MAT 320 Introduction to Analysis

Practice Questions for Midterm 2

The questions below are meant to give you some practice for the material, not to mimic an actual exam. (Some of these questions are harder than the exam questions would be.) There will be 5 questions on the exam (worth 10 points each).

1. Let a and b be positive numbers. Show that

$$\lim_{n\to\infty}(a^n + b^n)^{\frac{1}{n}} = \max\{a, b\}.$$

Hint: Set $x_n = (a^n + b^n)^{1/n}$ and without losing generality assume $0 < a \le b$. Establish the inequality $b \le x_n \le 2^{1/n} b$ and use it to complete the proof.

2. Let (x_n) and (y_n) be two convergent sequences. Define the sequence (z_n) by $z_n = \max\{x_n, y_n\}$. Prove that the sequence (z_n) converges. **Hint:** consider two cases, 1) $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n$, and 2) $\lim_{n\to\infty} x_n \ne \lim_{n\to\infty} y_n$.

3. Let the sequence (x_n) be defined by

$$x_1 = \frac{3}{2} \qquad \text{and} \qquad x_{n+1} = 2 - \frac{1}{x_n} \text{ for } n \ge 1.$$

Prove that (x_n) converges and find its limit.

4. Suppose that $x_n \ge 0$ for all $n \in \mathbb{N}$, and $((-1)^n x_n)$ is a convergent sequence. Show that the sequence (x_n) is convergent. What's its limit?

5. Let $x_n = \sqrt{n}$. Show that $\lim_{n\to\infty} |x_{n+1} - x_n| = 0$, but that (x_n) is not a Cauchy sequence.

6. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous at $p \in \mathbb{R}$. If $f(p) > A$, show that there is a $\delta > 0$ such that $f(x) > A$ whenever $|x - p| < \delta$.

7. Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ such that f is discontinuous at every point, but $|f|$ is continuous at every point.

8. If $f : [0, 1] \to \mathbb{R}$ is continuous and has only rational values, must f be constant? Explain.

9. Give an example of a bounded continuous function $f : \mathbb{R} \to \mathbb{R}$ which has no maximum or minimum. Can you give such an example for $f : [0, \infty) \to \mathbb{R}$? For $f : [0, 1] \to \mathbb{R}$?

10. Show that $f(x) = 1/x^2$ is uniformly continuous on $[1, \infty)$, but not uniformly continuous on $[0, \infty)$.