MAT 131 Midterm 2 Solutions

1. Determine the inflection point(s) of the function $f(x) = x^5 - 10x^4 + 30x^3 - 20x + 40$.

   Answer The inflections points are where $f''(x)$ changes its sign. Thus we must first look for the zeroes of $f''(x)$.

   $f'(x) = 5x^4 - 40x^3 + 90x^2 - 20$ and

   $$f''(x) = 20x^3 - 120x^2 + 180x = 20x(x^2 - 6x + 9) = 20x(x-3)^2 = 0$$

   implies that either $x = 0$ or $x = 3$.

   Then we must check whether f'' changes sign at these points. For $x < 0$, e.g. $x = -1$, we have $f''(-1) = -320 < 0$; for $0 < x < 3$, e.g. $x = 1$, we have $f''(1) = 80 > 0$; for $x > 3$, e.g. $x = 4$, we have $f''(4) = 80 > 0$. Therefore f'' changes sign at $x = 0$ but not at $x = 3$. Thus the only inflection point is $x = 0$.

2. Compute the derivatives:

   (a) $g(t) = \dfrac{1 + 2\ln t}{3 - t^2}$.

   Answer Use the Quotient Rule.

   $$\begin{aligned}\frac{d}{dt}\left(\frac{1+2\ln t}{3-t^2}\right) &= \frac{(3-t^2)\,\frac{d}{dt}(1+2\ln t) - (1+2\ln t)\,\frac{d}{dt}(3-t^2)}{(3-t^2)^2} \\ &= \frac{(3-t^2)\,\frac{2}{t} - (1+2\ln t)(-2t)}{(3-t^2)^2} \\ &= \frac{6t^{-1} + 4t\ln t}{(3-t^2)^2}\end{aligned}$$

   (b) $h(x) = \sin x \cos x$.

   Answer Use the Product Rule.

   $$h'(x) = \frac{d}{dx}(\sin x \cos x) = (\cos x)(\cos x) + (\sin x)(-\sin x) = \cos^2 x - \sin^2 x.$$

3. Compute the derivative of $f(x) = \dfrac{x^2 + 2}{x}$ *directly from the definition*. Use a derivative rule to check your answer. (You will not get credit for using *only* a derivative rule.)

Answer

$$\begin{aligned} f'(a) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{\frac{(x+h)^2+2}{x+h} - \frac{x^2+2}{x}}{h} \\ &= \lim_{h\to 0} \frac{1}{h} \frac{(x)(x^2+2xh+h^2+2)-(x+h)(x^2+2)}{x(x+h)} \\ &= \lim_{h\to 0} \frac{(x^3+2x^2h+xh^2+2x)-(x^3+hx^2+2x+2h)}{hx(x+h)} \\ &= \lim_{h\to 0} \frac{x^2h+xh^2-2h}{hx(x+h)} \\ &= \lim_{h\to 0} \frac{x^2+xh-2}{x(x+h)} \\ &= \frac{x^2-2}{x^2} \end{aligned}$$

Using the Power Rule, $f(x) = x + 2x^{-1}$ so $f'(x) = 1 - 2x^{-2}$, which agrees with the formula we derived.

4. When does the tangent line to the curve $y = e^{e^x - e^{-x}}$ at point $(0,1)$ intersect the x-axis?

   Answer Use the Chain Rule.

$$\begin{aligned} y &= e^u & u &= e^x - e^{-x} \\ \frac{dy}{du} &= e^u & \frac{du}{dx} &= e^x + e^{-x} \end{aligned}$$

   So $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = e^u(e^x + e^{-x}) = e^{e^x - e^{-x}}(e^x + e^{-x})$. At $x = 0$, then, we get $y'(0) = e^{1-1}(1+1) = 2$. So the tangent line through $(0,1)$ has slope $m = 2$, and therefore has equation

$$y - 1 = 2(x-0).$$

   This line intersects $y = 0$ when $x = -\frac{1}{2}$.

5. Consider the national debt $f(t)$ (in billions of US dollars) as a function of time t (in years). The federal deficit $g(t)$ is the rate of change of the debt. Below is a table of values of the federal deficit (in billions of US dollars) over the past 9 years.

