# MAT 320 FALL 2008
# Take-home exam

Deadline: Dec 22, 2008, before Final

You should try to answer 6 questions. The questions are in no particular order. You may use the textbook and your class notes and homework papers during the exam, but no other auxiliary material. Please take this test seriously and solve the problems by yourself. **You are not allowed to ask anyone in math or physics department for help. When you have finished, remember to write out and sign the following pledge on the examination booklet: I pledge my honor that I have not violated the honor code during this examination.**

NAME : ID :

**1**

Show that if $f : \mathbb{R} \to \mathbb{R}$ is a continuous function and if there is a constant $c > 0$ such that $|f(x) - f(y)| > c|x - y|$ for all $x, y \in \mathbb{R}$, then $f(\mathbb{R}) = \mathbb{R}$.

**2.** Choose one of the following two questions

(i) Let $I := [a, b]$ and $f : I \to \mathbb{R}$ be differentiable at $c \in I$. Show that for every $\epsilon > 0$ there exists $\delta > 0$ such that if $0 < |x - y| < \delta$ and $a \le x \le c \le y \le b$, then

$$\left| \frac{f(x) - f(y)}{x - y} - f'(c) \right| < \epsilon$$

(ii)

Let $I \subset \mathbb{R}$ be an open interval, let $f : I \to \mathbb{R}$ be differentiable on $I$. Suppose $f''(a)$ exists at $a \in I$. Show that for every $\epsilon > 0$ there exists $\delta > 0$ such that if $0 < |h| < \delta$

$$\left| \frac{f(a + h) + f(a - h) - 2f(a)}{h^2} - f''(a) \right| < \epsilon$$

Hint: Refer to section 6.3 when necessary.

**3.** Suppose $f : [a, b] \to \mathbb{R}$ is a continuous function and that $f$ is differentiable in $(a, b)$. Prove that there exists $\xi \in (a, b)$ such that

$$f(b) - f(a) = \xi \ln \frac{b}{a} \cdot f'(\xi).$$

**4**

(i)If $f(x)$ is continuous and non-negative on an interval $[a, b]$ and $f(c) > 0$ for some point in $[a, b]$, prove that $\int_a^b f(x)dx > 0$.

(ii)If $f$ and $g$ are continuous and if $\int_a^b f = \int_a^b g$. Show that there exists $c \in [a, b]$ such that $f(c) = g(c)$.

**5.** Define the function $f : [0, 1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} x^2 + 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q}, \end{cases}$$

Is $f$ Riemann integrable over $[0, 1]$? Prove your conclusion.

**6.** Show that the sequence $(x^2 e^{-nx})$ converges uniformly on $[0, \infty)$.

**7.** Suppose that $f(x)$ and $f_n(x)$ for $n = 1, 2, \cdots$ are continuous functions on $[0, 1]$ and that the sequence $f_n(x)$ converges uniformly on the rational numbers in $[0, 1]$ to $f(x)$. Does the sequence $f_n(x)$ converges uniformly to $f(x)$ at all points of $[0, 1]$? Justify your answer.

**8.**

Let $I \subset \mathbb{R}$ be an open interval. Suppose $f : I \to \mathbb{R}$ is convex on $I$. Prove the left and right derivatives of $f$ exist at every point on $I$.