MAT 131 HW solutions (5.3–5.4)

1 Section 5.3

2. $\int_1^3 (1+2x-4x^3)\,dx = (x+x^2-x^4)\big|_1^3 = (3+9-81)-(1+1-1) = -69-1 = -70$

6. $\int_1^8 x^{1/3}\,dx = (\frac{3}{4}x^{4/3})\big|_1^8 = \frac{3}{4}8^{4/3} - \frac{3}{4}1^{4/3} = \frac{3}{4}\cdot 16 - \frac{3}{4} = 11.25$

11. $\int_{-2}^{-1}\left(4y^3+\frac{2}{y^3}\right)dy = (y^4-y^{-2})\big|_{-2}^{-1} = ((-1)^4-(-1)^{-2}) - ((-2)^4 - (-2)^{-2}) = 0-(16-\frac{1}{4}) = -15.75$

20. $\int_0^1 \frac{4}{t^2+1}\,dt = (4\arctan t)\big|_0^1 = 4\arctan 1 - 4\arctan 0 = \pi$

27. $\int_1^e \frac{x^2+x+1}{x}\,dx = \int_1^e (x+1+x^{-1})\,dx = (\frac{1}{2}x^2+x+\ln x)\big|_1^e = (e^2/2 + e+1) - (1/2+1+0) = \frac{e^2+2e-1}{2}$

29. The function $-x^{-1}$ is only an antiderivative for x^{-2} on the interval $(0,\infty)$ or $(-\infty,0)$, not both. Because x^{-2} has a singularity at $x=0$, we cannot evaluate the integral over a domain containing $x=0$.

38. $\frac{d}{dx}(x\sin x+\cos x) = x\cos x+\sin x-\sin x = x\cos x$, and therefore $\int x\cos x\,dx = x\sin x+\cos x + C$.

41. $\int (1-t)(2+t^2)\,dt = \int (2-2t+t^2-t^3)\,dt = 2t-t^2+\frac{t^3}{3}-\frac{t^4}{4}+C$

47. $\int_5^{10} w'(t)\,dt = w(10)-w(5)$, the amount of weight the child has gained between ages 5 and 10.

52. $f(x)$ is the derivative of the height of the trail, $f(x)=h'(x)$, so that $\int_3^5 f(x)\,dx = \int_3^5 h'(x)\,dx = h(5)-h(3)$, the difference in height at $x=5$ vs. at $x=3$.

55. (a) $x(3)-x(0) = \int_0^3 v(t)\,dt = \int_0^3 (3t-5)\,dt = (\frac{3t^2}{2}-5t)\Big|_0^3 = -\frac{3}{2}$ meters.

(b) $d = \int_0^3 |3t-5|\,dt$. The easiest way to evaluate this integral is geometrically; it's the area of a triangle with height 5 and base $\frac{5}{3}$, plus the area of a triangle with height 4 and base $\frac{4}{3}$. Thus it's $d = \frac{1}{2}\cdot 5\cdot \frac{5}{3} + \frac{1}{2}\cdot 4\cdot\frac{4}{3} = \frac{25}{6} + \frac{16}{6} = \frac{41}{6}$ meters.

58. (a) $v(t) = v(0) + \displaystyle\int_0^t a(\tau)\,d\tau = 5 + \int_0^t (t+4)\,dt = 5 + 4t + t^2/2.$

(b) $v(t)$ is always positive in the interval $[0, 10]$, so that speed equals velocity, and thus

$$d = \int_0^{10} |v(t)|\,dt = \int_0^{10} v(t)\,dt = \int_0^{10} (5+4t+t^2/2)\,dt$$

