H})}\le\Delta_0, \quad |a-1|\le\frac{1}{2}.$$

We remark that Assumption 1.1 (b) sevres as the bootstrap assumption in [9], and the proof of [9, Theorem 1.1] is carried out by showing that the estimates in Assumption 1.1 (b) can be improved. The argument in [9] is based on some decomposition results provided by [9, Propositions 6.2–6.3] without proof. We reformulate these results in the following theorem.

Theorem 1.1. *Let Assumption 1.1 hold.*

(1) *There holds the decomposition* $\nabla\!\!\!/(na\underline{\zeta})=\mathcal{D}_tP+E$*, with* P *and* E *appropriate* S *tangent tensor fields verifying*

$$\mathcal{N}_1(P)+\|E\|_{\mathcal{P}^0}\lesssim\Delta_0^2+\mathcal{R}_0.$$

(2) *Denote by* F *either* $\hat{\chi}$ *or* ζ*. If* $F=\hat{\chi}$*, let* $M=\nabla\!\!\!/ tr\chi$*; if* $F=\zeta$*, let* $M=(\mu,0)$*. There holds the decomposition*

$$\nabla\!\!\!/(naF)=\mathcal{D}_tP+E+\nabla\!\!\!/\mathcal{D}^{-1}(naM)$$

with P *and* E *appropriate tensor fields verifying*

$$\mathcal{N}_1(P)+\|E\|_{\mathcal{P}^0}\lesssim\Delta_0^2+\mathcal{R}_0, \quad \lim_{t\to0} r\|P\|_{L_x^\infty}=0,$$

(3) *Let* $\check{\rho}=\rho-\frac{1}{2}\hat{\chi}\cdot\underline{\hat{\chi}}$ *and* $\check{\sigma}=\sigma-\frac{1}{2}\hat{\chi}\wedge\underline{\hat{\chi}}$*. There holds the decomposition*

$$an(\check{\rho},\check{\sigma})=\mathcal{D}_tp'+e' \quad \textit{with} \quad \mathcal{N}_1(p')+\|e'\|_{\mathcal{P}^0}\lesssim\Delta_0^2+\mathcal{R}_0,$$

where $p'={}^\star\mathcal{D}_1^{-1}\underline{\beta}$*, and* e' *denotes a pair of functions on* $\mathcal{H}$*.*

In the statement of Theorem 1.1, the operator $\mathcal{D}_t$ denotes $\frac{d}{dt}$ along null geodesics initiating from p. The operator ${}^\star\mathcal{D}_1$ is defined by ${}^\star\mathcal{D}_1:(f,f^*)\to-\nabla\!\!\!/ f+(\nabla\!\!\!/ f^*)^\star$ for any pair of functions (f,f^*). It is the adjoint of the operator $\mathcal{D}_1: F\to(\text{div}F, \text{curl}\,F)$ defined for any S-tangent 1-form F. We will also need the operator $\mathcal{D}_2: F\to\text{div}F$ defined for any S tangent symmetric traceless 2-tensor field F and its adjoint operator ${}^\star\mathcal{D}_2$ which takes S-tangent 1-form F to the S-tangent symmetric traceless 2-tensor $-\frac{1}{2}\widehat{\mathcal{L}_F\gamma}$, where $\widehat{\mathcal{L}_F\gamma}_{ab}=\nabla\!\!\!/_AF_B+\nabla\!\!\!/_BF_A-(\text{div}F)\gamma_{AB}$. We may refer to [9, Proposition 3.4] for properties of these operators. For the definition of the norm $\|F\|_{\mathcal{P}^0}$ we refer to [9, Eq. (4.2)].

The main purpose of this paper is to prove Theorem 1.1. Since all the results in [9, Sections 2–6.1] were derived under Assumption 1.1 only, we may use those results freely. For the purpose of further reference, we recall the following results which were obtained in [9] (see (2.67), (SobM1), Proposition 2.7, Remark 2.1 and Proposition 3.2 in [9] for the derivation):

- Under Assumption 1.1, if $\mathcal{R}_0>0$ is sufficiently small, there holds

$$\|\underline{A}\|_{L_x^4L_t^\infty}+\|\underline{A}\|_{L^6}+\mathcal{N}_1(\underline{A})+\|r^{\frac{1}{2}}M\|_{L_x^2L_t^\infty}+\|M\|_{L^2}+\|R_0\|_{L^2}\lesssim\Delta_0^2+\mathcal{R}_0, \tag{1.2}$$

  where ι,κ are regarded as elements of $\underline{A}$. Since $a^m\underline{A}$ with $m\in\mathbb{N}$ verify all the estimates for $\underline{A}$ in (1.2), symbolically we can also denote them as $\underline{A}$ whenever there occurs no confusion.
- Under Assumption 1.1, if $\mathcal{R}_0>0$ is sufficiently small, then for all $1/2\le\alpha_0<1$ there hold

$$\|\Lambda^{-\alpha_0}\check{\rho}\|_{L_t^\infty L_x^2}\lesssim\Delta_0^2+\mathcal{R}_0, \tag{1.3}$$

$$\underline{K}_{\alpha_0}:=\|\Lambda^{-\alpha_0}(K-r^{-2})\|_{L_t^\infty L_x^2}\lesssim\Delta_0^2+\mathcal{R}_0, \tag{1.4}$$

  where Λ^a, $a\in\mathbb{R}$, denotes the fractional power of the operator $\Lambda^2=r^{-2}Id-\triangle_\gamma$ on S_t, see [9, Section 3].

2. Sketch of the proof for Theorem 1.1

We will adapt the approach developed in [2] for null hypersurfaces with geodesic foliation. In the case of time foliation, the proof of Theorem 1.1 is rather involved because of the occurrence of the nontrivial lapse factor "*an*". In order to make the proof more readable, we give a brief outline of the arguments. We will use the structure equations on the Ricci coefficients which have been collected in [9, Section 2].

We first use Eq.(2.6), Eq.(2.8) and Eq.(2.9) in [9] which give symbolically

$$\begin{aligned} \operatorname{div}(an\hat{\chi}) &= anM + an\beta + l.o.t, \\ \mathcal{D}_1(an\zeta) &= anM - an(\check{\rho}, -\check{\sigma}) + l.o.t. \end{aligned}$$

where and in the following we will use *l.o.t.* to represent lower order terms which are easy to handle. Thus we can write

$$\begin{aligned} \nabla\!\!\!/(an\hat{\chi}) &= \nabla\!\!\!/\mathcal{D}_2^{-1}(anM) + \nabla\!\!\!/\mathcal{D}_2^{-1}(an\beta) + \nabla\!\!\!/\mathcal{D}_2^{-1}(l.o.t), \\ \nabla\!\!\!/(an\zeta) &= \nabla\!\!\!/\mathcal{D}_1^{-1}(anM) + \nabla\!\!\!/\mathcal{D}_1^{-1}\left(an(\check{\rho}, -\check{\sigma})\right) + \nabla\!\!\!/\mathcal{D}_1^{-1}(l.o.t). \end{aligned}$$

The main idea is to prove that there hold the decompositions

$$\nabla\!\!\!/\mathcal{D}_2^{-1}(an\beta), \nabla\!\!\!/\mathcal{D}_1^{-1}\left(an(\check{\rho}, -\check{\sigma})\right) = \mathcal{D}_t P + E, \tag{2.1}$$

where P and E are appropriate S tangent tensor fields such that

$$\mathcal{N}_1(P) + \|E\|_{\mathcal{P}^0} \lesssim \Delta_0^2 + \mathcal{R}_0.$$

According to [9, Eqs.(2.11)-(2.13)], symbolically we have

$$\mathcal{D}_t(\check{\rho}, -\check{\sigma}) = \mathcal{D}_1(an\beta) + l.o.t, \quad \mathcal{D}_t\underline{\beta} = {}^\star\mathcal{D}_1(an\rho, an\sigma) + l.o.t \tag{2.2}$$

which enables us to write $an\beta$ and $an(\check{\rho}, -\check{\sigma})$ as

$$an\beta = \mathcal{D}_1^{-1}\mathcal{D}_t(\check{\rho}, -\check{\sigma}) + \mathcal{D}_1^{-1}(l.o.t), \quad an(\rho, \sigma) = {}^\star\mathcal{D}_1^{-1}\mathcal{D}_t\underline{\beta} + {}^\star\mathcal{D}_1^{-1}(l.o.t)$$

Hence, we can derive

$$\left\{ \begin{aligned} \nabla\!\!\!/(an\beta) &= \nabla\!\!\!/\mathcal{D}_1^{-1}\mathcal{D}_t(\check{\rho}, -\check{\sigma}) + \nabla\!\!\!/\mathcal{D}_1^{-1}(l.o.t), \\ \nabla\!\!\!/\left(an(\rho, \sigma)\right) &= \nabla\!\!\!/{}^\star\mathcal{D}_1^{-1}\mathcal{D}_t\underline{\beta} + \nabla\!\!\!/{}^\star\mathcal{D}_1^{-1}(l.o.t). \end{aligned} \right. \tag{2.3}$$

Symbolically, we have from (2.3) that

$$\left\{ \begin{aligned} \nabla\!\!\!/(an\beta) &= \mathcal{D}_t\nabla\!\!\!/\mathcal{D}_1^{-1}(\check{\rho}, -\check{\sigma}) + [\nabla\!\!\!/\mathcal{D}_1^{-1}, \mathcal{D}_t](\check{\rho}, -\check{\sigma}) + \nabla\!\!\!/\mathcal{D}_1^{-1}(l.o.t), \\ \nabla\!\!\!/\left(an(\rho, \sigma)\right) &= \mathcal{D}_t\nabla\!\!\!/{}^\star\mathcal{D}_1^{-1}\underline{\beta} + [\nabla\!\!\!/{}^\star\mathcal{D}_1^{-1}, \mathcal{D}_t]\underline{\beta} + \nabla\!\!\!/{}^\star\mathcal{D}_1^{-1}(l.o.t). \end{aligned} \right. \tag{2.4}$$

Thus, in order to obtain (2.1), we need to prove the following results based on the decomposition in (2.4):

(†1) Denote by P'' the two terms $\nabla\!\!\!/\mathcal{D}_1^{-1}(\check{\rho}, -\check{\sigma})$, $\nabla\!\!\!/{}^\star\mathcal{D}_1^{-1}\underline{\beta}$, there holds

$$\mathcal{N}_1(P'') \lesssim \Delta_0^2 + \mathcal{R}_0.$$

(†2) Denote by E'' the error terms $\nabla\!\!\!/\mathcal{D}_1^{-1}(l.o.t)$ and $\nabla\!\!\!/{}^\star\mathcal{D}_1^{-1}(l.o.t)$ in (2.4), there holds

$$\|E''\|_{\mathcal{P}^0} \lesssim \Delta_0^2 + \mathcal{R}_0.$$

(†3) Let $C(\check{R})$ denote either $[\nabla\!\!\!/\mathcal{D}_1^{-1}, \mathcal{D}_t](\check{\rho}, -\check{\sigma})$ or $[\nabla\!\!\!/{}^\star\mathcal{D}_1^{-1}, \mathcal{D}_t]\underline{\beta}$, there holds the decomposition

$$C(\check{R}) = \mathcal{D}_t P' + E'$$

where P' and E' are appropriate S tangent tensor fields such that

$$\mathcal{N}_1(P') + \|E'\|_{\mathcal{P}^0} \lesssim \Delta_0^2 + \mathcal{R}_0.$$

In view of these results, the decomposition (2.1) follows with $P = P' + P''$ and $E = E' + E''$. It is much easier to show that the $\mathcal{P}^0$ norms of the two terms $\nabla\!\!\!/ \mathcal{D}_2^{-1}(l.o.t)$ and $\nabla\!\!\!/ \mathcal{D}_1^{-1}(l.o.t)$ are bounded by $\Delta_0^2 + \mathcal{R}_0$. Thus Theorem 1.1 (2) is obtained.

Next we sketch the proof of Theorem 1.1 (1). We use the Hodge system given by Eq.(2.7) and Eq.(2.16) in [9] which symbolically can be written as

$$\left\{ \begin{array}{l} \text{div}\underline{\zeta} = -\check{\rho} + L(a\delta + 2a\lambda) + l.o.t \\ \text{curl}\,\underline{\zeta} = -\check{\sigma}. \end{array} \right. \tag{2.5}$$

Hence $\nabla\!\!\!/(an\underline{\zeta})$ can be expressed as

$$\begin{aligned} \nabla\!\!\!/(an\underline{\zeta}) &= \nabla\!\!\!/ \mathcal{D}_1^{-1}(\mathcal{D}_t(a\delta + 2a\lambda)) + \nabla\!\!\!/ \mathcal{D}_1^{-1}(an(\rho,\sigma)) + \nabla\!\!\!/ \mathcal{D}_1^{-1}(l.o.t) \\ &= \mathcal{D}_t \nabla\!\!\!/ \mathcal{D}_1^{-1}(a\delta + 2a\lambda) + [\nabla\!\!\!/ \mathcal{D}_1^{-1}, \mathcal{D}_t](a\delta + 2a\lambda) \\ &\quad + \nabla\!\!\!/ \mathcal{D}_1^{-1}(an(\rho,\sigma)) + \nabla\!\!\!/ \mathcal{D}_1^{-1}(l.o.t). \end{aligned} \tag{2.6}$$

Therefore, if we can prove the following decomposition results:

(‡1) There holds the decomposition $\nabla\!\!\!/ \mathcal{D}_1^{-1}(an(\rho,\sigma)) = \mathcal{D}_t\tilde{P} + \tilde{E}$ with $\tilde{P}$ and $\tilde{E}$ appropriate S tangent tensors such that

$$\mathcal{N}_1(\tilde{P}) + \|\tilde{E}\|_{\mathcal{P}^0} \lesssim \Delta_0^2 + \mathcal{R}_0.$$

(‡2) The $\mathcal{P}^0$ norm of the term $\nabla\!\!\!/ \mathcal{D}_1^{-1}(l.o.t)$ is bounded by $\Delta_0^2 + \mathcal{R}_0$. Moreover

$$\mathcal{N}_1\left(\nabla\!\!\!/ \mathcal{D}_1^{-1}(a\delta + 2a\lambda)\right) \lesssim \Delta_0^2 + \mathcal{R}_0.$$

(‡3) For the commutator in (2.6), there holds

$$[\nabla\!\!\!/ \mathcal{D}_1^{-1}, \mathcal{D}_t](a\delta + 2a\lambda) = \mathcal{D}_t p + e$$

with p and e appropriate S tangent tensors such that

$$\mathcal{N}_1(p) + \|e\|_{\mathcal{P}^0} \lesssim \Delta_0^2 + \mathcal{R}_0,$$

then Theorem 1.1 (1) is obtained with $P = \tilde{P} + p + \nabla\!\!\!/ \mathcal{D}_1^{-1}(a\delta + 2a\lambda)$ and $E = \tilde{E} + e$.

The main purpose of Section 3 is to derive the decomposition for commutators, i.e. (†3) and (‡3), see Proposition 3.2. The preliminary estimates established in the proof of (†3) and (‡3) will also imply Theorem 1.1 (3), (†1), (†2), (‡1), (‡2) and all related error estimates. In Section 4, we will rely on the results in Section 3 and the structure equations on Ricci coefficients to complete the proof of Theorem 1.1.

3. Error estimates

Recall a few elliptic estimates that have been proved in [9].

Proposition 3.1. *Let $\mathcal{D}$ be one of the operators $\mathcal{D}_1$, $\mathcal{D}_2$ and ${}^\star\mathcal{D}_1$. Then for $1 < p \le 2$ and any S-tangent tensor field F on $\mathcal{H}$ there hold*

$$\|\mathcal{D}^{-1}F\|_{L^2(S)} \lesssim \|r^{2-\frac{2}{p}}F\|_{L^p(S)}, \tag{3.1}$$

$$\|\mathcal{D}^{-1}F\|_{\mathcal{P}^0} \lesssim \|r^{2-\frac{2}{p}}F\|_{L_t^2 L_x^p}. \tag{3.2}$$

Theorem 3.1 (Hodge-elliptic $\mathcal{P}^\sigma$-estimate)**.** *Let $\mathcal{D}$ denote either $\mathcal{D}_1$, $\mathcal{D}_2$ or their adjoint operators ${}^\star\mathcal{D}_1$ and ${}^\star\mathcal{D}_2$. Then for any S-tangent tensor fields ξ and F satisfying $\mathcal{D}\xi = F$ and any $0 \le \sigma < 1/2$,*

$$\|\nabla\!\!\!/\xi\|_{\mathcal{P}^\sigma} \lesssim \|F\|_{\mathcal{P}^\sigma} + \Delta_0\|\mathcal{D}^{-1}F\|^q_{L_t^b L_x^2}\|F\|^{1-q}_{L_t^2 L_x^2}, \tag{3.3}$$

where $1/2 \le \alpha_0 < q < 1 - \sigma$ and $b > 4$.

We will employ the following conventions:

- $\check{R}$ denotes either the pair $(\check{\rho}, -\check{\sigma})$ or $\underline{\beta}$
- $\mathcal{D}^{-1}\check{R}$ denotes either $\mathcal{D}_1^{-1}(\check{\rho}, -\check{\sigma})$ or ${}^\star\mathcal{D}_1^{-1}\underline{\beta}$
- $\mathcal{D}^{-2}\check{R}$ denotes either $\mathcal{D}_2^{-1}\mathcal{D}_1^{-1}(\check{\rho}, -\check{\sigma})$ or $\mathcal{D}_1^{-1}{}^\star\mathcal{D}_1^{-1}\underline{\beta}$

- $\mathcal{D}^{-1}\mathcal{D}_t\check{R}$ denotes either ${}^\star\mathcal{D}_1^{-1}\mathcal{D}_t\underline{\beta}$ or $\mathcal{D}_1^{-1}\mathcal{D}_t(\check{\rho},-\check{\sigma})$
- $C_0(\check{R})$ denotes $[\mathcal{D}_t,\mathcal{D}_1^{-1}](\check{\rho},-\check{\sigma})$ or $[\mathcal{D}_t,{}^\star\mathcal{D}_1^{-1}]\underline{\beta}$
- $\mathcal{D}^{-2}\mathcal{D}_t\check{R}$ denotes $\mathcal{D}_2^{-1}\mathcal{D}_1^{-1}\mathcal{D}_t(\check{\rho},-\check{\sigma})$ or $\mathcal{D}_1^{-1}{}^\star\mathcal{D}_1^{-1}\mathcal{D}_t\underline{\beta}$
- $\mathcal{D}^{-1}C_0(\check{R})$ denotes $\mathcal{D}_2^{-1}[\mathcal{D}_t,\mathcal{D}_1^{-1}](\check{\rho},-\check{\sigma})$ or $\mathcal{D}_1^{-1}[\mathcal{D}_t,{}^\star\mathcal{D}_1^{-1}]\underline{\beta}$
- $\mathcal{F}$ denotes $\mathcal{D}^{-1}\check{R}$ or $(a\delta+2a\lambda)$. [1]
- $\mathcal{D}^{-1}\mathcal{F}$ denotes either $\mathcal{D}^{-2}\check{R}$ or $\mathcal{D}_1^{-1}(a\delta+2a\lambda)$.

We will frequently employ the Sobolev type inequalities (Sob), (SobM1), (SobM2) and (SobIn), which can be found in [9, Section 2.3].

3.1. **Commutation formula.** We will study error terms which arise from commuting $\mathcal{D}_t$ with Hodge operators. Regard ι also as an element of A, for any S tangent tensor field F the commutator $[\mathcal{D}_t,\nabla\!\!\!\!/\,]F$ and its good part can be written symbolically as (see [1, Lemma 13.1.2]) for the derivation)

$$[\mathcal{D}_t,\nabla\!\!\!\!/\,]F = an\left((A+\frac{1}{r})\nabla\!\!\!\!/\,F+(A+\frac{1}{r})\cdot A\cdot F+\beta\cdot F\right), \tag{3.4}$$

$$[\mathcal{D}_t,\nabla\!\!\!\!/\,]_gF := an\left((A+\frac{1}{r})\nabla\!\!\!\!/\,F+(A+\frac{1}{r})\cdot A\cdot F\right). \tag{3.5}$$

Due to the nontrivial factor "an" in (3.4), the treatment in [7, Section 6] has to be modified. We rewrite equations [9, (2.11)-(2.13)] as

$$L(\check{\rho},-\check{\sigma})=\mathcal{D}_1\beta+r^{-1}\check{R}+A\cdot\tilde{R}, \tag{3.6}$$

$$\nabla\!\!\!\!/_L\underline{\beta}={}^\star\mathcal{D}_1(\rho,\sigma)+r^{-1}\check{R}+A\cdot\tilde{R}, \tag{3.7}$$

where

$$\tilde{R}:=R_0+\nabla\!\!\!\!/\,A+A\cdot\underline{A}+r^{-1}\underline{A}.$$

Corresponding to (3.6) and (3.7), we introduce the error terms

$$Err:=\mathcal{D}_1^{-1}\mathcal{D}_t(\check{\rho},-\check{\sigma})-an\beta \quad\text{and}\quad \widetilde{Err}:={}^\star\mathcal{D}_1^{-1}\mathcal{D}_t\underline{\beta}-an(\rho,\sigma). \tag{3.8}$$

Denote by $\mathfrak{F}$ either Err or $\widetilde{Err}$. Symbolically, $\mathfrak{F}$ has the form

$$\mathfrak{F}=\mathcal{D}^{-1}\{an(r^{-1}\check{R}+A\cdot\tilde{R})\}.$$

We then infer from (3.6) and (3.7) the symbolic expression

$$\mathcal{D}^{-1}\mathcal{D}_t\check{R}=anR_0+\mathfrak{F}. \tag{3.9}$$

By using (3.2) with $p=4/3$, (3.1) with $p=2$, and (1.2) we infer that

$$\|\mathfrak{F}\|_{\mathcal{P}^0}\lesssim\|r^{\frac{1}{2}}A\cdot\tilde{R}\|_{L_t^2L_x^{\frac{4}{3}}}+\|\check{R}\|_{L^2}\lesssim\Delta_0^2+\mathcal{R}_0. \tag{3.10}$$

We will consider the commutators

$$C(\check{R})=(C_1(\check{R}),C_2(\check{R}),C_3(\check{R})) \tag{3.11}$$

given in [2, Definition 6.3] which, by using the above conventions, can be written symbolically as

$$C_1(\check{R})=\nabla\!\!\!\!/\,\mathcal{D}^{-1}[\mathcal{D}_t,\mathcal{D}^{-1}]\check{R},$$
$$C_2(\check{R})=\nabla\!\!\!\!/\,[\mathcal{D}_t,\mathcal{D}^{-1}]\mathcal{D}^{-1}\check{R},$$
$$C_3(\check{R})=[\mathcal{D}_t,\nabla\!\!\!\!/\,]\mathcal{D}^{-2}\check{R}.$$

The purpose of this section is to prove

[1] For simplicity, we use $(a\delta+2a\lambda)$ to denote the pair of quantities $(a\delta+2a\lambda,0)$.