Year t	1996	1997	1998	1999	2000	2001	2002	2003	2004
Deficit $g(t)$	\$107	\$22	−\$69	−\$125	−\$236	−\$128	\$158	\$377	\$412

   (a) If the debt function $f(t)$ were a differentiable function of time, how would you express the deficit $g(t)$ in terms of $f(t)$?

   Answer The deficit is the rate of change of the debt, and therefore $g(t) = f'(t)$.

(b) For which years was the deficit function $g(t)$ increasing?

Answer The deficit is increasing for years 2000–2004.

(c) For which years was the debt function $f(t)$ increasing?

Answer The debt function is increasing when its derivative $g(t)$ is positive. This occurs for years 1996–1997 and for 2002–2004.

(d) For which years was the debt function $f(t)$ concave upward?

Answer The debt is concave up when its derivative $g(t)$ is increasing, so that the answer is the same as that for part (b): years 2000–2004.

6. Suppose that x and y satisfy the equation

$$y^5 - 3xy^2 = 2x^2.$$

Determine $\frac{dy}{dx}$ at the point $(-1, -1)$.

Answer Use implicit differentiation.

$$\begin{aligned}
\frac{d}{dx}(y^5 - 3xy^2) &= \frac{d}{dx}(2x^2) \\
5y^4 \frac{dy}{dx} - 3\frac{d}{dx}(xy^2) &= 4x \\
5y^4 \frac{dy}{dx} - 3y^2 - 6xy \frac{dy}{dx} &= 4x \\
(5y^4 - 6xy) \frac{dy}{dx} &= 4x + 3y^2 \\
\frac{dy}{dx} &= \frac{4x + 3y^2}{5y^4 - 6xy}
\end{aligned}$$

Therefore at $x = -1$, $y = -1$, we have $\frac{dy}{dx} = \frac{-4 + 3}{5 - 6} = 1$.

7. Suppose f and g are differentiable functions and that

$$f(1) = 1, \quad f'(1) = 3, \quad f'(3) = 1; \qquad g(1) = 3, \quad g'(1) = 1, \quad g'(3) = 3.$$

(a) Compute $h'(1)$ if $h(x) = f\big(g(x)\big)$

Answer Use the Chain Rule. $h'(x) = f'\big(g(x)\big)\, g'(x)$, so

$$h'(1) = f'\big(g(1)\big)g'(1) = f'(3)g'(1) = 1 \cdot 1 = 1$$

(b) Compute $j'(1)$ if $j(x) = \dfrac{x^2}{f(x)}$

Answer Use the Quotient Rule.

$$j'(x) = \frac{f(x)\,\frac{d}{dx}(x^2) - x^2\,\frac{d}{dx}f(x)}{f(x)^2} = \frac{2xf(x) - x^2 f'(x)}{f(x)^2}.$$

Therefore

$$j'(1) = \frac{2 \cdot 1 \cdot f(1) - 1^2 \cdot f'(1)}{f(1)^2} = \frac{2-3}{1^2} = -1$$

8. On the axes below, sketch the graph of a function $f(x)$ satisfying all of the following properties:

   (a) f is continuous on $[-2, 2]$.

   (b) $f' > 0$ on $[-2, 0)$ and on $(1, 2]$, while $f' < 0$ on $(0, 1)$.

   (c) $f'' > 0$ on $[-2, -1)$ and on $(1, 2]$, while $f'' < 0$ on $(-1, 1)$.

   (d) f is differentiable at every point except $x = 1$.

   Answer The information tells us that f is increasing on the interval $[-2, 0)$, then decreasing on the interval $(0, 1)$, then increasing again on $(1, 2]$. Since f is differentiable at $x = 0$, we must have $f'(0) = 0$. Since f is not differentiable at $x = 1$, we probably have some kind of kink or vertical tangent (since f is continuous).

   We then refine our graph to reflect the concavity: f is concave up on $[-2, -1)$, then concave down on $(-1, 1)$, then concave up again on $(1, 2]$. Therefore the graph must have an inflection point at $x = -1$. There is also an inflection point at $x = 1$, but now we are certain that $x = 1$ must be a kink.

   Below is one possible graph exhibiting all these features.