$$= (5t + 2t^2 + t^3/6)\Big|_0^{10} = \frac{1250}{3} \text{ meters}$$

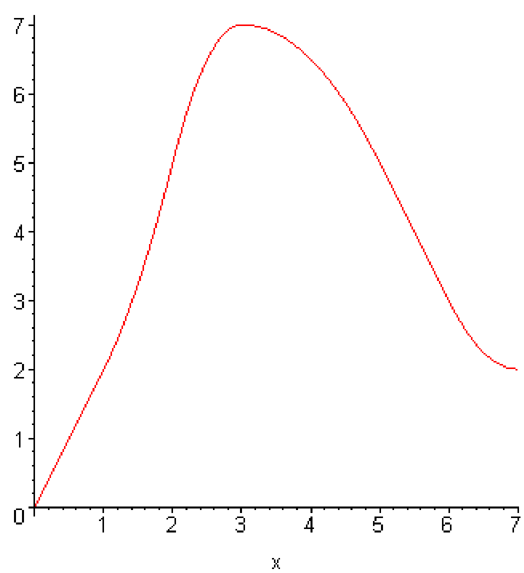
2 Section 5.4

3. (a) $g(0) = 0$, $g(1) = 2$, $g(2) = 5$, $g(3) = 7$, $g(6) = 3$.

(b) g is increasing on $[0, 3]$.

(c) g has a maximum value at $x = 3$.

(d) Here is a plot of g.



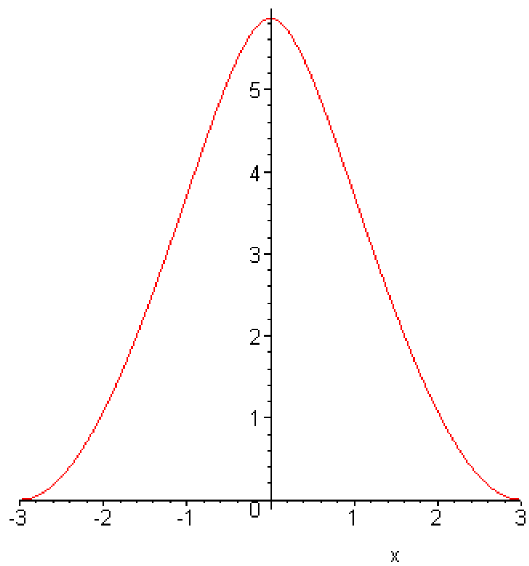
4. (a) $g(-3) = \int_{-3}^{-3} f(t)\,dt = 0$ and $g(3) = 0$ since the area from -3 to 0 cancels the area from 0 to 3, by symmetry.

   (b) $g(-2) \approx 1.1$, $g(-1) \approx 3.8$, $g(0) \approx 5.8$.

   (c) g is increasing when f is positive, on $[-3, 0]$.

   (d) g has a maximum when $x = 0$.

   (e) Here is an approximate graph of g.



   (e) Of course the graph of $g'(x)$ is the same as the graph of $f(x)$.

8. $g'(x) = \dfrac{d}{dx} \displaystyle\int_1^x \ln t\,dt = \ln x.$

11. If $H(u) = \int_2^u \arctan t\,dt$, then $H'(u) = \arctan u$ and $h(x) = H(1/x)$. Let $u = 1/x$. Then by the chain rule,

$$h'(x) = H'(u)u'(x) = -\frac{1}{x^2}\arctan u = -\frac{\arctan(1/x)}{x^2}$$

14. If $Y(u) = \int_0^u \sin^3 t\,dt$, then $y(x) = -Y(e^x)$, so $y'(x) = -Y'(e^x)e^x = -e^x \sin^3 e^x$.

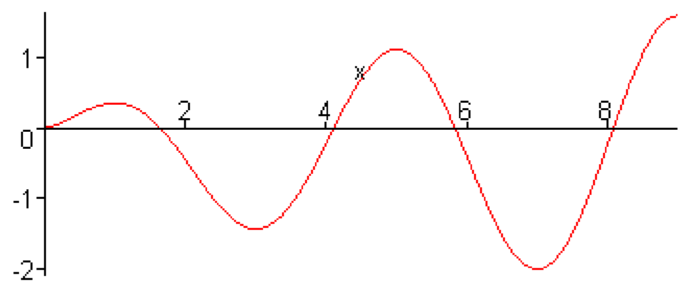
18. If $y = \displaystyle\int_0^x \frac{1}{1+t+t^2}\,dt$ then $y'(x) = (1+x+x^2)^{-1}$ and $y''(x) = -(1+x+x^2)^{-2}(1+2x)$, which vanishes at $x = -\frac{1}{2}$. For $x < -\frac{1}{2}$, $y''(x) > 0$, while for $x > -\frac{1}{2}$, $y''(x) < 0$. Thus y is concave up for $x < -\frac{1}{2}$.