Proposition 3.2. *There hold the following decomposition for commutators,*

$$C(\check{R}) = \mathcal{D}_t P + E,$$
$$[\mathcal{D}_t, \nabla\!\!\!/ \mathcal{D}_1^{-1}](a\delta + 2a\lambda) = \mathcal{D}_t P' + E',$$

where P, P' and E, E' are S tangent tensor fields verifying

$$\mathcal{N}_1(P) + \mathcal{N}_1(P') + \|E\|_{\mathcal{P}^0} + \|E'\|_{\mathcal{P}^0} \lesssim \Delta_0^2 + \mathcal{R}_0.$$
$$\lim_{t\to 0}\big(r\|P\|_{L^\infty(S)} + r\|P'\|_{L^\infty(S)}\big) = 0.$$

3.2. **Proof of Proposition 3.3: Part I.** In order to prove Proposition 3.2, let us consider the structure of commutators. We first use (3.4) to write

$$C_2(\check{R}), \nabla\!\!\!/[\mathcal{D}_t, \mathcal{D}_1^{-1}](a\delta + 2a\lambda) = \nabla\!\!\!/[\mathcal{D}_t, \mathcal{D}^{-1}]_g \mathcal{F} + \nabla\!\!\!/\mathcal{D}^{-1}(an\beta \cdot \mathcal{D}^{-1}\mathcal{F}), \tag{3.12}$$

$$C_3(\check{R}), [\mathcal{D}_t, \nabla\!\!\!/]\mathcal{D}_1^{-1}(a\delta + 2a\lambda) = [\mathcal{D}_t, \nabla\!\!\!/]_g \mathcal{D}^{-1}\mathcal{F} + an\beta \cdot \mathcal{D}^{-1}\mathcal{F}, \tag{3.13}$$

where, for any S tangent tensor F, we set

$$[\mathcal{D}_t, \mathcal{D}^{-1}]_g F := \mathcal{D}^{-1}\big(an(A + r^{-1}) \cdot \nabla\!\!\!/\mathcal{D}^{-1}F + an(A + r^{-1}) \cdot A \cdot \mathcal{D}^{-1}F\big). \tag{3.14}$$

The terms $\nabla\!\!\!/[\mathcal{D}_t, \mathcal{D}^{-1}]_g\mathcal{F}$ and $[\mathcal{D}_t, \nabla\!\!\!/]_g\mathcal{D}^{-1}\mathcal{F}$ are the "good" parts in the corresponding commutators and will be proved to be $\mathcal{P}^0$ bounded, see (3.17)-(3.20) below. The terms $\nabla\!\!\!/\mathcal{D}^{-1}(an\beta \cdot \mathcal{D}^{-1}\mathcal{F})$ and $an\beta \cdot \mathcal{D}^{-1}\mathcal{F}$ in (3.12) and (3.13) can not be bounded in $\mathcal{P}^0$ norm and will be further decomposed in Section 3.7 with the help of the results established in Sections 3.4–3.6.

Proposition 3.3. *For the error terms $C_0(\check{R})$, $C_1(\check{R})$, $C_2(\check{R})$, $C_3(\check{R})$, etc, there hold*

$$\|C_0(\check{R})\|_{\mathcal{P}^0} \lesssim \Delta_0^2 + \mathcal{R}_0, \tag{3.15}$$

$$\|C_1(\check{R})\|_{\mathcal{P}^0} \lesssim \Delta_0^2 + \mathcal{R}_0, \tag{3.16}$$

$$C_2(\check{R}) = \nabla\!\!\!/\mathcal{D}^{-1}(an\beta \cdot \mathcal{D}^{-2}\check{R}) + err, \tag{3.17}$$

$$C_3(\check{R}) = an\beta \cdot \mathcal{D}^{-2}\check{R} + err \tag{3.18}$$

$$\nabla\!\!\!/[\mathcal{D}_t, \mathcal{D}_1^{-1}](a\delta + 2a\lambda) = \nabla\!\!\!/\mathcal{D}^{-1}(an\beta \cdot \mathcal{D}^{-1}(a\delta + 2a\lambda)) + err \tag{3.19}$$

$$[\mathcal{D}_t, \nabla\!\!\!/]\mathcal{D}_1^{-1}(a\delta + 2a\lambda) = an\beta \cdot \mathcal{D}^{-1}(a\delta + 2a\lambda) + err \tag{3.20}$$

with

$$\|err\|_{\mathcal{P}^0} \lesssim \Delta_0^2 + \mathcal{R}_0.$$

We will prove (3.16) in Section 3.4. In this subsection we will establish (3.15) and (3.17)–(3.20) according to the following two steps.

Step 1. We first prove (3.15). In view of (3.4),

$$C_0(\check{R}) = \mathcal{D}^{-1}(an\{(A + r^{-1})(\nabla\!\!\!/\mathcal{D}^{-1}\check{R}) + (A + r^{-1}) \cdot A \cdot \mathcal{D}^{-1}\check{R} + \beta \cdot \mathcal{D}^{-1}\check{R}\}). \tag{3.21}$$

Using (3.2) with $p = 4/3$, [9, Proposition 3.4], also with the help of (1.2) and Hölder inequality, we can estimate the various terms in (3.21) to get

$$\|C_0(\check{R})\|_{\mathcal{P}^0} \lesssim \Delta_0^2 + \mathcal{R}_0 + \Delta_0 \cdot \mathcal{N}_1(\mathcal{D}^{-1}\check{R}). \tag{3.22}$$

By the definition of $\mathcal{N}_1(\mathcal{D}^{-1}\check{R})$ and [9, Proposition 3.4] it follows that

$$\mathcal{N}_1(\mathcal{D}^{-1}\check{R}) \lesssim \mathcal{R}_0 + \Delta_0^2 + \|\mathcal{D}^{-1}\mathcal{D}_t\check{R}\|_{L_t^2 L_x^2} + \|C_0(\check{R})\|_{L_t^2 L_x^2}.$$

While it follows from (3.9), (3.10) and (1.2) that

$$\|\mathcal{D}^{-1}\mathcal{D}_t\check{R}\|_{L^2} \lesssim \|\mathfrak{F}\|_{\mathcal{P}^0} + \|anR_0\|_{L^2} \lesssim \Delta_0^2 + \mathcal{R}_0.$$

Combining the above three inequalities and using the smallness of Δ_0 we obtain (3.15).

In the above proof, together with [9, Lemma 3.1] and (1.2) we have also verified the following

Proposition 3.4.

$$\|\mathcal{D}^{-1}\mathcal{D}_t\check{R}\|_{L^2} \lesssim \mathcal{R}_0 + \Delta_0^2, \tag{3.23}$$

$$\|[\mathcal{D}_t, \mathcal{D}^{-1}]\check{R}\|_{L^2} \lesssim \mathcal{R}_0 + \Delta_0^2, \tag{3.24}$$

$$\mathcal{N}_1(\mathcal{F}) \lesssim \mathcal{R}_0 + \Delta_0^2, \tag{3.25}$$

$$\mathcal{N}_1(\nabla\!\!\!/ \mathcal{D}^{-1}\mathcal{F}) \lesssim \mathcal{R}_0 + \Delta_0^2, \quad \mathcal{N}_2(\mathcal{D}^{-1}\mathcal{F}) \lesssim \mathcal{R}_0 + \Delta_0^2, \tag{3.26}$$

where $\mathcal{F}$ *denotes either* $\mathcal{D}^{-1}\check{R}$ *or* $(a\delta + 2a\lambda)$.

Step 2. We will prove (3.17)-(3.20). Let us first establish the following

Lemma 3.1. *Denote by* $\mathcal{D}^{-1}$ *one of the operators among* $\mathcal{D}_1^{-1}$, $\mathcal{D}_2^{-1}$ *and* ${}^\star\mathcal{D}_1^{-1}$. *For any* S_t *tangent tensor* H *and* $b > 4$ *there holds*

$$\|r^{-1-\frac{1}{b}}\mathcal{D}^{-1}(an\nabla\!\!\!/ \mathcal{D}^{-1}H)\|_{L_t^b L_x^2} \lesssim \mathcal{N}_1(\mathcal{D}^{-1}H)\|\zeta + \underline{\zeta}\|_{L_t^\infty L_x^4} + \mathcal{N}_1(r^{-\frac{1}{2}}\mathcal{D}^{-1}H). \tag{3.27}$$

Proof. Using [9, Proposition 3.4] and Proposition 3.1,

$$\begin{aligned}
&\|r^{-1-\frac{1}{b}}\mathcal{D}^{-1}(an\nabla\!\!\!/ \mathcal{D}^{-1}H)\|_{L_t^b L_x^2} \\
&\quad\lesssim \|r^{-1-\frac{1}{b}}\left(|\mathcal{D}^{-1}(\nabla\!\!\!/(an)\mathcal{D}^{-1}H)| + |\mathcal{D}^{-1}\nabla\!\!\!/(an\mathcal{D}^{-1}H)|\right)\|_{L_t^b L_x^2} \\
&\quad\lesssim \|r^{-\frac{1}{2}-\frac{1}{b}}\nabla\!\!\!/(an)\mathcal{D}^{-1}H\|_{L_t^b L_x^{4/3}} + \|r^{-1-\frac{1}{b}}\mathcal{D}^{-1}H\|_{L_t^b L_x^2}.
\end{aligned}$$

By using (SobIn), we obtain

$$\begin{aligned}
\|r^{-\frac{1}{2}-\frac{1}{b}}\nabla\!\!\!/(an)\mathcal{D}^{-1}H\|_{L_t^b L_x^{4/3}} &\lesssim \|r^{-\frac{1}{2}-\frac{1}{b}}\mathcal{D}^{-1}H\|_{L_t^b L_x^2}\|\nabla\!\!\!/(an)\|_{L_t^\infty L_x^4} \\
&\lesssim \mathcal{N}_1(\mathcal{D}^{-1}H)\|\zeta + \underline{\zeta}\|_{L_t^\infty L_x^4}
\end{aligned}$$

and $\|r^{-1-\frac{1}{b}}\mathcal{D}^{-1}H\|_{L_t^b L_x^2} \lesssim \mathcal{N}_1(r^{-1/2}\mathcal{D}^{-1}H)$. Thus (3.27) follows. □

The proof of (3.17)-(3.20) can be completed by using (3.25) combined with the following result.

Lemma 3.2. *Denote by* $\mathcal{D}^{-1}$ *either* $\mathcal{D}_1^{-1}$ *or* $\mathcal{D}_2^{-1}$. *For appropriate* S*-tangent tensor field* F *there hold*

$$\|[\mathcal{D}_t, \nabla\!\!\!/]_g\mathcal{D}^{-1}F\|_{\mathcal{P}^0} + \|\nabla\!\!\!/[\mathcal{D}^{-1}, \mathcal{D}_t]_g F\|_{\mathcal{P}^0} \lesssim \mathcal{N}_1(F), \tag{3.28}$$

$$\|[\mathcal{D}_t, \nabla\!\!\!/ \mathcal{D}^{-1}]_g F\|_{\mathcal{P}^0} \lesssim \mathcal{N}_1(F). \tag{3.29}$$

Proof. Observe that

$$\|[\mathcal{D}_t, \nabla\!\!\!/ \mathcal{D}^{-1}]_g F\|_{\mathcal{P}^0} \lesssim \|[\mathcal{D}_t, \nabla\!\!\!/]_g\mathcal{D}^{-1}F\|_{\mathcal{P}^0} + \|\nabla\!\!\!/[\mathcal{D}^{-1}, \mathcal{D}_t]_g F\|_{\mathcal{P}^0},$$

it suffices to prove (3.28) only.

In view of (3.4) and [9, Eq.(4.6)] we first derive for $4 < b < \infty$ that

$$\|[\mathcal{D}_t, \nabla\!\!\!/]_g\mathcal{D}^{-1}F\|_{\mathcal{P}^0} \lesssim \mathcal{N}_1(\nabla\!\!\!/ \mathcal{D}^{-1}F)\left(\|r^{\frac{1}{2}}\nabla\!\!\!/(anA)\|_{L_t^2 L_x^2} + \|r^{-\frac{1}{b}}anA\|_{L_t^b L_x^2}\right)$$

$$+ \mathcal{N}_2(\mathcal{D}^{-1}F)\big(\|anA\cdot A\|_{\mathcal{P}^0} + \|r^{-\frac{1}{2}}anA\|_{\mathcal{P}^0}\big) \tag{3.30}$$

$$+ \|r^{-1}an\nabla\!\!\!/ \mathcal{D}^{-1}F\|_{\mathcal{P}^0}. \tag{3.31}$$

By [9, Eq.(4.10)] and (1.2), we obtain

$$\|r^{-1}anA\|_{\mathcal{P}^0} \lesssim \Delta_0^2 + \mathcal{R}_0, \quad \|anA\cdot A\|_{\mathcal{P}^0} \lesssim \Delta_0\mathcal{N}_1(A) \lesssim \Delta_0^2 + \mathcal{R}_0.$$

Then the term in (3.30) can be bounded by $(\Delta_0^2 + \mathcal{R}_0)\mathcal{N}_2(\mathcal{D}^{-1}F)$.

We next consider the term in (3.31). In view of [9, Eq.(4.4)] and Theorem 3.1,

$$\|r^{-1}an\nabla\!\!\!/ \mathcal{D}^{-1}F\|_{\mathcal{P}^0} \lesssim \|r^{-1}F\|_{\mathcal{P}^0} + \Delta_0\|r^{-1}\mathcal{D}^{-1}F\|_{L_t^b L_x^2}^q\|r^{-1}F\|_{L^2(\mathcal{H})}^{1-q}.$$

From Proposition 3.1 and (SobM1) it follows that

$$\|r^{-1}\mathcal{D}^{-1}F\|_{L_t^b L_x^2} \lesssim \|F\|_{L_t^b L_x^2} \lesssim \mathcal{N}_1(F),$$

By using [9, Eq.(4.12)], we deduce

$$\|r^{-1}an\nabla\!\!\!/\mathcal{D}^{-1}F\|_{\mathcal{P}^0} \lesssim \mathcal{N}_1(F) + \Delta_0\mathcal{N}_1(F).$$

By (1.2), it is easy to check

$$\|r^{\frac{1}{2}}\nabla\!\!\!/(anA)\|_{L_t^2 L_x^2} + \|r^{-\frac{1}{b}}anA\|_{L_t^b L_x^2} \lesssim \Delta_0^2 + \mathcal{R}_0.$$

Consequently, we conclude that

$$\|[\mathcal{D}_t, \nabla\!\!\!/]_g\mathcal{D}^{-1}F\|_{\mathcal{P}^0} \lesssim (\Delta_0^2 + \mathcal{R}_0)(\mathcal{N}_1(\nabla\!\!\!/\mathcal{D}^{-1}F) + \mathcal{N}_2(\mathcal{D}^{-1}F)) + \mathcal{N}_1(F). \tag{3.32}$$

We then infer from (3.32) and [9, Lemma 3.1] that

$$\|[\mathcal{D}_t, \nabla\!\!\!/]_g\mathcal{D}^{-1}F\|_{\mathcal{P}^0} \lesssim \mathcal{N}_1(F). \tag{3.33}$$

Next, we prove for S-tangent tensor fields F on $\mathcal{H}$ the inequality

$$\|[\mathcal{D}_t, \mathcal{D}^{-1}]_g F\|_{L_t^b L_x^2} \lesssim \mathcal{N}_1(F) \quad \text{with } 4 < b < \infty. \tag{3.34}$$

Indeed, by using Proposition 3.1 with $p = 4/3$, (SobM1), (1.2), (3.27) and [9, Lemma 3.1] we can derive with the help of (3.14) that

$$\begin{aligned}
\|[\mathcal{D}_t, \mathcal{D}^{-1}]_g F\|_{L_t^b L_x^2} &\lesssim \|r^{1/2}A\cdot\nabla\!\!\!/\mathcal{D}^{-1}F\|_{L_t^b L_x^{4/3}} + \|r^{1/2}A\cdot A\cdot\mathcal{D}^{-1}F\|_{L_t^b L_x^{4/3}}\\
&\quad + \|r^{-1}\mathcal{D}^{-1}(an\nabla\!\!\!/\mathcal{D}^{-1}F)\|_{L_t^b L_x^2} + \|r^{-1/2}A\cdot\mathcal{D}^{-1}F\|_{L_t^b L_x^{4/3}}\\
&\lesssim \|A\|_{L_t^b L_x^2}\|\nabla\!\!\!/\mathcal{D}^{-1}F\|_{L_t^\infty L_x^4}\\
&\quad + \|\mathcal{D}^{-1}F\|_{L_t^\infty L_x^4}\times(\|A\cdot A\|_{L_t^b L_x^2} + \|\|r^{-\frac{1}{2}}A\|_{L_t^b L_x^2})\\
&\quad + \mathcal{N}_1(r^{-1/2}\mathcal{D}^{-1}F) + \Delta_0\mathcal{N}_1(\mathcal{D}^{-1}F)\\
&\lesssim \mathcal{N}_1(\nabla\!\!\!/\mathcal{D}^{-1}F) + \mathcal{N}_2(\mathcal{D}^{-1}F) \lesssim \mathcal{N}_1(F).
\end{aligned}$$

The combination of (3.34), (3.33) and (3.3) gives

$$\begin{aligned}
\|\nabla\!\!\!/[\mathcal{D}^{-1}, \mathcal{D}_t]_g F\|_{\mathcal{P}^0} &\lesssim \|[\mathcal{D}_t, \nabla\!\!\!/]_g\mathcal{D}^{-1}F\|_{\mathcal{P}^0} + \Delta_0\|[\mathcal{D}_t, \mathcal{D}^{-1}]_g F\|^q_{L_t^b L_x^2}\cdot\|[\mathcal{D}_t, \nabla\!\!\!/]_g\mathcal{D}^{-1}F\|^{1-q}_{L^2}\\
&\lesssim \mathcal{N}_1(F).
\end{aligned}$$

where in the above inequalities, $b > 4$ and $1/2 < q < 1$. □

3.3. **Proof of Proposition 3.3: Part II.** We will complete the proof of Proposition 3.3 by studying error type terms in Proposition 3.6. Let us first prove a useful result, Proposition 3.5.

We start with the following result whose proof will be given in the Appendix, where P_k denote the geometric Littlewood-Paley (GLP) projections and $F_k := P_k F$. For the definition and basic properties of GLP projections P_k we refer to [9, Section 4] and [3, 4].

Lemma 3.3. *For any S-tangent tensor field F and any $2 \le q \le \infty$, we have for $k > 0$ the inequalities*

$$\|r^{-\frac{1}{2}-\frac{1}{q}}P_k F\|_{L_t^q L_x^2} \lesssim 2^{-\frac{1}{2}k-\frac{1}{q}k}\mathcal{N}_1(F), \tag{3.35}$$

$$\|r^{-\frac{1}{q}}F_k\|_{L_t^q L_x^4} \lesssim 2^{-\frac{k}{q}}\mathcal{N}_1(F). \tag{3.36}$$

Proposition 3.5. *Let $\mathcal{D}^{-1}$ denote either $\mathcal{D}_1^{-1}$ or ${}^\star\mathcal{D}_1^{-1}$. For any S-tangent tensor fields F and G on $\mathcal{H}$ there holds*

$$\|r^{-\frac{1}{b}}\mathcal{D}^{-1}(anF\cdot\nabla\!\!\!/G)\|_{L_t^b L_x^2} \lesssim \mathcal{N}_1(F)\mathcal{N}_1(G), \quad \textit{with } 4 < b < \infty.$$

Proof. This result is analogous to [7, Proposition 6.5] which was proved with the help of the GLP decomposition $\sum_k P_k^2 + U(\infty) = Id$ and the inequalities in [7, Lemma 5.1, (6.25), (6.26)]. In view of Lemma 5.1 in the Appendix, all these key inequalities are still available for the current situation. Thus, according to the argument in [7, Page 310], we need only to establish for $\mathcal{I}_{lnm} := \|r^{-\frac{1}{b}} P_l^2 \mathcal{D}^{-1}(anF_n \cdot \nabla\!\!\!/ G_m)\|_{L_t^b L_x^2}$ the estimate [2]

$$\sum_{l,m,n>0} \mathcal{I}_{lnm} \lesssim \mathcal{N}_1(F)\mathcal{N}_1(G).$$

We can achieve this by repeating the argument in [7, Pages 310–311] except for the case $l < n < m$.

When $l < n < m$, we write

$$anF_n \cdot \nabla\!\!\!/ G_m = \nabla\!\!\!/(anF_n \cdot G_m) - \nabla\!\!\!/(an)F_n \cdot G_m - an\nabla\!\!\!/ F_n \cdot G_m,$$

thus we need to consider the three terms

$$\begin{aligned}
\mathcal{I}^1_{lnm} &:= \|r^{-\frac{1}{b}} P_l^2 \mathcal{D}^{-1}(an\nabla\!\!\!/ F_n \cdot G_m)\|_{L_t^b L_x^2},\\
\mathcal{I}^2_{lnm} &:= \|r^{-\frac{1}{b}} P_l^2 \mathcal{D}^{-1}\nabla\!\!\!/(anF_n \cdot G_m)\|_{L_t^b L_x^2},\\
\mathcal{I}^3_{lnm} &:= \|r^{-\frac{1}{b}} P_l^2 \mathcal{D}^{-1}(\nabla\!\!\!/(an)F_n \cdot G_m)\|_{L_t^b L_x^2}.
\end{aligned}$$

Using $C^{-1} < an < C$ and following the same procedure in [7], we can get

$$\sum_{0<l<n<m} (\mathcal{I}^1_{lnm} + \mathcal{I}^2_{lnm}) \lesssim \mathcal{N}_1(F)\mathcal{N}_1(G). \tag{3.37}$$

For the term $\mathcal{I}^3_{lnm}$, by using [9, Lemma 4.3] with $p = 4/3$ and [9, Proposition 4.1(1)] together with (SobM1), and (3.36), we obtain

$$\begin{aligned}
\mathcal{I}^3_{lnm} &= \|r^{-\frac{1}{b}} P_l^2 \mathcal{D}^{-1}(an(\zeta + \underline{\zeta})F_n \cdot G_m)\|_{L_t^b L_x^2(\mathcal{H})}\\
&\lesssim 2^{-\frac{l}{2}} \|r^{\frac{1}{2}-\frac{1}{b}}(\zeta + \underline{\zeta})anF_n \cdot G_m\|_{L_t^b L_x^{4/3}}\\
&\lesssim 2^{-\frac{l}{2}} \|r^{\frac{1}{2}-\frac{1}{b}} F_n \cdot G_m\|_{L_t^b L_x^2} \|A\|_{L_t^\infty L_x^4}\\
&\lesssim 2^{-\frac{l}{2}} \|r^{-\frac{1}{b}+\frac{1}{2}} G_m\|_{L_t^b L_x^4} \|F_n\|_{L_t^\infty L_x^4}\\
&\lesssim 2^{-\frac{l}{2}-\frac{m}{b}} \mathcal{N}_1(G) \cdot \mathcal{N}_1(F),
\end{aligned}$$

Therefore $\sum_{0<l<n<m} \mathcal{I}^3_{lnm} \lesssim \mathcal{N}_1(G)\mathcal{N}_1(F)$. The proof is complete. □

Let $\mathcal{D}$ be one of the operators $\mathcal{D}_1, {}^\star\mathcal{D}_1$ or $\mathcal{D}_2$. In the following result, we use Proposition 3.5 to estimate the error type terms.

Proposition 3.6. *For S-tangent tensors G on $\mathcal{H}$ verifying $\mathcal{N}_1(G) < \infty$, set*

$$\begin{aligned}
\mathcal{E}_1(G) &:= r^{-1}\mathcal{D}^{-1}(anA \cdot G) \ \ or \ \ \mathcal{D}^{-1}(anA \cdot A \cdot G),\\
\mathcal{E}_2(G) &:= \mathcal{D}^{-1}(an\nabla\!\!\!/ A \cdot G) \ \ or \ \ \mathcal{D}^{-1}(an\nabla\!\!\!/ G \cdot A).
\end{aligned}$$

Then for $4 < b < \infty$ there holds

$$\|r^{-\frac{1}{b}}\mathcal{E}_1(G)\|_{L_t^b L_x^2} + \|r^{-\frac{1}{b}}\mathcal{E}_2(G)\|_{L_t^b L_x^2} \lesssim (\Delta_0^2 + \mathcal{R}_0)\mathcal{N}_1(G).$$

Proof. Using Proposition 3.1 with $p = 4/3$, (SobM1) and (1.2), we obtain

$$\begin{aligned}
\|r^{-\frac{1}{b}}\mathcal{D}^{-1}(anA \cdot A \cdot G)\|_{L_t^b L_x^2} &\lesssim \|r^{\frac{1}{2}-\frac{1}{b}} A \cdot A \cdot G\|_{L_t^b L_x^{4/3}} \lesssim \|A\|^2_{L_t^\infty L_x^4} \|G\|_{L_t^b L_x^4}\\
&\lesssim (\Delta_0^2 + \mathcal{R}_0)^2 \mathcal{N}_1(G).
\end{aligned}$$

[2]Since the estimates for low frequency term can be treated similarly, we will omit the details.

Similarly, using also (SobIn) we have

$$\begin{aligned}\|r^{-1-\frac{1}{b}}\mathcal{D}^{-1}(an A\cdot G)\|_{L_t^bL_x^2} &\lesssim \|r^{-\frac{1}{b}}A\cdot G\|_{L_t^bL_x^2}\lesssim \|r^{-\frac{1}{b}}A\|_{L_t^bL_x^4}\|G\|_{L_t^\infty L_x^4}\\ &\lesssim (\Delta_0^2+\mathcal{R}_0)\mathcal{N}_1(G).\end{aligned}$$

For $\mathcal{E}_2(G)$, we infer from Proposition 3.5 and (1.2) that

$$\|r^{-\frac{1}{b}}\mathcal{E}_2(G)\|_{L_t^bL_x^2}\lesssim \mathcal{N}_1(G)\mathcal{N}_1(A)\lesssim (\Delta_0^2+\mathcal{R}_0)\mathcal{N}_1(G).$$

We thus obtain the desired estimates. □

By analyzing the expression of β and $C_0(F):=[\mathcal{D}_t,\mathcal{D}^{-1}]F$, we have

Corollary 3.1. *For any S-tangent tensor field F and $4<b<\infty$ there hold*

$$\|r^{-\frac{1}{b}}\mathcal{D}^{-1}(an\beta\cdot F)\|_{L_t^bL_x^2}\lesssim \mathcal{N}_1(F)(\Delta_0^2+\mathcal{R}_0), \tag{3.38}$$

$$\|r^{-\frac{1}{b}}C_0(F)\|_{L_t^bL_x^2}\lesssim \mathcal{N}_1(r^{-\frac{1}{2}}\mathcal{D}^{-1}F). \tag{3.39}$$

Proof. Using the Codazzi equation (see [9, Eq.(2.6)])

$$an\beta = an\nabla\!\!\!/ A + an(A\cdot A + r^{-1}A),$$

we infer

$$\mathcal{D}^{-1}(an\beta\cdot F)=\mathcal{E}_1(F)+\mathcal{E}_2(F).$$

Whence (3.38) follows from Proposition 3.6.

Similarly, using (3.4) we can write

$$C_0(F)=\mathcal{E}_1(\mathcal{D}^{-1}F)+\mathcal{E}_2(\mathcal{D}^{-1}F)+r^{-1}\mathcal{D}^{-1}(an\nabla\!\!\!/\mathcal{D}^{-1}F).$$

For the last term, using (3.27) with $H=F$, we infer

$$\|r^{-1-\frac{1}{b}}\mathcal{D}^{-1}(an\nabla\!\!\!/\mathcal{D}^{-1}F)\|_{L_t^bL_x^2}\lesssim \mathcal{N}_1(r^{-\frac{1}{2}}\mathcal{D}^{-1}F)(\Delta_0^2+\mathcal{R}_0+1).$$

The desired estimate then follows from Proposition 3.6. □

Proof of (3.16) in Proposition 3.3. Let $\mathcal{D}^{-1}$ denote either $\mathcal{D}_2^{-1}$ or $\mathcal{D}_1^{-1}$. From Proposition 3.1, (3.39) and (3.25) we derive for $4<b<\infty$ that

$$\|r^{-\frac{1}{b}}\mathcal{D}^{-1}C_0(\check{R})\|_{L_t^bL_x^2}\lesssim \|r^{1-\frac{1}{b}}C_0(\check{R})\|_{L_t^bL_x^2}\lesssim \mathcal{N}_1(r^{\frac{1}{2}}\mathcal{D}^{-1}\check{R})\lesssim \Delta_0^2+\mathcal{R}_0. \tag{3.40}$$

Observe that $C_1(\check{R})$ can be written symbolically in the form $C_1(\check{R})=\nabla\!\!\!/\mathcal{D}^{-1}C_0(\check{R})$. By using the Hodge-elliptic estimate (3.3) with $1/2<q<1$ and $4<b<\infty$, (3.15), (3.24) and (3.40), we obtain

$$\begin{aligned}\|C_1(\check{R})\|_{\mathcal{P}^0} &\lesssim \|C_0(\check{R})\|_{\mathcal{P}^0}+\Delta_0\|\mathcal{D}^{-1}C_0(\check{R})\|^q_{L_t^bL_x^2}\|C_0(\check{R})\|^{1-q}_{L^2(\mathcal{H})}\\ &\lesssim \Delta_0^2+\mathcal{R}_0+\Delta_0\|\mathcal{D}^{-1}C_0(\check{R})\|^q_{L_t^bL_x^2}(\Delta_0^2+\mathcal{R}_0)^{1-q}\lesssim \Delta_0^2+\mathcal{R}_0.\end{aligned}$$

This is the desired estimate. □

3.4. **A preliminary estimate for** (ρ,σ)**.** Given a scalar function f, we recall that $\bar{f}$ represents its average over S_t. The following result gives the estimates on $\bar{\rho}$, $\bar{\sigma}$, $\bar{\check{\rho}}$ and $\bar{\check{\sigma}}$.

Lemma 3.4.

$$|r^{\frac{3}{2}}\bar{\rho}|+|r^{\frac{3}{2}}\bar{\sigma}|\lesssim \Delta_0^2+\mathcal{R}_0, \tag{3.41}$$

$$|r^{\frac{3}{2}}\bar{\check{\rho}}|+|r^{\frac{3}{2}}\bar{\check{\sigma}}|\lesssim \Delta_0^2+\mathcal{R}_0. \tag{3.42}$$

Proof. By [2, Eq. (41)], i.e.

$$\frac{d}{ds}\rho + \frac{3}{2}\mathrm{tr}\chi\rho = F \tag{3.43}$$

where $F = \mathrm{div}\beta - \frac{1}{2}\underline{\hat{\chi}}\cdot\alpha + (\zeta + 2\underline{\zeta})\cdot\beta$, the transport equation for $\bar{\rho}$ can be obtained as follows

$$\begin{aligned}\frac{d}{ds}(\bar{\rho}) &= \nabla\!\!\!/_L(r^{-2}\int_S \rho)\\ &= -\overline{an\mathrm{tr}\chi}(an)^{-1}\bar{\rho} + r^{-2}(an)^{-1}\int_S \left(\nabla\!\!\!/_L\rho + \mathrm{tr}\chi\rho\right) and\mu_\gamma\\ &= -(an)^{-1}\overline{an\mathrm{tr}\chi}\bar{\rho} + (an)^{-1}\overline{\left(-\frac{1}{2}an\mathrm{tr}\chi\rho + anF\right)},\end{aligned}$$

and

$$\begin{aligned}\frac{d}{ds}(r^3\bar{\rho}) &= \frac{3r^3}{2}\overline{an\mathrm{tr}\chi}(an)^{-1}\bar{\rho}\\ &\quad + r^3\left\{-(an)^{-1}\overline{an\mathrm{tr}\chi}\bar{\rho} + (an)^{-1}\overline{\left(-\frac{1}{2}an\mathrm{tr}\chi\rho + anF\right)}\right\}\\ &= -\frac{1}{2}r^3(an)^{-1}\overline{an\mathrm{tr}\chi(\rho-\bar{\rho})} + r^3(an)^{-1}\overline{anF}.\end{aligned}$$

Integrating the above identity in t and noting $\lim_{t\to 0} r^\theta\rho = 0$ for any $\theta > 0$, we can obtain

$$r^{\frac{3}{2}}\bar{\rho}(t) = r^{-\frac{3}{2}}\int_0^t r^3\left(-\frac{1}{2}\overline{an\mathrm{tr}\chi\cdot \mathcal{O}sc(\rho)} + \overline{anF}\right)dt'.$$

In view of [9, Eq.(2.23)], (1.2) and [9, Proposition 2.2], we obtain

$$r^{-\frac{3}{2}}\int_0^t \left|r^3\overline{an\mathrm{tr}\chi\cdot\mathcal{O}sc(\rho)}\right| \le \|r\mathcal{O}sc(\rho)\|_{L_t^2L_\omega^2}\|r'\|_{L^2(0,t]}r^{-\frac{3}{2}} \lesssim \|\rho\|_{L^2(\mathcal{H})} \lesssim \mathcal{R}_0.$$

By integration by part on $S = S_t$, we can obtain

$$\begin{aligned}r^2\overline{anF} &= \int_S\left(an(\mathrm{div}\beta - \frac{1}{2}\underline{\hat{\chi}}\cdot\alpha) + an(\zeta + 2\underline{\zeta})\cdot\beta\right)d\mu_\gamma\\ &= \int_S an(\underline{\zeta}\beta - \frac{1}{2}\underline{\hat{\chi}}\cdot\alpha)d\mu_\gamma = \int_S an\underline{A}\cdot R_0 d\mu_\gamma.\end{aligned}$$

Hence by (1.2)

$$\begin{aligned}\left|r^{-\frac{3}{2}}\int_0^t r'^3\overline{anF}dt'\right| &\le r^{-\frac{3}{2}}\|r'\|_{L^2(0,t]}\|(r')^2\underline{A}\cdot R_0\|_{L_t^2L_\omega^1}\\ &\le \|R_0\|_{L^2}\|r\underline{A}\|_{L_t^\infty L_\omega^2} \lesssim \Delta_0^2 + \mathcal{R}_0.\end{aligned}$$

Following the same procedure as above, we can obtain the same estimate for $\bar{\sigma}$ in view of [2, Eq. (42)]. Note that

$$|r(\overline{\hat{\chi}\cdot\underline{\hat{\chi}}}, \overline{\hat{\chi}\wedge\underline{\hat{\chi}}})| \lesssim \|r^{\frac{1}{2}}\underline{A}\|_{L_t^\infty L_\omega^2}^2 \lesssim (\Delta_0^2 + \mathcal{R}_0)^2 \tag{3.44}$$

The estimates (3.42) then follows from (3.41) and (3.44). □

Proposition 3.7. *For $4 < b < \infty$ there hold*

$$\|r^{-\frac{1}{b}-\frac{1}{2}}\mathcal{D}_1^{-1}(an(\check{\rho}, -\check{\sigma}))\|_{L_t^bL_x^2} \lesssim \Delta_0^2 + \mathcal{R}_0, \tag{3.45}$$

$$\|r^{-\frac{1}{b}-\frac{1}{2}}\mathcal{D}_1^{-1}(an(\rho, -\sigma))\|_{L_t^bL_x^2} \lesssim \Delta_0^2 + \mathcal{R}_0. \tag{3.46}$$

Proof. Let $H = (\check\rho - \bar{\check\rho}, -\check\sigma + \bar{\check\sigma})$. In view of $\mathcal{D}_1\mathcal{D}_1^{-1}H = H$,

$$\mathcal{D}_1^{-1}(an(\check\rho, -\check\sigma)) = \mathcal{D}_1^{-1}(an\mathcal{D}_1\mathcal{D}_1^{-1}H) + \mathcal{D}_1^{-1}(an(\bar{\check\rho}, -\bar{\check\sigma})).$$

By Proposition 3.1,

$$\|r^{-\frac{1}{b}-\frac{1}{2}}\mathcal{D}_1^{-1}(an(\bar{\check\rho}, -\bar{\check\sigma}))\|_{L_t^b L_x^2} \lesssim \|r^{\frac{3}{2}-\frac{1}{b}}an(\bar{\check\rho}, -\bar{\check\sigma})\|_{L_t^b L_\omega^2}.$$

Hence, in view of (3.42) and (1.2),

$$\begin{aligned}\|r^{\frac{3}{2}-\frac{1}{b}}(an(\bar{\check\rho}, -\bar{\check\sigma}))\|_{L_t^b L_\omega^2} &\lesssim \|r^{\frac{3}{2}}an(\bar{\check\rho}, -\bar{\check\sigma})\|_{L_t^\infty L_\omega^\infty}^{1-\frac{2}{b}}\|ran(\bar{\check\rho}, -\bar{\check\sigma})\|_{L_t^2 L_\omega^2}^{\frac{2}{b}}\\ &\lesssim \Delta_0^2 + \mathcal{R}_0. \qquad (3.47)\end{aligned}$$

By Leibnitz rule, Proposition 3.1 and (SobIn), we have

$$\begin{aligned}\|r^{-\frac{1}{b}-\frac{1}{2}}\mathcal{D}_1^{-1}(an\mathcal{D}_1\mathcal{D}_1^{-1}H)\|_{L_t^b L_x^2} &\lesssim \|r^{-\frac{1}{b}-\frac{1}{2}}\mathcal{D}_1^{-1}\mathcal{D}_1(an\mathcal{D}_1^{-1}H)\|_{L_t^b L_x^2}\\ &\quad + \|r^{-\frac{1}{b}-\frac{1}{2}}\mathcal{D}_1^{-1}(\nabla\!\!\!/(an)\mathcal{D}_1^{-1}H)\|_{L_t^b L_x^2}\\ &\lesssim \|r^{-\frac{1}{b}-\frac{1}{2}}\mathcal{D}_1^{-1}H\|_{L_t^b L_x^2} + \|r^{-\frac{1}{b}}(\zeta + \underline{\zeta})\mathcal{D}_1^{-1}H\|_{L_t^b L_x^{4/3}}\\ &\lesssim \mathcal{N}_1(\mathcal{D}_1^{-1}H) + \|r^{-\frac{1}{b}}\mathcal{D}_1^{-1}H\|_{L_t^b L_x^2}\|\zeta + \underline{\zeta}\|_{L_t^\infty L_x^4}\\ &\lesssim \mathcal{N}_1(\mathcal{D}_1^{-1}H)(1 + \Delta_0^2 + \mathcal{R}_0) \lesssim \Delta_0^2 + \mathcal{R}_0\end{aligned}$$

For the last two inequalities, we employed (1.2) and (3.25).

In view of the estimate for $\mathcal{E}_1(\underline{A})$ in Proposition 3.6, (3.46) can be obtained by combining (3.45) with the estimate for $r^{-1}\mathcal{D}^{-1}(anA \cdot \underline{A})$. □

3.5. **$L_t^b L_x^2$ estimates for $\mathcal{D}^{-1}E_1^G$.** For arbitrary S-tangent tensor field F, we denote by E_1^G either $[\mathcal{D}_t, \mathcal{D}_1^{-1}](\check\rho, -\check\sigma) \cdot F$ or $Err \cdot F$. In what follows, we will establish estimates for $\|\mathcal{D}^{-1}E_1^G\|_{L_t^b L_x^2}$ with $4 < b < \infty$, which will be employed for the Hodge-elliptic $\mathcal{P}^0$ estimates of error terms arising in the decomposition procedure in Section 3.7.

Proposition 3.8. *Let $\mathcal{D}$ denote either $\mathcal{D}_1$ or $\mathcal{D}_2$. For appropriate S-tangent tensor fields F and $4 < b < \infty$ there holds the estimate*

$$\|r^{-\frac{1}{b}}\mathcal{D}^{-1}E_1^G\|_{L_t^b L_x^2} \lesssim (\Delta_0^2 + \mathcal{R}_0)\mathcal{N}_1(F). \qquad (3.48)$$

More precisely,

$$\|r^{-\frac{1}{b}}\mathcal{D}^{-1}(Err \cdot F)\|_{L_t^b L_x^2} \lesssim (\Delta_0^2 + \mathcal{R}_0)\mathcal{N}_1(F), \qquad (3.49)$$

$$\|r^{-\frac{1}{b}}\mathcal{D}^{-1}([\mathcal{D}_t, \mathcal{D}_1^{-1}](\check\rho, -\check\sigma) \cdot F)\|_{L_t^b L_x^2} \lesssim (\Delta_0^2 + \mathcal{R}_0)\mathcal{N}_1(F). \qquad (3.50)$$

where Err is defined in (3.8).