19. (a) The local maxima of g occur when f goes from positive to negative, i.e., at $x = 1$ and $x = 5$. The local minima of g occur at $x = 3$ and $x = 7$.

(b) The absolute maximum occurs either at $x = 0$, $x = 1$, $x = 5$, or $x = 9$. Clearly $g(1) > g(0)$, and $g(5)$ is slightly larger than $g(1)$ (since the positive area from 3 to 5 is slightly larger than the negative area from 1 to 3). Similarly $g(9)$ is slightly larger than $g(5)$, since the positive area from 7 to 9 is larger than the negative area from 5 to 7. So the absolute maximum happens at $x = 9$.

(c) g is concave downward when f is decreasing, which happens on $[1/2, 2]$, on $[4, 6]$, and on $[8, 9]$.

(d) Here is a graph of g.



22. (a) $\displaystyle\int_a^b e^{-t^2}\,dt = \int_0^b e^{-t^2}\,dt - \int_0^a e^{-t^2}\,dt = \frac{\sqrt{\pi}}{2}\text{erf}(b) - \frac{\sqrt{\pi}}{2}\text{erf}(a).$

(b) If $y = e^{x^2}\text{erf}(x)$, then

$$y'(x) = 2xe^{x^2}\text{erf}(x) + e^{x^2}\text{erf}'(x) = 2xy(x) + e^{x^2}\frac{2}{\sqrt{\pi}}e^{-x^2} = 2xy(x) + \frac{2}{\sqrt{\pi}}$$

26. (a) We need to consider several cases. First, if $x < 0$, then $g(x) = \int_0^x f(t)\,dt = -\int_x^0 f(t)\,dt = -\int_x^0 0\,dt = 0$. Next, if $0 < x < 1$, then $g(x) - g(0) = \int_0^x t\,dt = \frac{x^2}{/}2$, so $g(x) = x^2/2$. Thus $g(1) = \frac{1}{2}$.

So if $1 < x < 2$, then $g(x) - g(1) = \int_1^x f(t)\,dt$, so $g(x) = \frac{1}{2} + \int_1^x (2-t)\,dt = \frac{1}{2} + 2x - \frac{1}{2}x^2 - (2 - 1/2) = 2x - \frac{x^2}{2} - 1$. Thus $g(2) = 1$.

Finally if $x > 2$ then $g(x) = g(2) + \int_2^x f(t)\,dt = 1 + \int_2^x 0\,dt = 1$.

Thus we can summarize

$$g(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{2}x^2 & 0 \le x < 1 \\ 2x - \frac{1}{2}x^2 - 1 & 1 \le x < 2 \\ 1 & x \ge 2 \end{cases}$$

27. We want a function f and a number a such that

$$6 + \int_a^x \frac{f(t)}{t^2}\,dt = 2\sqrt{x}.$$

We will first solve for x by differentiating both sides with respect to x (using the Fundamental Theorem of Calculus). We do this because it eliminates the dummy variable t and the unknown constant a. We obtain

$$\frac{f(x)}{x^2} = x^{-1/2}$$

for all x, and therefore $f(x) = x^{3/2}$ for all x.

Now plugging this function in, we get

$$\begin{aligned} 6 + \int_a^x t^{-1/2}\,dt &= 2\sqrt{x}, \\ 6 + 2t^{1/2}\Big|_a^x &= 2\sqrt{x}, \\ 6 + 2\sqrt{x} - 2\sqrt{a} &= 2\sqrt{x}, \\ 3 &= \sqrt{a}, \\ a &= 9. \end{aligned}$$