In order to prove Proposition 3.8, in view of [9, (2.11) and (2.12)] we may use the error type terms introduced in Proposition 3.6 to rewrite (3.8) as

$$Err = \mathcal{D}_1^{-1}(an\mathrm{tr}\chi(\check\rho, -\check\sigma)) + \mathcal{E}_1(\underline{A}) + \mathcal{E}_2(\underline{A}). \qquad (3.51)$$

Proof of Proposition 3.8. (3.50) can be obtained by using Proposition 3.1, (3.39), (3.25) and (SobM1) as follows,

$$\|r^{-\frac{1}{b}}\mathcal{D}^{-1}(C_0(\check R) \cdot F)\|_{L_t^b L_x^2} \lesssim \|r^{\frac{1}{2}-\frac{1}{b}}C_0(\check R)\|_{L_t^b L_x^2}\|F\|_{L_t^\infty L_x^4} \lesssim (\Delta_0^2 + \mathcal{R}_0)\mathcal{N}_1(F).$$

Similarly, by using Proposition 3.6, we have for $i = 1, 2$ that

$$\|r^{-\frac{1}{b}}\mathcal{D}^{-1}(\mathcal{E}_i(\underline{A}) \cdot F)\|_{L_t^b L_x^2} \lesssim \|r^{\frac{1}{2}-\frac{1}{b}}\mathcal{E}_i(\underline{A})\|_{L_t^b L_x^2}\|F\|_{L_t^\infty L_x^4} \lesssim (\Delta_0^2 + \mathcal{R}_0)\mathcal{N}_1(F).$$

Thus, in order to prove (3.49), in view of (3.51) it remains to show

$$\|r^{-\frac{1}{b}}\mathcal{D}^{-1}\big(\mathcal{D}_1^{-1}(an\mathrm{tr}\chi(\check\rho, -\check\sigma)) \cdot F\big)\|_{L_t^b L_x^2} \lesssim \mathcal{N}_1(F)\Delta_0.$$

For this estimate, we proceed as follows. Let $H = (\check\rho - \bar{\check\rho}, -\check\sigma + \bar{\check\sigma})$, then $H = \mathcal{D}_1\mathcal{D}_1^{-1}H$. Thus

$$\|r^{-\frac{1}{b}}\mathcal{D}^{-1}\big(\mathcal{D}_1^{-1}(an\mathrm{tr}\chi(\check\rho, -\check\sigma))\cdot F\big)\|_{L_t^bL_x^2} \lesssim \|r^{-\frac{1}{b}}\mathcal{D}^{-1}(\mathcal{D}_1^{-1}(an\mathrm{tr}\chi\mathcal{D}_1\mathcal{D}_1^{-1}H)F)\|_{L_t^bL_x^2} + \|r^{-\frac{1}{b}}\mathcal{D}^{-1}\big(\mathcal{D}_1^{-1}(an\mathrm{tr}\chi(\bar{\check\rho}, -\bar{\check\sigma}))\cdot F)\|_{L_t^bL_x^2}$$

By I_1 and I_2 we denote the two terms on the right of the above inequality. Using Proposition 3.1, (SobM1), (SobIn) and (3.25) it yields

$$\begin{aligned} I_1 &\lesssim \|r^{-\frac{1}{b}}\mathcal{D}^{-1}(an\mathrm{tr}\chi\mathcal{D}^{-1}H\cdot F)\|_{L_t^bL_x^2} + \|r^{-\frac{1}{b}}\mathcal{D}^{-1}(\mathcal{D}_1^{-1}(\nabla\!\!\!/(an\mathrm{tr}\chi)\mathcal{D}_1^{-1}H)\cdot F)\|_{L_t^bL_x^2}\\ &\lesssim \|r^{-\frac{1}{b}}\mathcal{D}^{-1}H\cdot F\|_{L_t^bL_x^2} + \|r^{-\frac{1}{b}+\frac{1}{2}}\mathcal{D}_1^{-1}(\nabla\!\!\!/(an\mathrm{tr}\chi)\mathcal{D}^{-1}H)\|_{L_t^bL_x^2}\|F\|_{L_t^\infty L_x^4}\\ &\lesssim \|F\|_{L_t^\infty L_x^4}\|r^{-\frac{1}{b}}\mathcal{D}^{-1}H\|_{L_t^bL_x^4} + \|r^{1-\frac{1}{b}}\nabla\!\!\!/(an\mathrm{tr}\chi)\|_{L_t^bL_x^2}\|\mathcal{D}^{-1}H\|_{L_t^\infty L_x^4}\|F\|_{L_t^\infty L_x^4}\\ &\lesssim \mathcal{N}_1(F)\mathcal{N}_1(\mathcal{D}^{-1}H) \lesssim (\Delta_0^2+\mathcal{R}_0)\mathcal{N}_1(F), \end{aligned}$$

where we employed

$$\|r^{\frac{1}{2}-\frac{1}{b}}\nabla\!\!\!/(an\mathrm{tr}\chi)\|_{L_t^bL_x^2} \lesssim \|r^{\frac{1}{2}}\nabla\!\!\!/(an\mathrm{tr}\chi)\|_{L_t^\infty L_x^2}^{1-\frac{2}{b}}\|\nabla\!\!\!/(an\mathrm{tr}\chi)\|_{L^2}^{\frac{2}{b}} \lesssim \Delta_0^2+\mathcal{R}_0.$$

By Proposition 3.1, (3.47) and (SobM1), we also have

$$\begin{aligned} I_2 &\lesssim \|r^{-\frac{1}{b}+\frac{1}{2}}\mathcal{D}^{-1}(an\mathrm{tr}\chi(\bar{\check\rho}, -\bar{\check\sigma}))\|_{L_t^bL_x^2}\|F\|_{L_t^\infty L_x^4}\\ &\lesssim \|r^{\frac{3}{2}-\frac{1}{b}}an\mathrm{tr}\chi(\bar{\check\rho}, -\bar{\check\sigma})\|_{L_t^bL_x^2}\|F\|_{L_t^\infty L_x^4}\\ &\lesssim \|r^{\frac{3}{2}-\frac{1}{b}}(\bar{\check\rho}, -\bar{\check\sigma})\|_{L_t^bL_\omega^2}\mathcal{N}_1(F)\\ &\lesssim (\Delta_0^2+\mathcal{R}_0)\mathcal{N}_1(F). \end{aligned}$$

The proof is complete. □

3.6. $L_t^bL_x^2$ **estimates for** $\nabla\!\!\!/_L\mathcal{D}^{-1}\mathcal{F}$**.** We will establish the following

Proposition 3.9. *Denote by* $\mathcal{D}^{-1}\mathcal{F}$ *either* $\mathcal{D}^{-2}\check R$ *or* $\mathcal{D}_1^{-1}(a\delta+2a\lambda)$*. For* $4<b<\infty$ *there holds*

$$\|r^{-\frac{1}{b}}\mathcal{D}_t\mathcal{D}^{-1}\mathcal{F}\|_{L_t^bL_x^2} \lesssim \mathcal{R}_0+\Delta_0^2.$$

We will prove Proposition 3.9 by considering two cases: $\mathcal{F}=\mathcal{D}^{-1}\check R$ or $\mathcal{F}=(a\delta+2a\lambda)$.

Case 1: $\mathcal{F}=\mathcal{D}^{-1}\check R$**.** We denote by $\mathcal{D}^{-1}\mathfrak{F}$ either $\mathcal{D}_2^{-1}Err$ or ${}^\star\mathcal{D}_1^{-1}\widetilde{Err}$. To prove Proposition 3.9, we will rely on (3.52) in the following result.

Proposition 3.10. *Let* $\mathfrak{F}=(Err, \widetilde{Err})$ *with* Err *and* $\widetilde{Err}$ *given by (3.8) and let* $4<b<\infty$*. There hold*

$$\|r^{-\frac{1}{b}}\mathcal{D}^{-1}\mathfrak{F}\|_{L_t^bL_x^2} \lesssim \Delta_0^2+\mathcal{R}_0, \tag{3.52}$$

$$\|\nabla\!\!\!/\mathcal{D}^{-1}\mathfrak{F}\|_{\mathcal{P}^0} \lesssim \Delta_0^2+\mathcal{R}_0. \tag{3.53}$$

Assuming (3.52), now we prove Proposition 3.9.

Proof of Proposition 3.9 for Case 1. In view of the formula

$$\mathcal{D}_t\mathcal{D}^{-2}\check R = [\mathcal{D}_t, \mathcal{D}^{-1}]\mathcal{D}^{-1}\check R + \mathcal{D}^{-1}[\mathcal{D}_t, \mathcal{D}^{-1}]\check R + \mathcal{D}^{-2}\mathcal{D}_t\check R,$$

we only need to show for $4<b<\infty$ that

$$\|r^{-\frac{1}{b}}[\mathcal{D}_t, \mathcal{D}^{-1}]\mathcal{D}^{-1}\check R\|_{L_t^bL_x^2} \lesssim \Delta_0^2+\mathcal{R}_0 \tag{3.54}$$

$$\|r^{-\frac{1}{b}}\mathcal{D}^{-1}[\mathcal{D}_t, \mathcal{D}^{-1}]\check R\|_{L_t^bL_x^2} \lesssim \Delta_0^2+\mathcal{R}_0 \tag{3.55}$$

$$\|r^{-\frac{1}{b}}\mathcal{D}^{-2}\mathcal{D}_t\check R\|_{L_t^bL_x^2} \lesssim \Delta_0^2+\mathcal{R}_0. \tag{3.56}$$

The estimate (3.55) was proved in (3.40). By using the fact that $\mathcal{N}_1(r^{-\frac{1}{2}}\mathcal{D}^{-2}\check{R}) \lesssim \mathcal{N}_2(\mathcal{D}^{-2}\check{R}) \lesssim \Delta_0^2 + \mathcal{R}_0$, the estimate (3.54) follows from (3.39) with $F = \mathcal{D}^{-1}\check{R}$.

It only remains to prove (3.56). Consider first the case $\mathcal{D}^{-2}\mathcal{D}_t\check{R} = \mathcal{D}_2^{-1}\mathcal{D}_1^{-1}\mathcal{D}_t(\check{\rho}, -\check{\sigma})$. By $an\beta = \nabla\!\!\!/(anA) + an(A\cdot A + r^{-1}A)$, (SobM1) and (1.2),

$$\begin{aligned}\|r^{-\frac{1}{b}}\mathcal{D}^{-1}(an\beta)\|_{L_t^bL_x^2} &\lesssim \|r^{-\frac{1}{b}}\mathcal{D}^{-1}(\nabla\!\!\!/(anA) + an(A\cdot A + r^{-1}A))\|_{L_t^bL_x^2}\\ &\lesssim \|r^{-\frac{1}{b}}A\|_{L_t^bL_x^2} + \|r^{-\frac{1}{b}+1}A\cdot A\|_{L_t^bL_x^2}\\ &\lesssim \mathcal{N}_1(A) + \mathcal{N}_1(A)^2 \lesssim \Delta_0^2 + \mathcal{R}_0.\end{aligned}$$

Then by (3.52), we obtain

$$\|r^{-\frac{1}{b}}\mathcal{D}^{-1}(Err + an\beta)\|_{L_t^bL_x^2} \lesssim \Delta_0^2 + \mathcal{R}_0.$$

In view of the definition of Err in (3.8), we have

$$\|r^{-\frac{1}{b}}\mathcal{D}^{-2}\mathcal{D}_t(\check{\rho}, -\check{\sigma})\|_{L_t^bL_x^2} \lesssim \Delta_0^2 + \mathcal{R}_0. \tag{3.57}$$

Using ${}^\star\mathcal{D}_1^{-1}\mathcal{D}_t\underline{\beta} = an(\rho,\sigma) + \widetilde{Err}$, Proposition 3.7 and (3.52),

$$\begin{aligned}\|r^{-\frac{1}{b}}\mathcal{D}^{-2}\mathcal{D}_t\underline{\beta}\|_{L_t^bL_x^2} &\lesssim \|r^{-\frac{1}{b}}\mathcal{D}_1^{-1}(an(\rho,\sigma))\|_{L_t^bL_x^2} + \|r^{-\frac{1}{b}}\mathcal{D}^{-1}\mathfrak{F}\|_{L_t^bL_x^2}\\ &\lesssim \Delta_0^2 + \mathcal{R}_0 \end{aligned}\tag{3.58}$$

With the help of (3.57) and (3.58), we thus obtain (3.56). □

To prove Proposition 3.10, we will rely on the following result.

Lemma 3.5. *Let $\mathcal{D}^{-1}$ denote one of the operators $\mathcal{D}_1^{-1}$, $\mathcal{D}_2^{-1}$ or ${}^\star\mathcal{D}_1^{-1}$. For any appropriate S-tangent tensor field G there holds*

$$\|\mathcal{D}^{-1}(an\check{\rho}\cdot G)\|_{L_t^bL_x^2} \lesssim \|\Lambda^{-\alpha_0}\check{\rho}\|_{L_t^\infty L_x^2}\mathcal{N}_1(G) \tag{3.59}$$

where $1/2 \le \alpha_0 < 1$ and $4 < b < \infty$.

Proof. In view of $\|\nabla\!\!\!/(an)\|_{L_x^4L_t^\infty} \lesssim \Delta_0^2 + \mathcal{R}_0$ and $an < C$, we can adapt the proof for [7, Lemma 4.4] to derive for any S tangent tensor fields F the estimates

$$\|\Lambda^{-\alpha}(an\check{\rho}\cdot F_m)\|_{L^2(S)} \lesssim \|\Lambda^{-\alpha_0}\check{\rho}\|_{L_t^\infty L_x^2}2^m r^{-1}\|P_mF\|_{L^2(S)}, \tag{3.60}$$

where $1 > \alpha > \alpha_0 \ge 1/2$.

By using [9, Proposition 3.1(3)], [9, Lemma 4.3] and a duality argument we can obtain

$$\|\mathcal{D}^{-1}\Lambda^\alpha P_lF\|_{L^2(S)} \lesssim 2^{(-1+\alpha)l}r^{1-\alpha}\|F\|_{L^2(S)}. \tag{3.61}$$

In view of [9, Proposition 3.1(3)] and (FBB) in [9], we also have

$$\|\Lambda^\alpha P_n^2G\|_{L_x^2} \lesssim \|\nabla\!\!\!/ P_n^2G\|_{L_x^2}^\alpha\|P_n^2G\|_{L_x^2}^{1-\alpha} \lesssim 2^{\alpha n}r^{-\alpha}\|P_nG\|_{L_x^2}. \tag{3.62}$$

Set $\Omega_{nl} := \mathcal{D}^{-1}P_l^2(an\check{\rho}\cdot P_n^2G)$, with $l, n \in \mathbb{N}$. We now prove

$$\sum_{l,n>0}\|\Omega_{nl}\|_{L_t^bL_x^2} \lesssim \|\Lambda^{-\alpha_0}\check{\rho}\|_{L_t^\infty L_x^2}\mathcal{N}_1(G), \tag{3.63}$$

and lower frequency terms can be treated similarly.

We first prove (3.63) for the case $0 < n < l$. With the help of (3.61) and (3.60),

$$\|\Omega_{nl}\|_{L_x^2} \lesssim 2^{-(1-\alpha)l}2^nr^{-\alpha}\|\Lambda^{-\alpha_0}\check{\rho}\|_{L_t^\infty L_x^2}\|P_nG\|_{L_x^2}.$$

Taking L_t^b norm for $4 < b < \infty$ and using (3.35) in Lemma 3.3 it yields

$$\|\Omega_{nl}\|_{L_t^bL_x^2} \lesssim 2^{-(1-\alpha)l}2^{n(\frac{1}{2}-\frac{1}{b})}r^{\frac{1}{2}+\frac{1}{b}-\alpha}\|\Lambda^{-\alpha_0}\check{\rho}\|_{L_t^\infty L_x^2}\mathcal{N}_1(G).$$

Since we can choose $\alpha_0 < \alpha < \frac{1}{2} + \frac{1}{b}$, we deduce

$$\sum_{0<n<l}\|\Omega_{nl}\|_{L_t^bL_x^2} \lesssim \|\Lambda^{-\alpha_0}\check{\rho}\|_{L_t^\infty L_x^2}\mathcal{N}_1(G).$$

For the case $0 < l < n$, we pair Ω_{nl} with any S tangent tensor F satisfying $\|F\|_{L^2(S)} \le 1$. By using (3.60), [9, Lemma 4.3] and (3.62) we obtain

$$\begin{aligned}\langle \Omega_{nl}, F\rangle &= \langle P_n^2 G, an\check{\rho} P_l^{2\star}\mathcal{D}^{-1}F\rangle = \langle \Lambda^\alpha P_n^2 G, \Lambda^{-\alpha}(an\check{\rho}P_l^{2\star}\mathcal{D}^{-1}F)\rangle\\ &\le \|\Lambda^\alpha P_n^2 G\|_{L_x^2}\|\Lambda^{-\alpha_0}\check{\rho}\|_{L_t^\infty L_x^2} 2^l r^{-1}\|P_l{}^\star\mathcal{D}^{-1}F\|_{L_x^2}\\ &\lesssim 2^{\alpha n} r^{-\alpha}\|\Lambda^{-\alpha_0}\check{\rho}\|_{L_t^\infty L_x^2}\|P_n G\|_{L_x^2}.\end{aligned}$$

Hence, by (3.35) in Lemma 3.3

$$\|\Omega_{nl}\|_{L_t^b L_x^2} \lesssim 2^{\alpha n - (\frac{1}{2}+\frac{1}{b})n} r^{-\alpha+\frac{1}{2}+\frac{1}{b}}\|\Lambda^{-\alpha_0}\check{\rho}\|_{L_t^\infty L_x^2}\mathcal{N}_1(G).$$

Consequently $\sum_{0<l<n}\|\Omega_{nl}\|_{L_t^b L_x^2} \lesssim \|\Lambda^{-\alpha_0}\check{\rho}\|_{L_t^\infty L_x^2}\mathcal{N}_1(G)$ and (3.59) thus follows. □

We are ready to prove Proposition 3.10.

Proof of Proposition 3.10. (3.53) can be derived by using (3.52), Theorem 3.1 and (3.10).

Now we prove (3.52). By setting $F = 1$ in (3.49) we obtain for $4 < b < \infty$ that

$$\|r^{-\frac{1}{b}}\mathcal{D}_2^{-1} Err\|_{L_t^b L_x^2} \lesssim \Delta_0^2 + \mathcal{R}_0.$$

Thus we only need to consider $\mathcal{D}_1^{-1}\widetilde{Err}$. Recall the definition of $\widetilde{Err}$ given in (3.8). In view of [9, (2.13)] we can rewrite $\widetilde{Err}$ symbolically as

$$\widetilde{Err} = {}^\star\mathcal{D}_1^{-1}\left(an\mathrm{tr}\chi\underline{\beta} + an\underline{A}\cdot(\nabla\!\!\!/ A + A\cdot A + r^{-1}A)\right) + {}^\star\mathcal{D}_1^{-1}(an(\zeta\cdot\rho - \zeta^\star\sigma)). \tag{3.64}$$

By Propositions 3.1 and 3.6, we have

$$\|r^{-\frac{1}{b}}\mathcal{D}^{-2}(an\underline{A}\cdot(\nabla\!\!\!/ A + r^{-1}A + A\cdot A))\|_{L_t^b L_x^2} \lesssim \Delta_0^2 + \mathcal{R}_0,$$

where $\mathcal{D}^{-2}$ denotes $\mathcal{D}_1^{-1\star}\mathcal{D}_1^{-1}$. In order to complete the proof of (3.52), in view of (3.64) we need to derive the estimates on

$$\begin{aligned}J_1 &:= \|r^{-\frac{1}{b}}\mathcal{D}^{-2}(an\mathrm{tr}\chi\underline{\beta})\|_{L_t^b L_x^2},\\ J_2 &:= \|r^{-\frac{1}{b}}\mathcal{D}^{-2}(an\underline{\zeta}^\star\check{\sigma})\|_{L_t^b L_x^2},\\ J_3 &:= \|r^{-\frac{1}{b}}\mathcal{D}^{-2}(an\underline{\zeta}\cdot\check{\rho})\|_{L_t^b L_x^2}.\end{aligned}$$

Since $\underline{\beta} = {}^\star\mathcal{D}_1{}^\star\mathcal{D}_1^{-1}\underline{\beta}$, we may use Proposition 3.1, (SobIn), (1.2) and (3.25) to derive

$$\begin{aligned}J_1 &\lesssim \|r^{-\frac{1}{b}}\mathcal{D}^{-2}\left(({}^\star\mathcal{D}_1(an\mathrm{tr}\chi{}^\star\mathcal{D}_1^{-1}\underline{\beta})) - \nabla\!\!\!/(an\mathrm{tr}\chi){}^\star\mathcal{D}_1^{-1}\underline{\beta}\right)\|_{L_t^b L_x^2}\\ &\lesssim \|r^{1-\frac{1}{b}}(an\mathrm{tr}\chi{}^\star\mathcal{D}_1^{-1}\underline{\beta})\|_{L_t^b L_x^2} + \|r^{\frac{3}{2}-\frac{1}{b}}\nabla\!\!\!/(an\mathrm{tr}\chi){}^\star\mathcal{D}_1^{-1}\underline{\beta}\|_{L_t^b L_x^{4/3}}\\ &\lesssim \|r^{-\frac{1}{b}}{}^\star\mathcal{D}_1^{-1}\underline{\beta}\|_{L_t^b L_x^2} + \|r^{1-\frac{1}{b}}{}^\star\mathcal{D}_1^{-1}\underline{\beta}\|_{L_t^b L_x^4}\|r^{\frac{1}{2}}\nabla\!\!\!/(an\mathrm{tr}\chi)\|_{L_t^\infty L_x^2}\\ &\lesssim \mathcal{N}_1(r^{1/2\star}\mathcal{D}_1^{-1}\underline{\beta})(\|r^{\frac{1}{2}}\nabla\!\!\!/(an\mathrm{tr}\chi)\|_{L_t^\infty L_x^2} + 1) \lesssim \Delta_0^2 + \mathcal{R}_0.\end{aligned}$$

In order to estimate J_2, we use [9, (2.9)] to write $\check{\sigma} = curl\zeta$. Thus by using Propositions 3.1 and 3.6, we can obtain

$$J_2 = \|r^{-\frac{1}{b}}\mathcal{D}^{-1}\mathcal{E}_2(\zeta)\|_{L_t^b L_x^2} \lesssim \mathcal{N}_1(\zeta)\mathcal{N}_1(\underline{\zeta}) \lesssim (\Delta_0^2 + \mathcal{R}_0)^2.$$

Finally, by using Proposition 3.1 and (3.59), we have

$$\begin{aligned}J_3 &\lesssim \|r^{-\frac{1}{b}}\mathcal{D}^{-2}(an\check{\rho}\cdot\underline{\zeta})\|_{L_t^b L_x^2} \lesssim \|r^{-\frac{1}{b}+1}\mathcal{D}^{-1}(an\check{\rho}\cdot\underline{\zeta})\|_{L_t^b L_x^2}\\ &\lesssim \mathcal{N}_1(\underline{\zeta})\|\Lambda^{-\alpha_0}\check{\rho}\|_{L_t^\infty L_x^2} \lesssim (\Delta_0^2 + \mathcal{R}_0)^2,\end{aligned}$$

where we used (1.2) and (1.3) to obtain the last inequality. □

Case 2: $\mathcal{F} = (a\delta + 2a\lambda)$. We give a slightly stronger result than Proposition 3.9 for Case 2.

Proposition 3.11. *For $4 < b < \infty$ there holds*

$$\|r^{-\frac{1}{2}-\frac{1}{b}}\mathcal{D}_t\mathcal{D}_1^{-1}(a\delta+2a\lambda)\|_{L_t^b L_x^2} \lesssim \Delta_0^2+\mathcal{R}_0. \tag{3.65}$$

Proof. It is easy to see

$$\begin{aligned}\|r^{-\frac{1}{2}-\frac{1}{b}}\mathcal{D}_t\mathcal{D}_1^{-1}(a\delta+2a\lambda)\|_{L_t^b L_x^2} &\lesssim \|r^{-\frac{1}{2}-\frac{1}{b}}\mathcal{D}_1^{-1}\mathcal{D}_t(a\delta+2a\lambda)\|_{L_t^b L_x^2}\\ &+\|r^{-\frac{1}{2}-\frac{1}{b}}[\mathcal{D}_t,\mathcal{D}_1^{-1}](a\delta+2a\lambda)\|_{L_t^b L_x^2}.\end{aligned} \tag{3.66}$$

By [9, Eq.(2.7) and Eq.(2.16)], we have

$$an\underline{\zeta} = \mathcal{D}_1^{-1}\mathcal{D}_t(a\delta+2a\lambda) - \mathcal{D}_1^{-1}(an(\check{\rho},\check{\sigma})) + \mathcal{D}_1^{-1}\mathrm{err}_1 \tag{3.67}$$

where, symbolically, $\mathrm{err}_1 := an(a\not\pi\mathrm{tr}\chi + A\cdot A)$. Therefore

$$\begin{aligned}&\|r^{-\frac{1}{2}-\frac{1}{b}}\mathcal{D}_1^{-1}\mathcal{D}_t(a\delta+2a\lambda)\|_{L_t^b L_x^2}\\ &\lesssim \|r^{-\frac{1}{2}-\frac{1}{b}}(an\underline{\zeta})\|_{L_t^b L_x^2} + \|r^{-\frac{1}{2}-\frac{1}{b}}\mathcal{D}_1^{-1}(an(\check{\rho},\check{\sigma}))\|_{L_t^b L_x^2} &(3.68)\\ &+\|r^{-\frac{1}{2}-\frac{1}{b}}\mathcal{D}_1^{-1}\mathrm{err}_1\|_{L_t^b L_x^2}. &(3.69)\end{aligned}$$

By (SobIn) and Proposition 3.7, the two terms on the right of (3.68) can be bounded by $\mathcal{N}_1(\underline{\zeta})+\Delta_0^2+\mathcal{R}_0 \lesssim \Delta_0^2+\mathcal{R}_0$. For the term in (3.69), in view of [9, Eq.(2.23)] and (SobIn) we may use Proposition 3.1 and (1.2) to deduce

$$\begin{aligned}\|r^{-\frac{1}{2}-\frac{1}{b}}\mathcal{D}_1^{-1}\mathrm{err}_1\|_{L_t^b L_x^2} &\lesssim \|r^{\frac{1}{2}-\frac{1}{b}}\mathrm{err}_1\|_{L_t^b L_x^2}\\ &\lesssim \|r^{\frac{1}{2}-\frac{1}{b}}a\not\pi\mathrm{tr}\chi\|_{L_t^b L_x^2} + \|r^{\frac{1}{2}-\frac{1}{b}}A\cdot A\|_{L_t^b L_x^2}\\ &\lesssim \|r^{-\frac{1}{2}-\frac{1}{b}}\not\pi\|_{L_t^b L_x^2} + \|A\cdot A\|_{L_t^\infty L_x^2}\\ &\lesssim \mathcal{N}_1(\not\pi)+\|A\|_{L_t^\infty L_x^4}^2 \lesssim \Delta_0^2+\mathcal{R}_0.\end{aligned} \tag{3.70}$$

Thus we obtain

$$\|r^{-\frac{1}{2}-\frac{1}{b}}\mathcal{D}_1^{-1}\mathcal{D}_t(a\delta+2a\lambda)\|_{L_t^b L_x^2} \lesssim \Delta_0^2+\mathcal{R}_0. \tag{3.71}$$

Repeating the derivation of (3.39) and using [9, Lemma 3.1], we have

$$\begin{aligned}\|r^{-\frac{1}{2}-\frac{1}{b}}[\mathcal{D}_1^{-1},\mathcal{D}_t](a\delta+2a\lambda)\|_{L_t^b L_x^2} &\lesssim (\Delta_0^2+\mathcal{R}_0)\mathcal{N}_1(r^{-1}\mathcal{D}_1^{-1}(a\delta+2a\lambda))\\ &\lesssim (\Delta_0^2+\mathcal{R}_0)\mathcal{N}_1(a\not\pi) \lesssim (\Delta_0^2+\mathcal{R}_0)^2.\end{aligned}$$

Therefore (3.65) is proved. □

For the term $\mathrm{err}_1 := an(a\not\pi\mathrm{tr}\chi + A\cdot A)$ appearing in the proof of the above proposition, we have the following simple but useful estimates.

Lemma 3.6. *For $err_1 := an(a\not\pi tr\chi + A\cdot A)$ there hold the estimate*

$$\|err_1\|_{\mathcal{P}^0} + \|\nabla\!\!\!/\,\mathcal{D}^{-1}err_1\|_{\mathcal{P}^0} \lesssim \Delta_0^2+\mathcal{R}_0. \tag{3.72}$$

Proof. By using [9, Eq.(4.10)] and (1.2), we have

$$\|anA\cdot A\|_{\mathcal{P}^0} \lesssim \Delta_0\mathcal{N}_1(A) \lesssim \Delta_0^2+\mathcal{R}_0 \tag{3.73}$$

$$\|a^2 n\mathrm{tr}\chi\not\pi\|_{\mathcal{P}^0} \lesssim \mathcal{N}_1(a\not\pi) \lesssim \Delta_0^2+\mathcal{R}_0. \tag{3.74}$$

Thus the first inequality of (3.72) follows immediately. The second one in (3.72) follows from the combination of (3.73), (3.74), (3.70) and Theorem 3.1. □

3.7. **Decomposition for commutators.** In order to complete the proof of Proposition 3.2, it remains to decompose the "bad" terms

$$an\beta\cdot\mathcal{D}^{-1}\mathcal{F}, \qquad \nabla\!\!\!/\,\mathcal{D}^{-1}(an\beta\cdot\mathcal{D}^{-1}\mathcal{F}),$$

which have not been treated in Proposition 3.3, where $\mathcal{F}$ represents either $\mathcal{D}^{-1}\check{R}$ or $(a\delta+2a\lambda)$. We will complete this step by applying Theorem 3.2 below to $F=\mathcal{D}^{-1}\mathcal{F}$ which, in view of (3.26) and Proposition 3.9, satisfies the assumptions in this result.

Theorem 3.2. *Assume that F is an S-tangent tensor field of appropriate order on $\mathcal{H}$ verifying $\mathcal{N}_2(F)<\infty$ and $\|r^{-\frac{1}{b}}\nabla\!\!\!/_L F\|_{L_t^b L_x^2}<\infty$ with $4<b<\infty$. Then we have*

(i) *There exists a 1-form E_0 such that*[3]

$$an\beta=\mathcal{D}_t\mathcal{D}^{-1}\check{R}+E_0 \quad \textit{with} \quad \|E_0\|_{\mathcal{P}^0}\lesssim\Delta_0^2+\mathcal{R}_0 \tag{3.75}$$

(ii) *There exists a decomposition $an\beta\cdot F=\mathcal{D}_tQ+E$, where Q and E are tensor fields of the same type as $an\beta\cdot F$ satisfying*

$$\lim_{t\to 0}\|Q\|_{L_x^\infty}=0 \tag{3.76}$$

and the estimates

$$\mathcal{N}_1(Q)\lesssim\Delta_0\mathcal{N}_2(F), \quad \|E\|_{\mathcal{P}^0}\lesssim\Delta_0\left(\mathcal{N}_2(F)+\|r^{-\frac{1}{b}}\nabla\!\!\!/_L F\|_{L_t^bL_x^2}\right). \tag{3.77}$$

(iii) *There exist tensor fields $\bar{Q}$ and $\bar{E}$ verifying (3.77) and*

$$\lim_{t\to 0}\|\bar{Q}\|_{L_x^\infty}<\infty \tag{3.78}$$

such that

$$\nabla\!\!\!/\,\mathcal{D}^{-1}(an\beta\cdot F)=\mathcal{D}_t\bar{Q}+\bar{E}, \tag{3.79}$$

where $\mathcal{D}$ denote either $\mathcal{D}_1$ or $\mathcal{D}_2$.

Proof. In view of (3.8), we have

$$an\beta=\mathcal{D}_t\mathcal{D}^{-1}\check{R}+C_0(\check{R})+Err. \tag{3.80}$$

This proves (i) with $E_0:=Err+C_0(\check{R})$ which, in view of (3.10) and (3.15), satisfies $\|E_0\|_{\mathcal{P}^0}\lesssim\Delta_0^2+\mathcal{R}_0$.

Next we prove (ii). We have from (3.80) that

$$an\beta\cdot F=(\mathcal{D}_t\mathcal{D}_1^{-1}\check{R}+Err+C_0(\check{R}))\cdot F=\mathcal{D}_t(\mathcal{D}_1^{-1}\check{R}\cdot F)+E_1^B+E_1^G,$$

where

$$E_1^B:=-\mathcal{D}_1^{-1}\check{R}\cdot\mathcal{D}_tF \quad \text{and} \quad E_1^G:=(Err+C_0(\check{R}))\cdot F.$$

By using [9, Eq.(4.7)], (3.15) and (3.10) we have

$$\|E_1^G\|_{\mathcal{P}^0}\lesssim\mathcal{N}_2(F)\left(\|Err\|_{\mathcal{P}^0}+\|C_0(\check{R})\|_{\mathcal{P}^0}\right)\lesssim(\Delta_0^2+\mathcal{R}_0)\mathcal{N}_2(F). \tag{3.81}$$

By using [9, Eq.(4.6)] and (3.25) we also have

$$\begin{aligned}\|E_1^B\|_{\mathcal{P}^0}&\lesssim\mathcal{N}_1(\mathcal{D}^{-1}\check{R})(\|r^{-\frac{1}{b}}\nabla\!\!\!/_LF\|_{L_t^bL_x^2}+\|r^{\frac{1}{2}}\nabla\!\!\!/\,\mathcal{D}_tF\|_{L_t^2L_x^2})\\&\lesssim(\mathcal{R}_0+\Delta_0^2)(\mathcal{N}_2(F)+\|r^{-\frac{1}{b}}\nabla\!\!\!/_LF\|_{L_t^bL_x^2}).\end{aligned}$$

Now we set

$$Q_1:=\mathcal{D}_1^{-1}\check{R}\cdot F \quad \text{and} \quad E_1:=E_1^B+E_1^G. \tag{3.82}$$

From the above estimates it follows that

$$\|E_1\|_{\mathcal{P}^0}\lesssim(\Delta_0^2+\mathcal{R}_0)(\mathcal{N}_2(F)+\|r^{-\frac{1}{b}}\nabla\!\!\!/_LF\|_{L_t^bL_x^2}).$$

---

[3]In Theorem 3.2 and the following proof, $\check{R}=(\check{\rho},-\check{\sigma})$ and $C_0(\check{R})=[\mathcal{D}_t,\mathcal{D}_1^{-1}](\check{\rho},-\check{\sigma})$.

In order to estimate $\mathcal{N}_1(Q_1)$, we consider $\|E_1\|_{L^2}$ first. By using Hölder's inequality, (Sob) and (SobM1), we can obtain

$$\|E_1^B\|_{L^2} = \|\mathcal{D}^{-1}\check{R}\cdot \nabla\!\!\!\!/_L F\|_{L^2} \lesssim \|\mathcal{D}^{-1}\check{R}\|_{L_t^\infty L_x^4}\|\nabla\!\!\!\!/_L F\|_{L_t^2 L_x^4}$$
$$\lesssim \mathcal{N}_1(\mathcal{D}^{-1}\check{R})(\|\nabla\!\!\!\!/\mathcal{D}_t F\|_{L^2} + \|r^{-\frac{1}{2}}\nabla\!\!\!\!/_L F\|_{L^2}),$$

and by using $\|E_1^G\|_{L^2(\mathcal{H})} \lesssim \|E_1^G\|_{\mathcal{P}^0}$ and (3.81) we can obtain

$$\|E_1^G\|_{L^2} \lesssim (\Delta_0^2 + \mathcal{R}_0)\mathcal{N}_2(F).$$

Therefore

$$\|E_1\|_{L^2} \lesssim (\Delta_0^2 + \mathcal{R}_0)\mathcal{N}_2(F). \tag{3.83}$$

Now we show

$$\mathcal{N}_1(Q_1) \lesssim \mathcal{N}_2(F)(\Delta_0^2 + \mathcal{R}_0). \tag{3.84}$$

By (Sob), (SobM1) and (3.25) we have

$$\|r^{-1}Q_1\|_{L^2(\mathcal{H})} \lesssim \|\mathcal{D}^{-1}\check{R}\|_{L_t^\infty L_x^4}\|r^{-1}F\|_{L_t^2 L_x^4} \lesssim (\Delta_0^2 + \mathcal{R}_0)\mathcal{N}_2(F).$$

With the help of $\mathcal{D}_t Q_1 = an\beta\cdot F - E_1$, (3.83), (1.2) and (SobM2), we can obtain

$$\|\nabla\!\!\!\!/_L Q_1\|_{L^2} \lesssim \|\beta\cdot F\|_{L_t^2 L_x^2} + \|E_1\|_{L^2} \lesssim (\Delta_0^2 + \mathcal{R}_0)\mathcal{N}_2(F).$$

Similar to [2, Section 6.12], we get $\|\nabla\!\!\!\!/ Q_1\|_{L_t^2 L_x^2} \lesssim (\Delta_0^2 + \mathcal{R}_0)\mathcal{N}_2(F)$. Hence (3.84) follows and the proof of (3.77) is complete.

By (3.82) and (SobM2) we have

$$\|Q_1\|_{L_x^\infty} \le \|F\|_{L_x^\infty}\|\mathcal{D}_1^{-1}\check{R}\|_{L_x^\infty} \lesssim r^{\frac{1}{2}}\|\mathcal{D}_1^{-1}\check{R}\|_{L_x^\infty}\mathcal{N}_2(F). \tag{3.85}$$

Since $\mathcal{N}_2(F) < \infty$ and $\lim_{t\to 0}\|\mathcal{D}_1^{-1}\check{R}\|_{L_x^\infty} < \infty$, we can obtain (3.76) by letting $t \to 0$ in (3.85). This completes the proof of (ii).

Finally we prove (iii) by using the iteration procedure in [2, Section 6.12]. Let $Q_0 := \mathcal{D}F$, then we can apply (ii) to construct recursively two sequences $\{Q_i\}$ and $\{E_i\}$ of S-tangent tensor fields such that

$$an\beta\cdot\mathcal{D}^{-1}Q_{i-1} = \mathcal{D}_t Q_i + E_i, \tag{3.86}$$

where $\mathcal{D}^{-1}$ denote either $\mathcal{D}_1^{-1}$ or $\mathcal{D}_2^{-1}$ and

$$\mathcal{N}_1(Q_i) \le C(\Delta_0^2 + \mathcal{R}_0)\mathcal{N}_2(\mathcal{D}^{-1}Q_{i-1}), \tag{3.87}$$

$$\|E_i\|_{\mathcal{P}^0} \le C(\Delta_0^2 + \mathcal{R}_0)\left(\mathcal{N}_2(\mathcal{D}^{-1}Q_{i-1}) + \|r^{-\frac{1}{b}}\nabla\!\!\!\!/_L\mathcal{D}^{-1}Q_{i-1}\|_{L_t^b L_x^2}\right). \tag{3.88}$$

Such Q_i and $E_i = E_i^B + E_i^G$ can be constructed as in the proof of (ii). Then for $i = 1, 2\cdots$,

$$Q_i = \mathcal{D}_1^{-1}\check{R}\cdot\mathcal{D}^{-1}Q_{i-1}, \qquad Q_0 = \mathcal{D}F, \tag{3.89}$$

$$E_i^B := -\mathcal{D}_1^{-1}\check{R}\cdot\mathcal{D}_t\mathcal{D}^{-1}Q_{i-1}, \qquad E_i^G := (Err + C_0(\check{R}))\cdot\mathcal{D}^{-1}Q_{i-1}. \tag{3.90}$$

In particular, Q_1 and E_1 are given by (3.82).

With the above definitions of Q_k and E_k, we have from (3.86) that

$$\nabla\!\!\!\!/\mathcal{D}^{-1}(an\beta\cdot F) = \mathcal{D}_t\bar{Q}_k + \nabla\!\!\!\!/\mathcal{D}^{-1}(\mathcal{D}_t Q_k) + \bar{E}_k,$$

where

$$\bar{Q}_k = \nabla\!\!\!\!/\mathcal{D}^{-1}(Q_1 + ... + Q_{k-1}) + Q_2 + ... + Q_k, \tag{3.91}$$
$$\bar{E}_k = [\nabla\!\!\!\!/\mathcal{D}^{-1}, \mathcal{D}_t]_g(Q_1 + ... + Q_{k-1}) + \nabla\!\!\!\!/\mathcal{D}^{-1}(E_1 + ... + E_k)$$
$$+ E_2 + ... + E_k.$$

By using [9, Lemma 3.1], it is easy to see from (3.87) that

$$\mathcal{N}_1(Q_k) \le (C(\Delta_0^2 + \mathcal{R}_0))^k\mathcal{N}_2(F). \tag{3.92}$$

Moreover we have

Proposition 3.12. *For $\{Q_k\}_{k=1}^\infty$ and $\{E_k\}_{k=1}^\infty$ constructed above there hold*

$$\|r^{-\frac{1}{b}}\nabla\!\!\!/_L\mathcal{D}^{-1}Q_k\|_{L_t^bL_x^2}\lesssim(\Delta_0^2+\mathcal{R}_0)(\mathcal{N}_2(\mathcal{D}^{-1}Q_{k-1})+\|\nabla\!\!\!/_L\mathcal{D}^{-1}Q_{k-1}\|_{L_t^bL_x^2}),\tag{3.93}$$

$$\|\nabla\!\!\!/\mathcal{D}^{-1}E_k\|_{\mathcal{P}^0}\lesssim\|E_k\|_{\mathcal{P}^0}+(\Delta_0^2+\mathcal{R}_0)(\mathcal{N}_2(\mathcal{D}^{-1}Q_{k-1})+\|\nabla\!\!\!/_L\mathcal{D}^{-1}Q_{k-1}\|_{L_t^bL_x^2}).\tag{3.94}$$

We will prove this result at the end of this section. We observe that [9, Lemma 3.1], (3.93), (3.87) and (3.88) clearly imply

$$\|E_k\|_{\mathcal{P}^0}\le(C(\Delta_0^2+\mathcal{R}_0))^k\left(\mathcal{N}_2(F)+\|r^{-\frac{1}{b}}\nabla\!\!\!/_LF\|_{L_t^bL_x^2}\right).\tag{3.95}$$

It then follows from (3.92), (3.94), (3.95) and (3.29) that

$$\mathcal{N}_1(\bar{Q}_k-\bar{Q}_j)\le\mathcal{N}_2(F)\sum_{j\le m\le k-1}(C(\Delta_0^2+\mathcal{R}_0))^m\lesssim(C(\Delta_0^2+\mathcal{R}_0))^j\mathcal{N}_2(F)$$

and

$$\begin{aligned}\|\bar{E}_k-\bar{E}_j\|_{\mathcal{P}^0}&\le\big(\mathcal{N}_2(F)+\|r^{-\frac{1}{b}}\nabla\!\!\!/_LF\|_{L_t^bL_x^2}\big)\sum_{j\le m\le k-1}(C(\Delta_0^2+\mathcal{R}_0))^m\\&\lesssim(C(\Delta_0^2+\mathcal{R}_0))^j\big(\mathcal{N}_2(F)+\|r^{-\frac{1}{b}}\nabla\!\!\!/_LF\|_{L_t^bL_x^2}\big).\end{aligned}$$

Therefore $\{\bar{Q}_k\}$ forms a Cauchy sequence relative to the norm $\mathcal{N}_1(\cdot)$, while $\{\bar{E}_k\}$ forms a Cauchy sequence relative to the $\mathcal{P}^0$ norm. Denote by $\bar{Q}$ and $\bar{E}$ their corresponding limits, we have

$$\mathcal{N}_1(\bar{Q})\lesssim(\Delta_0^2+\mathcal{R}_0)\mathcal{N}_2(F)\quad\text{and}\quad\|\bar{E}\|_{\mathcal{P}^0}\lesssim(\Delta_0^2+\mathcal{R}_0)\big(\mathcal{N}_2(F)+\|r^{-\frac{1}{b}}\nabla\!\!\!/_LF\|_{L_t^bL_x^2}\big).$$

By using [9, Proposition 3.4] we can see

$$\|\nabla\!\!\!/\mathcal{D}^{-1}(an\beta\cdot F)-\mathcal{D}_t\bar{Q}_k-\bar{E}_k\|_{L^2}=\|\nabla\!\!\!/\mathcal{D}^{-1}(\mathcal{D}_tQ_k)\|_{L^2}\lesssim\mathcal{N}_1(Q_k).$$

Letting $k\to+\infty$, by (3.92) we get

$$\|\nabla\!\!\!/\mathcal{D}^{-1}(an\beta\cdot F)-\mathcal{D}_t\bar{Q}-\bar{E}\|_{L_t^2L_x^2}=0.$$

Hence $\nabla\!\!\!/\mathcal{D}^{-1}(an\beta\cdot F)=\mathcal{D}_t\bar{Q}+\bar{E}$. This completes the proof of (3.79) in (iii). It remains to derive (3.78) whose proof is deferred to the Appendix. □

We conclude this section with the proof of Proposition 3.12.

Proof of Proposition 3.12. We first prove (3.94). By using (3.3) we have for $4<b<\infty$ and $1/2<q<1$ that

$$\|\nabla\!\!\!/\mathcal{D}^{-1}E_k\|_{\mathcal{P}^0}\lesssim\|E_k\|_{\mathcal{P}^0}+(\Delta_0^2+\mathcal{R}_0)\|\mathcal{D}^{-1}E_k\|^q_{L_t^bL_x^2}\|E_k\|^{1-q}_{L^2}.$$

Thus it suffices to show for $4<b<\infty$ that

$$\|r^{-\frac{1}{b}}\mathcal{D}^{-1}E_k\|_{L_t^bL_x^2}\lesssim(\Delta_0^2+\mathcal{R}_0)\left(\mathcal{N}_2(\mathcal{D}^{-1}Q_{k-1})+\|\nabla\!\!\!/_L\mathcal{D}^{-1}Q_{k-1}\|_{L_t^bL_x^2}\right).\tag{3.96}$$

By the constructions of Q_k and E_k, it suffices to show it for $k=1$. In view of $E_1=E_1^G+E_1^B$, this can be accomplished by applying Proposition 3.8 for $\|r^{-\frac{1}{b}}\mathcal{D}^{-1}E_1^G\|_{L_t^bL_x^2}$ and the estimate

$$\begin{aligned}\|r^{-\frac{1}{b}}\mathcal{D}^{-1}E_1^B\|_{L_t^bL_x^2}&\lesssim\|r^{\frac{1}{2}-\frac{1}{b}}E_1^B\|_{L_t^bL_x^{4/3}}\lesssim\|\mathcal{D}_1^{-1}\check{R}\|_{L_t^\infty L_x^4}\|\mathcal{D}_tF\|_{L_t^bL_x^2}\\&\lesssim\mathcal{N}_1(\mathcal{D}^{-1}\check{R})\|\mathcal{D}_tF\|_{L_t^bL_x^2}\lesssim(\Delta_0^2+\mathcal{R}_0)\|\nabla\!\!\!/_LF\|_{L_t^bL_x^2}.\end{aligned}$$

which follows from Proposition 3.1, Hölder inequality, (SobM1) and (3.25).

In order to prove (3.93), we first note that

$$\|r^{-\frac{1}{b}}\nabla\!\!\!/_L\mathcal{D}^{-1}Q_k\|_{L_t^bL_x^2}\lesssim\|r^{-\frac{1}{b}}[\mathcal{D}_t,\mathcal{D}^{-1}]Q_k\|_{L_t^bL_x^2}+\|r^{-\frac{1}{b}}\mathcal{D}^{-1}\mathcal{D}_tQ_k\|_{L_t^bL_x^2}.\tag{3.97}$$

By using (3.39) and (3.87), the first term on the right side of (3.97) can be estimated as

$$\begin{aligned}\|r^{-\frac{1}{b}}C_0(Q_k)\|_{L_t^bL_x^2} &\lesssim \mathcal{N}_1(r^{-\frac{1}{2}}\mathcal{D}^{-1}Q_k) \lesssim \mathcal{N}_2(r^{\frac{1}{2}}\mathcal{D}^{-1}Q_k)\\ &\lesssim \mathcal{N}_1(Q_k) \lesssim (\Delta_0^2+\mathcal{R}_0)\mathcal{N}_2(\mathcal{D}^{-1}Q_{k-1}),\end{aligned}$$

while by using (3.86), (3.96) and (3.38), the second term can be estimated as

$$\begin{aligned}\|r^{-\frac{1}{b}}\mathcal{D}^{-1}\mathcal{D}_tQ_k\|_{L_t^bL_x^2} &\lesssim \|r^{-\frac{1}{b}}\mathcal{D}^{-1}(an\beta\cdot\mathcal{D}^{-1}Q_{k-1}-E_k)\|_{L_t^bL_x^2}\\ &\lesssim \|r^{-\frac{1}{b}}\mathcal{D}^{-1}(an\beta\cdot\mathcal{D}^{-1}Q_{k-1})\|_{L_t^bL_x^2}+\|r^{-\frac{1}{b}}\mathcal{D}^{-1}E_k\|_{L_t^bL_x^2}\\ &\lesssim (\Delta_0^2+\mathcal{R}_0)(\mathcal{N}_2(\mathcal{D}^{-1}Q_{k-1})+\|\nabla\!\!\!/_L\mathcal{D}^{-1}Q_{k-1}\|_{L_t^bL_x^2}).\end{aligned}$$

Therefore (3.93) is proved. □

4. **Proof of Theorem 1.1**

We first prove Theorem 1.1(3). We use (3.7) and set

$$p' := {}^\star\mathcal{D}_1^{-1}\underline{\beta}, \quad e' = [{}^\star\mathcal{D}_1^{-1},\mathcal{D}_t]\underline{\beta} - \widetilde{Err} + anA\cdot\underline{A}.$$

Note that (3.15) implies $\|[\mathcal{D}_t, {}^\star\mathcal{D}_1^{-1}]\check{R}\|_{\mathcal{P}^0} \lesssim \Delta_0^2+\mathcal{R}_0$. This together with (3.10), [9, Eq.(4.10)] and (1.2) gives the decomposition result in Theorem 1.1 (3). In the following we will give the proof of Theorem 1.1 (1) and (2).

4.1. **Decomposition for $\nabla\!\!\!/(na\underline{\zeta})$.** From (3.8) we have

$$\begin{aligned}\nabla\!\!\!/\mathcal{D}_1^{-1}\left(an(\rho,\sigma)\right) &= \nabla\!\!\!/\mathcal{D}_1^{-1\star}\mathcal{D}_1^{-1}\mathcal{D}_t\underline{\beta} - \nabla\!\!\!/\mathcal{D}_1^{-1}\widetilde{E}rr\\ &= \mathcal{D}_t\nabla\!\!\!/\mathcal{D}^{-2}\check{R} + C(\check{R}) + \nabla\!\!\!/\mathcal{D}^{-1}\mathfrak{F}.\end{aligned}$$

By Proposition 3.2, there exists P and E such that $C(\check{R}) = \mathcal{D}_tP+E$. Let $\tilde{P} = P+\nabla\!\!\!/\mathcal{D}^{-2}\check{R}$ and $\tilde{E} = \nabla\!\!\!/\mathcal{D}^{-1}\mathfrak{F}+E$. It follows from (3.26) and (3.53) that

$$\nabla\!\!\!/\mathcal{D}_1^{-1}\left(an(\rho,\sigma)\right) = \mathcal{D}_t\tilde{P}+\tilde{E}, \quad \mathcal{N}_1(\tilde{P})+\|\tilde{E}\|_{\mathcal{P}^0} \lesssim \Delta_0^2+\mathcal{R}_0.$$

In view of (3.67), we obtain

$$\begin{aligned}\nabla\!\!\!/(an\underline{\zeta}) &= \nabla\!\!\!/\mathcal{D}_1^{-1}(\mathcal{D}_t(a\delta+2a\lambda)) + \nabla\!\!\!/\mathcal{D}_1^{-1}(an(\rho,\sigma)) + \nabla\!\!\!/\mathcal{D}_1^{-1}\mathrm{err}_1,\\ &= \mathcal{D}_t\nabla\!\!\!/\mathcal{D}_1^{-1}(a\delta+2a\lambda) + [\nabla\!\!\!/\mathcal{D}_1^{-1},\mathcal{D}_t](a\delta+2a\lambda) + \nabla\!\!\!/\mathcal{D}_1^{-1}\mathrm{err}_1 + \mathcal{D}_t\tilde{P}+\tilde{E}.\end{aligned} \tag{4.1}$$

According to Proposition 3.2, there exists P' an E' such that

$$[\nabla\!\!\!/\mathcal{D}_1^{-1},\mathcal{D}_t](a\delta+2a\lambda) = \mathcal{D}_tP'+E', \text{ with } \mathcal{N}_1(P')+\|E'\|_{\mathcal{P}^0} \lesssim \Delta_0^2+\mathcal{R}_0.$$

Let $P'' = P' + \nabla\!\!\!/\mathcal{D}_1^{-1}(a\delta+2a\lambda)$, we conclude that

$$\mathcal{D}_t\nabla\!\!\!/\mathcal{D}_1^{-1}(a\delta+2a\lambda) + [\nabla\!\!\!/\mathcal{D}_1^{-1},\mathcal{D}_t](a\delta+2a\lambda) = \mathcal{D}_tP''+E'.$$

By (3.26) we have $\mathcal{N}_1(P'') \lesssim \Delta_0^2+\mathcal{R}_0$. Hence we obtain the decomposition

$$\nabla\!\!\!/(an\underline{\zeta}) = \mathcal{D}_tP_3+E_3,$$

where $E_3 = E'+\nabla\!\!\!/\mathcal{D}_1^{-1}\mathrm{err}_1+\tilde{E}$ and $P_3 = P''+\tilde{P}$. Applying Lemma 3.6 for $\|\nabla\!\!\!/\mathcal{D}_1^{-1}\mathrm{err}_1\|_{\mathcal{P}^0}$, we can conclude from the above estimates that

$$\|E_3\|_{\mathcal{P}^0} \lesssim \Delta_0^2+\mathcal{R}_0, \qquad \mathcal{N}_1(P_3) \lesssim \Delta_0^2+\mathcal{R}_0.$$

4.2. **Decomposition for $\nabla\!\!\!/(na\hat{\chi})$.** We first have from [9, Eq.(2.6)] that

$$\begin{aligned}\text{div}(an\hat{\chi}) &= \frac{1}{2}an\nabla\!\!\!/\text{tr}\chi + \frac{1}{2}an\text{tr}\chi\zeta - an\beta + an\underline{\zeta}\hat{\chi}\\ &= anM + anA\cdot A + r^{-1}an\zeta - an\beta\end{aligned}$$

Since (3.8) gives $an\beta = \mathcal{D}_1^{-1}\mathcal{D}_t(\check{\rho}, -\check{\sigma}) - \mathfrak{F}$ with $\mathfrak{F} = Err$, we can obtain

$$\text{div}(an\hat{\chi}) = anM + anA\cdot A + r^{-1}an\zeta - \mathcal{D}_1^{-1}\mathcal{D}_t(\check{\rho}, -\check{\sigma}) + \mathfrak{F}$$

which implies

$$an\hat{\chi} = -\mathcal{D}_2^{-1}\mathcal{D}_1^{-1}\mathcal{D}_t(\check{\rho}, -\check{\sigma}) + \mathcal{D}_2^{-1}(\mathfrak{F} + anM + anA\cdot A + r^{-1}an\zeta).$$

Setting $\mathcal{D}^{-2} = \mathcal{D}_2^{-1}\mathcal{D}_1^{-1}$ and $\mathcal{D}^{-1} = \mathcal{D}_2^{-1}$, we obtain after taking covariant derivatives

$$\nabla\!\!\!/(an\hat{\chi}) = -\nabla\!\!\!/\mathcal{D}^{-2}\mathcal{D}_t\check{R} + F + \nabla\!\!\!/\mathcal{D}^{-1}(anM), \tag{4.2}$$

where $F = \nabla\!\!\!/\mathcal{D}^{-1}(\mathfrak{F} + anA\cdot A + r^{-1}an\zeta)$ and $M = \nabla\!\!\!/\text{tr}\chi$.

By using (3.53) and [9, Eq.(6.2), Eq.(4.9)] we obtain

$$\|F\|_{\mathcal{P}^0} \lesssim \Delta_0^2 + \mathcal{R}_0. \tag{4.3}$$

For the first term on the right hand side of (4.2), we may use the notations in (3.11) to write

$$\nabla\!\!\!/\mathcal{D}^{-2}\mathcal{D}_t(\check{R}) = \mathcal{D}_t(\nabla\!\!\!/\mathcal{D}^{-2}\check{R}) + C(\check{R}).$$

where, by Proposition 3.2, there exist tensors P' and E' so that $C(\check{R}) = \mathcal{D}_tP' + E'$ and

$$\mathcal{N}_1(P') + \|E'\|_{\mathcal{P}^0} \lesssim \Delta_0^2 + \mathcal{R}_0, \quad \lim_{t\to 0} r\|P'\|_{L_x^\infty} = 0. \tag{4.4}$$

Thus (4.2) becomes

$$\nabla\!\!\!/(an\hat{\chi}) = \mathcal{D}_tP + \nabla\!\!\!/\mathcal{D}^{-1}(anM) + E \tag{4.5}$$

where $P = \nabla\!\!\!/\mathcal{D}^{-2}\check{R} + P'$ and $E = F + E'$. By using (3.26),(4.3) and (4.4)

$$\mathcal{N}_1(P) + \|E\|_{\mathcal{P}^0} \lesssim \Delta_0^2 + \mathcal{R}_0, \quad \lim_{t\to 0} r\|P\|_{L_x^\infty} = 0. \tag{4.6}$$

4.3. **Decomposition for $\nabla\!\!\!/(an\zeta)$.** By using [9, Eq.(2.8) and Eq.(2.9)] we have

$$\begin{aligned}\text{div}(an\zeta) &= an\underline{\zeta}\cdot\zeta - an\mu - an\check{\rho} + \frac{1}{2}a^2n\delta\text{tr}\chi,\\ \text{curl}\,(an\zeta) &= an\check{\sigma} + an(\zeta + \underline{\zeta})\wedge\zeta,\end{aligned}$$

which symbolically can be written as $\mathcal{D}_1(an\zeta) = anA\cdot A - an(\mu, 0) - an(\check{\rho}, -\check{\sigma}) + an(a\delta\text{tr}\chi, 0)$. Hence

$$an\zeta = -\mathcal{D}_1^{-1}(an(\check{\rho}, -\check{\sigma})) + \mathcal{D}_1^{-1}(anA\cdot\underline{A}) - \mathcal{D}_1^{-1}(an(\mu, 0)) + \mathcal{D}_1^{-1}(an(r^{-1}\underline{A}, 0)).$$

Let J be the involution $(\rho, \sigma) \to (-\rho, \sigma)$ and let $\mathfrak{F} = \widetilde{Err}$ given by (3.8). Then

$$\begin{aligned}\nabla\!\!\!/(an\zeta) =&\nabla\!\!\!/\mathcal{D}_1^{-1}\cdot J\cdot{}^\star\mathcal{D}_1^{-1}\mathcal{D}_t\underline{\beta} + \nabla\!\!\!/\mathcal{D}_1^{-1}\cdot J\cdot\mathfrak{F} - \nabla\!\!\!/\mathcal{D}_1^{-1}(an\mu, 0)\\ &+ \nabla\!\!\!/\mathcal{D}_1^{-1}(anA\cdot A) + \nabla\!\!\!/\mathcal{D}_1^{-1}(r^{-1}an\underline{A}, 0).\end{aligned}$$

Set $\mathcal{D}^{-2} = \mathcal{D}_1^{-1}\cdot J\cdot{}^\star\mathcal{D}_1^{-1}$ and $\mathcal{D}^{-1} = \mathcal{D}_1^{-1}$. By using (3.11) we get

$$\nabla\!\!\!/(an\zeta) = \mathcal{D}_t\nabla\!\!\!/\mathcal{D}^{-2}\underline{\beta} + C(\check{R}) + \nabla\!\!\!/\mathcal{D}^{-1}(anM) + F,$$

where $M = (\mu, 0)$ and $F = \nabla\!\!\!/\mathcal{D}^{-1}\big(\mathfrak{F} + an(A\cdot\underline{A} + r^{-1}\underline{A})\big)$. By (3.53) and [9, Eq.(6.2)], we can derive $\|F\|_{\mathcal{P}^0} \lesssim \Delta_0^2 + \mathcal{R}_0$. In view of Proposition 3.2, we have $C(\check{R}) = \mathcal{D}_t\tilde{P} + \tilde{E}$ for some S tangent tensor fields $\tilde{P}$ and $\tilde{E}$. Let$E = \tilde{E} + F$ and $P = \tilde{P} + \nabla\!\!\!/\mathcal{D}^{-2}\underline{\beta}$, we thus obtain

$$\nabla\!\!\!/(an\zeta) = \mathcal{D}_tP + \nabla\!\!\!/\mathcal{D}^{-1}(anM) + E$$

and

$$\mathcal{N}_1(P) + \|E\|_{\mathcal{P}^0} \leq \Delta_0^2 + \mathcal{R}_0, \quad \lim_{t\to 0} r\|P\|_{L_x^\infty} = 0. \tag{4.7}$$

5. Appendix

In this section we will first prove Lemma 3.3 and derive a few useful consequences. We then use these results to complete the proof of (3.78) in Theorem 3.2. We will frequently employ inequalities (FB), (FBB), (FBD), (FD), (WB) and (SB) in [9].

5.1. **Proof of Lemma 3.3.** We will regard κ and ι as elements of A. By using (1.2) and [9, Lem 2.8], or more precisely the estimates

$$\|\underline{K}\|_{L_t^2 L_x^2} + \|\beta\|_{L_t^2 L_x^2} + \mathcal{N}_1(A) \lesssim \Delta_0^2 + \mathcal{R}_0,$$

we can adapt the approach in [4, Lemma 5.3] and [6, Chapter 9] to derive the following estimates for the commutator $[P_k, \mathcal{D}_t]$ where P_k denote the GLP projections.

Lemma 5.1. *For any S tangent tensor fields F and any $q < 2$ sufficiently close to $q = 2$, there hold*

$$\|r^{\frac{1}{2}-\frac{1}{q}}[P_k, \mathcal{D}_t]F\|_{L_t^q L_x^2} + 2^{-k}\|r^{\frac{3}{2}-\frac{1}{q}}\nabla\!\!\!/[P_k, \mathcal{D}_t]F\|_{L_t^q L_x^2} \lesssim 2^{-\frac{k}{2}+}\mathcal{N}_1(F), \tag{5.1}$$

$$\|r^{-\frac{1}{2}}[P_k, \mathcal{D}_t]F\|_{L_t^1 L_x^2} + 2^{-k}\|r^{\frac{1}{2}}\nabla\!\!\!/[P_k, \mathcal{D}_t]F\|_{L_t^1 L_x^2} \lesssim 2^{-k+}\mathcal{N}_1(F). \tag{5.2}$$

Now we are ready to prove Lemma 3.3.

Proof of Lemma 3.3. The result is trivial when $q = 2$. So we only need to consider the case $q > 2$. It is easy to see

$$\|r^{-\frac{1}{2}-\frac{1}{q}}P_k F\|_{L_t^q L_x^2} \lesssim \|r^{-1}P_k F\|_{L_t^2 L_x^2}^{\frac{2}{q}} \|r^{-\frac{1}{2}}P_k F\|_{L_t^\infty L_x^2}^{\frac{q-2}{q}}.$$

By integrating along an arbitrary null geodesic through the vertex of $\mathcal{H}$ we have

$$\begin{aligned}\|r^{-\frac{1}{2}}P_k F\|_{L_x^2 L_t^\infty}^2 &\lesssim \|P_k(\mathcal{D}_t F)\|_{L_t^2 L_x^2}\|r^{-1}P_k F\|_{L_t^2 L_x^2} + \|r^{-1}[P_k, \mathcal{D}_t]F \cdot P_k F\|_{L_t^1 L_x^1} \\ &\quad + \|r^{-1}P_k F\|_{L_t^2 L_x^2}^2.\end{aligned} \tag{5.3}$$

Let $1 \le q' < 2$ be such that $1/q' + 1/q = 1$. With the help of Lemma 5.1 we can obtain

$$\begin{aligned}\|r^{-1}[P_k, \mathcal{D}_t]F \cdot P_k F\|_{L_t^1 L_x^1} &\lesssim \|r^{\frac{1}{2}-\frac{1}{q}}[P_k, \mathcal{D}_t]F\|_{L_t^{q'} L_x^2}\|r^{-\frac{1}{q}-\frac{1}{2}}P_k F\|_{L_t^q L_x^2} \\ &\lesssim 2^{-\frac{k}{2}+}\mathcal{N}_1(F)\|r^{-\frac{1}{q}-\frac{1}{2}}P_k F\|_{L_t^q L_x^2}\end{aligned} \tag{5.4}$$

Combining the above estimates it yields

$$\begin{aligned}\|r^{-\frac{1}{2}-\frac{1}{q}}P_k F\|_{L_t^q L_x^2}^q &\lesssim \|r^{-1}P_k F\|_{L_t^2 L_x^2}^2\Big(\|P_k(\mathcal{D}_t F)\|_{L_t^2 L_x^2}\|r^{-1}P_k F\|_{L_t^2 L_x^2} \\ &\quad + 2^{-\frac{k}{2}+}\mathcal{N}_1(F)\|r^{-\frac{1}{q}-\frac{1}{2}}P_k F\|_{L_t^q L_x^2} + \|r^{-1}P_k F\|_{L_t^2 L_x^2}^2\Big)^{\frac{q}{2}-1}.\end{aligned} \tag{5.5}$$

This together with (FB), gives for any $0 \le \alpha < \frac{1/2-1/q}{q/2+1}$

$$\|r^{-\frac{1}{2}-\frac{1}{q}}P_k F\|_{L_t^q L_x^2} \lesssim 2^{-k(\frac{1}{2}+\frac{1}{q})}(1 + 2^{-\alpha k})\mathcal{N}_1(F).$$

which in particular implies (3.35).

In order to derive (3.36), we combine (5.3), (5.4) and (3.35) with $2 < q < \infty$, and use (FB) to obtain

$$\|r^{-\frac{1}{2}}P_k F\|_{L_t^\infty L_x^2} \lesssim 2^{-\frac{1}{2}k}\mathcal{N}_1(F).$$

With the help of (Sob) and (FBB) we have

$$\begin{aligned}\|r^{-\frac{1}{q}}F_k\|_{L_t^q L_x^4} &\lesssim \|r^{\frac{1}{2}-\frac{1}{q}}\nabla\!\!\!/ F_k\|_{L_t^q L_x^2}^{\frac{1}{2}}\|r^{-\frac{1}{2}-\frac{1}{q}}F_k\|_{L_t^q L_x^2}^{\frac{1}{2}} + \|r^{-\frac{1}{2}-\frac{1}{q}}F_k\|_{L_t^q L_x^2} \\ &\lesssim (2^{\frac{k}{2}} + 1)\|r^{-\frac{1}{2}-\frac{1}{q}}P_k F\|_{L_t^q L_x^2}\end{aligned}$$

which together with (3.35) gives (3.36). □

Lemma 5.2. *Let $\mathcal{D}^{-1}$ denote one of the operators $\mathcal{D}_1^{-1}$, $\mathcal{D}_2^{-1}$ and ${}^\star\mathcal{D}_1^{-1}$. Then for appropriate S tangent tensor field G and $k>0$ there hold*

$$\|\mathcal{D}^{-1}P_k^2 G\|_{L_t^\infty L_x^2} \lesssim r^{\frac{3}{2}} 2^{-\frac{3k}{2}} \mathcal{N}_1(G), \tag{5.6}$$

$$\|\nabla\!\!\!\!/ \mathcal{D}^{-1}P_k^2 G\|_{L_t^\infty L_x^4} \lesssim \mathcal{N}_1(G)\Big(1+2^{-\frac{k}{2}} r^{\frac{1}{2}} \|\underline{K}\|_{L_x^2}^{\frac{1}{2}}\Big). \tag{5.7}$$

Proof. By using [9, Lemma 4.3] and (3.35) we can obtain (5.6). Now we consider (5.7). For any S tangent tensor fields F, we define

$$\|F\|_{H_x^1} = \|\nabla\!\!\!\!/ F\|_{L^2(S)} + \|r^{-1}F\|_{L^2(S)}.$$

By the Böchner identity in [3] there holds

$$\|\nabla\!\!\!\!/^2 F\|_{L_x^2} \lesssim \|\Delta\!\!\!\!/ F\|_{L_x^2} + \|\underline{K}\cdot F\|_{L_x^2} + \|\underline{K}\|_{L_x^2}^{\frac{1}{2}} \|\nabla\!\!\!\!/ F\|_{L_x^4} + r^{-1}\|F\|_{H_x^1},$$

and by Sobolev embedding (see [3, Section 4] or [6, (3.38)]) we have

$$\|F\|_{L_x^\infty} \lesssim r^{\frac{1}{p}} \|\nabla\!\!\!\!/^2 F\|_{L_x^2}^{\frac{1}{p}} \|F\|_{H_x^1}^{\frac{p-1}{p}} + \|F\|_{H_x^1}, \quad 2<p<\infty. \tag{5.8}$$

It is easy to observe from [1, page 38] that symbolically

$$\Delta\!\!\!\!/ = {}^\star\mathcal{D}\mathcal{D} \pm (\underline{K} + r^{-2} Id). \tag{5.9}$$

Hence with $2<p<\infty$,

$$\|\nabla\!\!\!\!/^2 F\|_{L_x^2} \lesssim \|{}^\star\mathcal{D}\mathcal{D}F\|_{L_x^2} + \|\underline{K}\|_{L_x^2}^{\frac{p}{p-1}} r^{\frac{1}{p-1}} \|F\|_{H_x^1} + \|\underline{K}\|_{L_x^2}\|\nabla\!\!\!\!/ F\|_{L_x^2} + r^{-1}\|F\|_{H_x^1}. \tag{5.10}$$

Applying this inequality to $F=\mathcal{D}^{-1}H$ and using [9, Proposition 3.4] it yields

$$\begin{aligned}\|\nabla\!\!\!\!/^2 \mathcal{D}^{-1}H\|_{L_x^2} &\lesssim \|{}^\star\mathcal{D}H\|_{L_x^2} + \|\underline{K}\|_{L_x^2}^{\frac{p}{p-1}} r^{\frac{1}{p-1}} \|\mathcal{D}^{-1}H\|_{H_x^1} + \|\underline{K}\|_{L_x^2}\|\nabla\!\!\!\!/ \mathcal{D}^{-1}H\|_{L_x^2}\\ &\quad + r^{-1}\|\mathcal{D}^{-1}H\|_{H_x^1}\\ &\lesssim \|{}^\star\mathcal{D}H\|_{L_x^2} + \|\underline{K}\|_{L_x^2}^{\frac{p}{p-1}} r^{\frac{1}{p-1}} \|H\|_{L_x^2} + \Big(\|\underline{K}\|_{L_x^2} + r^{-1}\Big)\|H\|_{L_x^2}.\end{aligned}$$

Now we set $H=P_k^2 G$. It then follows from (FBB) and [9, Proposition 3.4 and Lemma 4.3] that

$$\begin{aligned}\|\nabla\!\!\!\!/^2 \mathcal{D}^{-1}P_k^2 G\|_{L_x^2} &\lesssim \|{}^\star\mathcal{D}P_k^2 G\|_{L_x^2} + (\|\underline{K}\|_{L_x^2} + r^{-1})\|P_k^2 G\|_{L_x^2}\\ &\lesssim \Big((2^k+1)r^{-1} + \|\underline{K}\|_{L_x^2}\Big)\|P_k G\|_{L_x^2}.\end{aligned}$$

With the help of (Sob), [9, Proposition 3.4] and (3.35), we derive

$$\begin{aligned}\|\nabla\!\!\!\!/ \mathcal{D}^{-1}P_k^2 G\|_{L_x^4} &\lesssim \|\nabla\!\!\!\!/^2 \mathcal{D}^{-1}P_k^2 G\|_{L_x^2}^{\frac{1}{2}} \|\nabla\!\!\!\!/ \mathcal{D}^{-1}P_k^2 G\|_{L_x^2}^{\frac{1}{2}} + r^{-\frac{1}{2}}\|\nabla\!\!\!\!/ \mathcal{D}^{-1}P_k^2 G\|_{L_x^2}\\ &\lesssim (1+2^{-\frac{k}{2}} r^{\frac{1}{2}} \|\underline{K}\|_{L_x^2}^{\frac{1}{2}})\mathcal{N}_1(G)\end{aligned}$$

which is the desired estimate. □

5.2. **Proof of (3.78) in Theorem 3.2.** We first prove (3.78) by assuming the following two results in which the definition of the Besov norms $B_{2,1}^a$ can be found in [9, Eq.(4.3)].

Lemma 5.3. *Denote by $\mathcal{D}^{-1}G$ one of the terms $\mathcal{D}_1^{-1}G$, $\mathcal{D}_2^{-1}G$ and ${}^\star\mathcal{D}_1^{-1}G$ for appropriate S tangent tensor G. Let $F=\mathcal{D}^{-1}\check{R}\cdot\mathcal{D}^{-1}G$, we have*

$$\|\nabla\!\!\!\!/ F\|_{B_{2,1}^0} \lesssim \mathcal{N}_1(G)\left(\Delta_0^2 + \mathcal{R}_0 + c_0 r^{\frac{1}{2}}(\|\mathcal{D}^{-1}\check{R}\|_{L_x^\infty} + \|\check{R}\|_{L_x^2} + r^2\|\nabla\!\!\!\!/ \check{R}\|_{L_x^\infty})\right)$$

where c_0 depends on $\|r\underline{K}\|_{L_x^\infty} + \underline{K}_{\alpha_0}$.

Lemma 5.4. *Let $\mathcal{D}^{-1}$ denotes one of the operators $\mathcal{D}_1^{-1}$, $\mathcal{D}_2^{-1}$ and ${}^\star\mathcal{D}_1^{-1}$. For S tangent tensor H, there hold*

$$\|\nabla\!\!\!/ \mathcal{D}^{-1} H\|_{B^1_{2,1}} \lesssim \|{}^\star\mathcal{D} H\|_{B^0_{2,1}} + \|H\|_{L^2_\omega} + c_0 r^{\frac{1}{2}} \mathcal{N}_1(H), \tag{5.11}$$

$$\|\nabla\!\!\!/ \mathcal{D}^{-1} H\|_{L^\infty_x} \le \|{}^\star\mathcal{D} H\|_{B^0_{2,1}} + (c_0 r^\theta + 1)(\|\nabla\!\!\!/ H\|_{L^2_x} + \|H\|_{L^2_\omega} + c_0 r^{\frac{1}{2}} \mathcal{N}_1(H)). \tag{5.12}$$

where c_0 is depending on the quantity $r\|\underline{K}\|_{L^\infty_x} + \underline{K}_{\alpha_0} + r\|\nabla\!\!\!/ \underline{K}\|_{L^2_x}$, and $\theta > 0$ is very close to 0.

Let $\{Q_i\}$ be the sequence of S-tangent tensor defined in the proof of Theorem 3.2. We set $Q = \sum_{i=1}^\infty Q_i$ and $\tilde{Q} = \nabla\!\!\!/ \mathcal{D}^{-1} Q$. It is easy to see $\bar{Q} = \tilde{Q} + Q - Q_1$ since $\bar{Q}$ is the limit of the Cauchy sequence $\{\bar{Q}_k\}$ contained in (3.91). Since (3.76) implies $\lim_{t\to 0} \|Q_1\|_{L^\infty_\omega} = 0$, (3.78) can be proved by establishing

$$\lim_{t\to 0} \|Q\|_{L^\infty_x} = 0, \quad \text{and} \quad \lim_{t\to 0} \|\tilde{Q}\|_{L^\infty_x} < \infty. \tag{5.13}$$

In view of (3.89), we have from (SobM2), [9, Lemma 3.1] and (3.92) that

$$\begin{aligned} \|Q_i\|_{L^\infty_\omega} &\le \|\mathcal{D}_1^{-1} \check{R}\|_{L^\infty_\omega} \|\mathcal{D}^{-1} Q_{i-1}\|_{L^\infty_\omega} \lesssim r^{\frac{1}{2}} \|\mathcal{D}_1^{-1} \check{R}\|_{L^\infty_\omega} \mathcal{N}_2(\mathcal{D}^{-1} Q_{i-1}) \\ &\lesssim r^{\frac{1}{2}} \mathcal{N}_1(Q_{i-1}) \|\mathcal{D}_1^{-1} \check{R}\|_{L^\infty_\omega} \lesssim r^{\frac{1}{2}} (C(\Delta_0^2 + \mathcal{R}_0))^{i-1} \mathcal{N}_2(F) \|\mathcal{D}_1^{-1} \check{R}\|_{L^\infty_\omega}. \end{aligned}$$

Summing over $i \ge 1$, we have for sufficiently small $(\Delta_0^2 + \mathcal{R}_0)$ that

$$\|Q\|_{L^\infty_\omega} \lesssim r^{\frac{1}{2}} \mathcal{N}_2(F) \|\mathcal{D}_1^{-1} \check{R}\|_{L^\infty_\omega}.$$

Noting that $\lim_{t\to 0} \|\mathcal{D}_1^{-1} \check{R}\|_{L^\infty_\omega} < \infty$ and $\mathcal{N}_2(F) < \infty$, we obtain $\lim_{t\to 0} \|Q\|_{L^\infty_x} = 0$.

It remains to prove the second part of (5.13). By (5.12) we have

$$\|\tilde{Q}\|_{L^\infty_x} \le \|\nabla\!\!\!/ Q\|_{B^0_{2,1}} + (c_0 r^\theta + 1) \left(\|\nabla\!\!\!/ Q\|_{L^2_x} + \|Q\|_{L^2_\omega} + c_0 r^{\frac{1}{2}} \mathcal{N}_1(Q) \right). \tag{5.14}$$

Recall the definition of Q_i in (3.89). For each $Q_i = \mathcal{D}^{-1} \check{R} \cdot \mathcal{D}^{-1} Q_{i-1}$, we use Lemma 5.3 to obtain

$$\begin{aligned} \|\nabla\!\!\!/ Q_i\|_{B^0_{2,1}} &\lesssim c_0 \mathcal{N}_1(Q_{i-1}) r^{\frac{1}{2}} \left(\|\mathcal{D}^{-1} \check{R}\|_{L^\infty_x} + \|\check{R}\|_{L^2_x} + r^2 \|\mathcal{D} \check{R}\|_{L^\infty_x} \right) \\ &\quad + \mathcal{N}_1(Q_{i-1})(\Delta_0^2 + \mathcal{R}_0). \end{aligned}$$

In view of (3.92), summing over $i \ge 1$ gives

$$\begin{aligned} \|\nabla\!\!\!/ Q\|_{B^0_{2,1}} &\lesssim \sum_{i\ge 1} \|\nabla\!\!\!/ Q_i\|_{B^0_{2,1}} \\ &\lesssim c_0 \sum_{i\ge 1} \left(C(\Delta_0^2 + \mathcal{R}_0) \right)^{i-1} \mathcal{N}_2(F) r^{\frac{1}{2}} \left(\|\mathcal{D}^{-1} \check{R}\|_{L^\infty_x} + \|\check{R}\|_{L^2_x} + r^2 \|\mathcal{D} \check{R}\|_{L^\infty_x} \right) \\ &\quad + \sum_{i\ge 1} (C(\Delta_0^2 + \mathcal{R}_0))^{i-1} \mathcal{N}_2(F)(\Delta_0^2 + \mathcal{R}_0). \end{aligned}$$

Hence

$$\begin{aligned} \|\nabla\!\!\!/ Q\|_{B^0_{2,1}} &\lesssim c_0 \mathcal{N}_2(F) r^{\frac{1}{2}} \left(\|\mathcal{D}^{-1} \check{R}\|_{L^\infty_x} + \|\check{R}\|_{L^2_x} + r^2 \|\mathcal{D} \check{R}\|_{L^\infty_x} \right) \\ &\quad + \mathcal{N}_2(F)(\Delta_0^2 + \mathcal{R}_0). \end{aligned} \tag{5.15}$$

By (3.89), (SobM1), [9, Lemma 3.1], (3.92) and (3.25), we also have

$$\begin{aligned} \|Q_i\|_{L^2_\omega} &\lesssim r^{-1} \|\mathcal{D}_1^{-1} \check{R}\|_{L^4_x} \|\mathcal{D}^{-1} Q_{i-1}\|_{L^4_x} \lesssim \mathcal{N}_1(\mathcal{D}^{-1} \check{R}) r^{-1} \mathcal{N}_1(\mathcal{D}^{-1} Q_{i-1}) \\ &\lesssim \mathcal{N}_1(Q_{i-1})(\Delta_0^2 + \mathcal{R}_0) \lesssim (C(\Delta_0^2 + \mathcal{R}_0))^{i-1} (\Delta_0^2 + \mathcal{R}_0) \mathcal{N}_2(F). \end{aligned}$$

Therefore

$$\|Q\|_{L^2_\omega} \lesssim (\Delta_0^2 + \mathcal{R}_0) \mathcal{N}_2(F). \tag{5.16}$$

It is easy to derive from $Q=\sum_{i\geq 1} Q_i$ and (3.92) that

$$\mathcal{N}_1(Q)\lesssim(\Delta_0^2+\mathcal{R}_0)\mathcal{N}_2(F). \tag{5.17}$$

Thus, in view of (5.14), we can combine the estimates (5.15), (5.16) and (5.17) to obtain

$$\lim_{t\to 0}\|\tilde{Q}\|_{L_x^\infty}\lesssim\mathcal{N}_2(F)(\Delta_0^2+\mathcal{R}_0)<\infty$$

which is as desired.

Proof of Lemma 5.3. We first estimate $\|\nabla\!\!\!/ F\|_{L_x^2}$. We have

$$\begin{aligned}\|\nabla\!\!\!/ F\|_{L_x^2}&=\|\nabla\!\!\!/(\mathcal{D}^{-1}\check{R}\cdot\mathcal{D}^{-1}G)\|_{L_x^2}\\&\lesssim\|\nabla\!\!\!/\mathcal{D}^{-1}\check{R}\|_{L_x^2}\|\mathcal{D}^{-1}G\|_{L_x^\infty}+\|\mathcal{D}^{-1}\check{R}\|_{L_x^4}\|\nabla\!\!\!/ D^{-1}G\|_{L_x^4}.\end{aligned}$$

From (SobM2), (SobM1), [9, Proposition 3.4 and Lemma 3.1] and (3.25) it follows

$$\begin{aligned}\|\nabla\!\!\!/ F\|_{L_x^2}&\lesssim r^{\frac12}\|\check{R}\|_{L_x^2}\mathcal{N}_2(\mathcal{D}^{-1}G)+\mathcal{N}_1(\mathcal{D}^{-1}\check{R})\mathcal{N}_1(\nabla\!\!\!/\mathcal{D}^{-1}G)\\&\lesssim\left(r^{\frac12}\|\check{R}\|_{L_x^2}+\mathcal{N}_1(\mathcal{D}^{-1}\check{R})\right)\mathcal{N}_1(G)\\&\lesssim\left(r^{\frac12}\|\check{R}\|_{L_x^2}+\Delta_0^2+\mathcal{R}_0\right)\mathcal{N}_1(G).\end{aligned}$$

Next we prove

$$\sum_{k>0}\|P_k\nabla\!\!\!/(\mathcal{D}^{-1}\check{R}\cdot\mathcal{D}^{-1}G)\|_{L_x^2}\lesssim c_0\mathcal{N}_1(G)r^{\frac12}\left(\|\mathcal{D}^{-1}\check{R}\|_{L_x^\infty}+\|\check{R}\|_{L_x^2}+r^2\|\nabla\!\!\!/\check{R}\|_{L_x^\infty}\right). \tag{5.18}$$

To this end, we employ the GLP decomposition to write $G=\sum_{m>0}P_m^2G+P_{\leq 0}G+U(\infty)G$. For simplicity of presentation, we will consider only the high frequency term $\sum_{m>0}P_m^2G$. The other two terms can be treated similarly as in *Case 2.*

Case 1: $k<m$. By using (FBB) and (5.6) we can obtain

$$\|P_k\nabla\!\!\!/(\mathcal{D}^{-1}\check{R}\cdot\mathcal{D}^{-1}G_m)\|_{L_x^2}\lesssim 2^{k-\frac{3m}{2}}r^{\frac12}\mathcal{N}_1(G)\|\mathcal{D}^{-1}\check{R}\|_{L_x^\infty}.$$

Thus

$$\sum_{k>0}\sum_{m>k}\|P_k\nabla\!\!\!/(\mathcal{D}^{-1}\check{R}\cdot\mathcal{D}^{-1}G_m)\|_{L_x^2}\lesssim r^{\frac12}\mathcal{N}_1(G)\|\mathcal{D}^{-1}\check{R}\|_{L_x^\infty}.$$

Case 2: $k>m$. We decompose further that

$$P_k\nabla\!\!\!/(\mathcal{D}^{-1}\check{R}\cdot\mathcal{D}^{-1}G_m)=P_k\nabla\!\!\!/(\sum_{n>0}P_n^2+P_{\leq 0})(\mathcal{D}^{-1}\check{R}\cdot\mathcal{D}^{-1}G_m). \tag{5.19}$$

For simplicity we consider only the high frequency terms, and the low frequency terms can be treated similarly. We can adapt the proof for [7, Proposition 4.5] to obtain for any S tangent tensor field F and $1>\alpha>\alpha_0\geq 1/2$ the inequality

$$\begin{aligned}\|P_k\nabla\!\!\!/ P_n^2F\|_{L_x^2}\lesssim&\Big(2^{\min(k,n)}2^{-2|n-k|}r^{-1}+2^{\min(k,n)}2^{-(1-\alpha)\max(k,n)}\underline{K}_{\alpha_0}r^{-\alpha}\\&+2^{-|k-n|}\|\underline{K}\|_{L_x^2}^\alpha\underline{K}_{\alpha_0}\Big)\|P_nF\|_{L_x^2}.\end{aligned} \tag{5.20}$$

Let $\mathcal{I}_{nm}=\|P_n(\mathcal{D}^{-1}\check{R}\cdot\mathcal{D}^{-1}G_m)\|_{L_x^2}$, we have

$$\begin{aligned}&\|P_k\nabla\!\!\!/ P_n^2(\mathcal{D}^{-1}\check{R}\cdot\mathcal{D}^{-1}G_m)\|_{L_x^2}\\&\lesssim\Big(2^{\min(k,n)}2^{-2|n-k|}r^{-1}+2^{\min(k,n)}2^{-(1-\alpha)\max(k,n)}\underline{K}_{\alpha_0}r^{-\alpha}\\&+2^{-|k-n|}\|\underline{K}\|_{L_x^2}^\alpha\underline{K}_{\alpha_0}\Big)\mathcal{I}_{nm}.\end{aligned} \tag{5.21}$$

Now we estimate $\mathcal{I}_{nm}$. We first consider the case $n > m > 0$. By [9, Proposition 4.1(3)] we have

$$\begin{aligned}\mathcal{I}_{nm} &\lesssim r^2 2^{-2n} \|\tilde{P}_n \not\Delta(\mathcal{D}^{-1}\check{R} \cdot \mathcal{D}^{-1} G_m)\|_{L_x^2} \\ &\lesssim r^2 2^{-2n} \Big(\|\not\Delta \mathcal{D}^{-1}\check{R} \cdot \mathcal{D}^{-1} G_m\|_{L_x^2} + \|\tilde{P}_n(\not\nabla \mathcal{D}^{-1}\check{R} \cdot \not\nabla \mathcal{D}^{-1} G_m)\|_{L_x^2} \\ &\quad + \|\not\Delta \mathcal{D}^{-1} G_m \cdot \mathcal{D}^{-1}\check{R}\|_{L_x^2} \Big).\end{aligned}$$

By using (5.6) and (5.9) we can obtain

$$\begin{aligned}&\|\not\Delta \mathcal{D}^{-1}\check{R} \cdot \mathcal{D}^{-1} G_m\|_{L_x^2} \\ &\quad\lesssim \left(\|\not\nabla \check{R}\|_{L_x^\infty} + \|\underline{K}\|_{L_x^\infty} \|\mathcal{D}^{-1}\check{R}\|_{L_x^\infty} + r^{-2} \|\mathcal{D}^{-1}\check{R}\|_{L_x^\infty} \right) 2^{-\frac{3m}{2}} r^{\frac{3}{2}} \mathcal{N}_1(G).\end{aligned}$$

With the help of (WB), [9, Proposition 3.4] and (5.7) we also have

$$\begin{aligned}\|\tilde{P}_n(\not\nabla \mathcal{D}^{-1}\check{R} \cdot \not\nabla \mathcal{D}^{-1} G_m)\|_{L_x^2} &\lesssim 2^{\frac{n}{2}} r^{-\frac{1}{2}} \|\not\nabla \mathcal{D}^{-1}\check{R}\|_{L_x^2} \|\not\nabla \mathcal{D}^{-1} G_m\|_{L_x^4} \\ &\lesssim 2^{\frac{n}{2}} r^{-\frac{1}{2}} \|\check{R}\|_{L_x^2} \mathcal{N}_1(G) \left(1 + 2^{-\frac{m}{2}} r^{\frac{1}{2}} \|\underline{K}\|_{L_x^2}\right).\end{aligned}$$

Furthermore, by employing (5.9), (FBB) and (3.35), (5.6), we have

$$\begin{aligned}&\|\not\Delta \mathcal{D}^{-1} G_m \cdot \mathcal{D}^{-1}\check{R}\|_{L_x^2} \\ &\quad\lesssim \|\mathcal{D}^{-1}\check{R}\|_{L_x^\infty} \left(\|{}^\star\mathcal{D} G_m\|_{L_x^2} + \|\underline{K}\|_{L_x^\infty} \|\mathcal{D}^{-1} G_m\|_{L_x^2} + r^{-2} \|\mathcal{D}^{-1} G_m\|_{L_x^2} \right) \\ &\quad\lesssim \|\mathcal{D}^{-1}\check{R}\|_{L_x^\infty} \left(2^{\frac{m}{2}} r^{-\frac{1}{2}} + \|\underline{K}\|_{L_x^\infty} 2^{-\frac{3m}{2}} r^{\frac{3}{2}} + r^{-\frac{1}{2}} 2^{-\frac{3m}{2}} \right) \mathcal{N}_1(G).\end{aligned}$$

Therefore

$$\begin{aligned}\mathcal{I}_{nm} &\lesssim r^2 2^{-2n} \mathcal{N}_1(G) \Big(2^{\frac{m}{2}} r^{-\frac{1}{2}} \|\mathcal{D}^{-1}\check{R}\|_{L_x^\infty} + 2^{\frac{n}{2}} r^{-\frac{1}{2}} \|\check{R}\|_{L_x^2} \left(1 + 2^{-\frac{m}{2}} r^{\frac{1}{2}} \|\underline{K}\|_{L_x^2}\right) \\ &\quad + \|\not\nabla \check{R}\|_{L_x^\infty} 2^{-\frac{3m}{2}} r^{\frac{3}{2}} + \|\underline{K}\|_{L_x^\infty} \|\mathcal{D}^{-1}\check{R}\|_{L_x^\infty} 2^{-\frac{3m}{2}} r^{\frac{3}{2}} \Big). \end{aligned} \tag{5.22}$$

Combining this with (5.21) and summing over $k, n, m > 0$ for the cases $k > n > m$ and $n > k > m$, we thus obtain

$$\begin{aligned}&\sum_{k,n,m>0, k>m, n>m} \|P_k \not\nabla P_n^2 (\mathcal{D}^{-1}\check{R} \cdot \mathcal{D}^{-1} G_m)\|_{L_x^2} \\ &\qquad\lesssim c_0 \mathcal{N}_1(G) r^{\frac{1}{2}} \left(\|\mathcal{D}^{-1}\check{R}\|_{L_x^\infty} + \|\check{R}\|_{L_x^2} + r^2 \|\not\nabla \check{R}\|_{L_x^\infty} \right)\end{aligned}$$

with c_0 depending on $\|r^2 \underline{K}\|_{L_x^\infty} + \underline{K}_{\alpha_0} + \|\underline{K}\|_{L_x^2}$.

It remains to estimate $\mathcal{I}_{nm}$ when $k > m > n$. By (5.6) we have

$$\mathcal{I}_{nm} \lesssim \|\mathcal{D}^{-1}\check{R} \cdot \mathcal{D}^{-1} G_m\|_{L_x^2} \lesssim 2^{-\frac{3m}{2}} r^{\frac{3}{2}} \mathcal{N}_1(G) \|\mathcal{D}^{-1}\check{R}\|_{L_x^\infty}. \tag{5.23}$$

This together with (5.20) gives

$$\sum_{k,n,m>0, k>m>n} \|P_k \not\nabla P_n^2 (\mathcal{D}^{-1}\check{R} \cdot \mathcal{D}^{-1} G_m)\|_{L_x^2} \le c_0 \mathcal{N}_1(G) r^{\frac{1}{2}} \|\mathcal{D}^{-1}\check{R}\|_{L_x^\infty}. \tag{5.24}$$

The proof is thus complete. □

In order to prove Lemma 5.4, we need the commutation formula for $[\not\nabla, \not\Delta]F$ for any S tangent tensor F which symbolically is given by (see [3] or [7, page 300])

$$[\not\nabla, \not\Delta]F = \not\nabla(K \cdot F) + K \cdot \not\nabla F. \tag{5.25}$$

Proof of Lemma 5.4. Assuming (5.11), we first prove (5.12). For simplicity we set $\tilde{H} = \not\nabla \mathcal{D}^{-1} H$. We have from [4, Proposition 3.20 (x)] that

$$\|\tilde{H}\|_{L_x^\infty} \lesssim \sum_{k>0} 2^k r^{-1} \|P_k \tilde{H}\|_{L_x^2} + r^\theta c_0 (\|\not\nabla \tilde{H}\|_{L_x^2} + \|r^{-1} \tilde{H}\|_{L_x^2}), \tag{5.26}$$

where c_0 depends on $\|\underline{K}\|_{L_x^2}$, and $\theta > 0$ is very close to 0.

In view of (5.11), the first term on the right of (5.26) can be bounded by

$$\|\tilde{H}\|_{B^1_{2,1}} \lesssim \|{}^\star\mathcal{D}H\|_{B^0_{2,1}} + \|H\|_{L^2_\omega} + r^{\frac{1}{2}} c_0 \mathcal{N}_1(H). \tag{5.27}$$

We then estimate $\|\nabla\!\!\!/ \tilde{H}\|_{L^2_x}$. By applying (5.10) to $F = \mathcal{D}^{-1}H$ and using [9, Proposition 3.4] and (SobM1) we can obtain

$$\|\nabla\!\!\!/ \tilde{H}\|_{L^2_x} \lesssim \|{}^\star\mathcal{D}H\|_{L^2_x} + r^{-1}\|H\|_{L^2_x} + r^{\frac{1}{2}} c_0 \mathcal{N}_1(H), \tag{5.28}$$

with c_0 depending on $\|\underline{K}\|_{L^2_x}$.

Combining (5.28), (5.27) and (5.26) and using [9, Proposition 3.4] we thus obtain (5.12).

Next we prove (5.11). Using the GLP projections, we need to prove

$$\sum_{k,m>0} 2^k r^{-1} \|P_k \nabla\!\!\!/ P_m^2 \mathcal{D}^{-1}H\|_{L^2_x} \lesssim \|{}^\star\mathcal{D}H\|_{B^0_{2,1}} + \|H\|_{L^2_\omega} + c_0 r^{\frac{1}{2}} \mathcal{N}_1(H), \tag{5.29}$$

$$\sum_{k,m>0} 2^k r^{-1} \|P_k \nabla\!\!\!/ P_{\le 0} \mathcal{D}^{-1}H\|_{L^2_x} \lesssim \|H\|_{L^2_\omega} + c_0 r^{\frac{1}{2}} \mathcal{N}_1(H), \tag{5.30}$$

where c_0 depends on $\|\underline{K}\|_{L^2_x} + r\|\nabla\!\!\!/ \underline{K}\|_{L^2_x} + r\|\underline{K}\|_{L^\infty_x}$.

The proof of (5.30) is similar to the treatment in *Case 1* for

$$\mathcal{I}_{km} := 2^k r^{-1} \|P_k \nabla\!\!\!/ P_m^2 \mathcal{D}^{-1}H\|_{L^2_x}.$$

Thus we will give the proof of (5.29) only.

Case 1: $k > m$. By (FD) we first have

$$\mathcal{I}_{km} \le 2^{-k} r \left(\|P_k \nabla\!\!\!/ \Delta\!\!\!/ P_m^2 \mathcal{D}^{-1}H\|_{L^2_x} + \|P_k [\Delta\!\!\!/, \nabla\!\!\!/] P_m^2 \mathcal{D}^{-1}H\|_{L^2_x} \right). \tag{5.31}$$

Let us denote the two terms on the right by $\mathcal{I}^1_{km}$ and $\mathcal{I}^2_{km}$ respectively. In view of (5.9), (FBB) and [9, Lemma 4.3] we have

$$\begin{aligned} \mathcal{I}^1_{km} &\lesssim 2^{-k} r \|P_k \nabla\!\!\!/ P_m^2 ({}^\star\mathcal{D}H + \underline{K}\mathcal{D}^{-1}H + r^{-2}\mathcal{D}^{-1}H)\|_{L^2_x} \\ &\lesssim 2^{m-k}\|P_m {}^\star\mathcal{D}H\|_{L^2_x} + 2^{-k} r\|H\|_{L^2_x} + 2^{-k} r \|P_k \nabla\!\!\!/ P_m^2 (\underline{K} \cdot \mathcal{D}^{-1}H)\|_{L^2_x} \end{aligned} \tag{5.32}$$

We only need to consider the last term of (5.32) which, by using (5.20, can be estimated as

$$\begin{aligned} 2^{-k} r \|P_k \nabla\!\!\!/ P_m^2 (\underline{K} \cdot \mathcal{D}^{-1}H)\|_{L^2_x} \lesssim \Big(& 2^{-3|m-k|} + 2^{-|m-k|} 2^{-(1-\alpha)k} \underline{K}_{\alpha_0} r^{1-\alpha} \\ & + 2^{-k} 2^{-|k-m|} r \|\underline{K}\|^\alpha_{L^2_x} \underline{K}_{\alpha_0} \Big) \|P_m (\underline{K}\mathcal{D}^{-1}H)\|_{L^2_x}. \end{aligned} \tag{5.33}$$

For the last two terms, in view of $k > m$, (SobM2) and [9, Lemma 3.1], we have

$$\begin{aligned} \sum_{k>m} & \left(2^{-|m-k|} 2^{-(1-\alpha)k} \underline{K}_{\alpha_0} r^{1-\alpha} + 2^{-k} 2^{-|k-m|} r \|\underline{K}\|^\alpha_{L^2_x} \underline{K}_{\alpha_0} \right) \|P_m (\underline{K}\mathcal{D}^{-1}H)\|_{L^2_x} \\ & \lesssim \underline{K}_{\alpha_0} \left(r^{1-\alpha} + r\|\underline{K}\|^\alpha_{L^2_x} \right) \|\underline{K}\|_{L^2_x} r^{\frac{1}{2}} \mathcal{N}_1(H). \end{aligned} \tag{5.34}$$

Let us decompose $\underline{K} = \sum_n P_n^2 \underline{K} + \bar{\underline{K}}$. It is easy to derive by Gauss-Bonnet Theorem that $\bar{\underline{K}} = 0$. Now consider the high frequency term only for the purpose of simplicity. With the help of [9, Proposition 3.4], the proof contained in [7, pages 299–300] implies for $m, n > 0$

$$\|P_m (\underline{K}_n \mathcal{D}^{-1}H)\|_{L^2_x} \lesssim 2^{-\frac{3}{4}|m-n|} \|P_n \underline{K}\|_{L^2_x} \left(\|\mathcal{D}^{-1}H\|_{L^\infty_x} + \|H\|_{L^2_x} \right). \tag{5.35}$$

Therefore the first term on the right of (5.33) can be estimated as follows

$$\begin{aligned}\sum_{k>m}\sum_{n>0}&2^{-3|m-k|}\|P_m(\underline{K}_n\mathcal{D}^{-1}H)\|_{L^2_x}\\ &\lesssim \sum_{k>m}\sum_{n>0}2^{-3|m-k|-\frac{3}{4}|m-n|}\|P_n\underline{K}\|_{L^2_x}\left(\|\mathcal{D}^{-1}H\|_{L^\infty_x}+\|H\|_{L^2_x}\right)\\ &\lesssim \|\underline{K}\|_{B^0_{2,1}}r^{\frac{1}{2}}\mathcal{N}_1(H)\end{aligned}$$

where we employed (SobM2), [9, Lemma 3.1] and (SobM1) to obtain the last inequality. It is easy to check by (FB) that $\|\underline{K}\|_{B^0_{2,1}}\lesssim\|\underline{K}\|_{L^2_x}+r\|\nabla\!\!\!/\underline{K}\|_{L^2_x}$. Consequently,

$$\sum_{k>m}\mathcal{I}^1_{km}\lesssim\|{}^\star\mathcal{D}H\|_{B^0_{2,1}}+c_0r^{\frac{1}{2}}\mathcal{N}_1(H).$$

Now we consider $\mathcal{I}^2_{km}$. With the help of (5.25), (SobM2), (FBB) and [9, Lemma 4.3] we can obtain

$$\begin{aligned}\mathcal{I}^2_{km}&\lesssim 2^{-k}r\left(\|P_k\nabla\!\!\!/(K\cdot P^2_m\mathcal{D}^{-1}H)\|_{L^2_x}+\|P_k(K\cdot\nabla\!\!\!/P^2_m\mathcal{D}^{-1}H)\|_{L^2_x}\right)\\ &\lesssim 2^{-k}r\Big(\|\nabla\!\!\!/\underline{K}\|_{L^2_x}\|P^2_m\mathcal{D}^{-1}H\|_{L^\infty_x}+\|\underline{K}\|_{L^\infty_x}\|\nabla\!\!\!/P^2_m\mathcal{D}^{-1}H\|_{L^2_x}\\ &\quad+r^{-2}\|P_k(\nabla\!\!\!/P^2_m\mathcal{D}^{-1}H)\|_{L^2_x}\Big)\\ &\lesssim 2^{-k}\left(r\|\nabla\!\!\!/\underline{K}\|_{L^2_x}r^{\frac{1}{2}}\mathcal{N}_2(\mathcal{D}^{-1}H)+r\|\underline{K}\|_{L^\infty_x}\|H\|_{L^2_x}+r^{-1}\|H\|_{L^2_x}\right).\end{aligned}$$

Using [9, Lemma 3.1]and (SobM1) we then get

$$\sum_{k>m>0}\mathcal{I}^2_{km}\lesssim\left(r\|\nabla\!\!\!/\underline{K}\|_{L^2_x}+r\|\underline{K}\|_{L^\infty_x}\right)r^{\frac{1}{2}}\mathcal{N}_1(H)+r^{-1}\|H\|_{L^2_x}.$$

Case 2: $k<m$. In this case we have from (FD) and (FBD) that

$$\begin{aligned}\mathcal{I}_{km}&\le 2^{k-2m}r\|P_k\nabla\!\!\!/\Delta\!\!\!/P^2_m\mathcal{D}^{-1}H\|_{L^2_x}\\ &\le 2^{k-2m}r\left(\|P_k\Delta\!\!\!/\nabla\!\!\!/P^2_m\mathcal{D}^{-1}H\|_{L^2_x}+\|P_k[\nabla\!\!\!/,\Delta\!\!\!/]P^2_m\mathcal{D}^{-1}H\|_{L^2_x}\right)\\ &\le 2^{3k-2m}r^{-1}\|P_k\nabla\!\!\!/P^2_m\mathcal{D}^{-1}H\|_{L^2_x}+2^{k-2m}r\|P_k[\nabla\!\!\!/,\Delta\!\!\!/]P^2_m\mathcal{D}^{-1}H\|_{L^2_x}.\end{aligned}\tag{5.36}$$

Let us denote by $\mathcal{I}^1_{km}$ the first term in the line of (5.36) and by $\mathcal{I}^2_{km}$ the second term. Consider $\mathcal{I}^1_{km}$ first. By (FBB), (FD) and (5.9) we have

$$\begin{aligned}\mathcal{I}^1_{km}&\lesssim 2^{4k-2m}r^{-2}\|P^2_m\mathcal{D}^{-1}H\|_{L^2_x}\lesssim 2^{4k-4m}\|P^2_m\Delta\!\!\!/\mathcal{D}^{-1}H\|_{L^2_x}\\ &\lesssim 2^{-4|k-m|}\left(\|P^2_m{}^\star\mathcal{D}H\|_{L^2_x}+\|P^2_m(\underline{K}\mathcal{D}^{-1}H)\|_{L^2_x}+r^{-2}\|P^2_m\mathcal{D}^{-1}H\|_{L^2_x}\right).\end{aligned}$$

From (FB), [9, Proposition 3.4], (SobM2) and [9, Lemma 3.1] it follows

$$\begin{aligned}\|P^2_m(\underline{K}\mathcal{D}^{-1}H)\|_{L^2_x}&\lesssim 2^{-m}r\left(\|\nabla\!\!\!/\underline{K}\|_{L^2_x}\|\mathcal{D}^{-1}H\|_{L^\infty_x}+\|\underline{K}\|_{L^\infty_x}\|H\|_{L^2_x}\right)\\ &\lesssim 2^{-m}r^{\frac{3}{2}}\mathcal{N}_1(H)\left(\|\nabla\!\!\!/\underline{K}\|_{L^2_x}+\|\underline{K}\|_{L^\infty_x}\right).\end{aligned}\tag{5.37}$$

Using [9, Lemma 4.3] we then obtain

$$\mathcal{I}^1_{km}\lesssim 2^{-4|k-m|}\left(\|P^2_m{}^\star\mathcal{D}H\|_{L^2_x}+r^{-1}2^{-m}\|H\|_{L^2_x}+2^{-m}r^{\frac{3}{2}}\mathcal{N}_1(H)(\|\nabla\!\!\!/\underline{K}\|_{L^2_x}+\|\underline{K}\|_{L^\infty_x})\right).$$

Therefore

$$\sum_{k,m>0,k<m}\mathcal{I}^1_{km}\lesssim\|{}^\star\mathcal{D}H\|_{B^0_{2,1}}+r^{-1}\|H\|_{L^2_x}+r^{\frac{1}{2}}\mathcal{N}_1(H)\left(r\|\nabla\!\!\!/\underline{K}\|_{L^2_x}+r\|\underline{K}\|_{L^\infty_x}\right).\tag{5.38}$$

Now consider $\mathcal{I}^2_{km}$. in view of (5.25) we have

$$\mathcal{I}^2_{km} \lesssim 2^{k-2m} r \Big(\|P_k \nabla\!\!\!/ (\underline{K} P_m^2 \mathcal{D}^{-1} H)\|_{L_x^2} + \|P_k(\underline{K} \nabla\!\!\!/ P_m^2 \mathcal{D}^{-1} H)\|_{L_x^2} + r^{-2} \|P_k \nabla\!\!\!/ P_m^2 \mathcal{D}^{-1} H\|_{L_x^2} \Big).$$

By (FBB) and [9, Lemma 4.3] we can derive

$$\mathcal{I}^2_{km} \lesssim 2^{k-2m} r \left(\|\underline{K}\|_{L_x^\infty} \|H\|_{L_x^2} + 2^{k-m} r^{-2} \|H\|_{L_x^2} \right).$$

Using (SobM1) we thus obtain

$$\sum_{k,m>0,m>k} \mathcal{I}^2_{km} \lesssim r^{-1} \|H\|_{L_x^2} + r^{\frac{1}{2}} \mathcal{N}_1(H) \cdot r \|\underline{K}\|_{L_x^\infty}.$$

Therefore

$$\sum_{k,m>0,m>k} \mathcal{I}_{km} \lesssim r^{-1} \|H\|_{L_x^2} + c_0 r^{\frac{1}{2}} \mathcal{N}_1(H)$$

where c_0 depends on $r(\|\underline{K}\|_{L_x^\infty} + \|\nabla\!\!\!/ \underline{K}\|_{L_x^2})$. □

References

[1] Christodoulou, D. and Klainerman, S., *The Global Nonlinear Stability of the Minkowski Space*, Princeton Mathematical Series 41, 1993.

[2] Klainerman, S. and Rodnianski, I., *Causal geometry of Einstein-vacuum spacetimes with finite curvature flux*, Invent. Math., 159(2005), no. 3, 437–529.

[3] Klainerman, S. and Rodnianski, I., *A geometric Littlewood-Paley theory*, Geom. Funct. Anal., 16 (2006), 126–163.

[4] Klainerman, S. and Rodnianski, I., *Sharp Trace theorems for null hypersurfaces on Einstein metrics with finite curvature flux*, Geom. Funct. Anal., 16 (2006), no. 1, 164–229.

[5] Klainerman, S. and Rodnianski, I., *On the breakdown criterion in general relativity*, J. Amer. Math. Soc., 23 (2010), no. 2, 345–382.

[6] Wang, Q., *Causal Geometry of Einstein-Vacuum spacetimes*, PhD thesis, Princeton University 2006.

[7] Wang, Q., *On the geometry of null cones in Einstein Vacuum Spacetimes*, Ann. Inst. H. Poincarè Anal. Non Linéaire, 26(2009), 285-328

[8] Wang, Q., *Improved breakdown criterion for solution of Einstein vacuum equation in CMC gauge*, 2010, arXiv:1004.2938.

[9] Wang, Q., *On Ricci coefficients of null hypersurfaces with time foliation in Einstein vacuum space-time: Part I*, preprint 2010.

Department of Mathematics, Stony Brook University, Stony Brook, NY 11794, USA
E-mail address: qwang@math.sunysb.edu