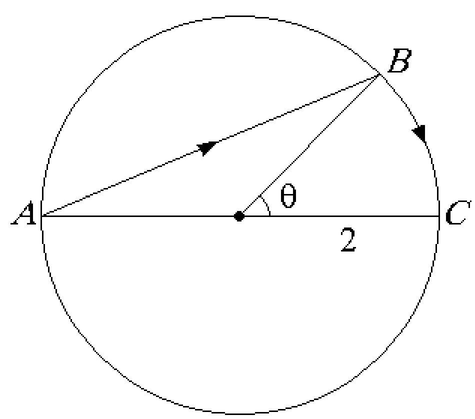
MAT 131 HW solutions (4.6–4.9)

1 Section 4.6

16. The vertices of the rectangle will be at points $(x, 0)$, $(-x, 0)$, $(x, 8-x^2)$, and $(-x, 8-x^2)$. Thus it will have area $A(x) = 2x(8-x^2) = 16x-2x^3$. The endpoints are $x = 0$ and $x = 2\sqrt{2}$; either one of these gives a line segment instead of a rectangle. The critical point occurs when $A'(x) = 16 - 6x^2 = 0$, i.e., at $x = \sqrt{8/3}$. We have $A(0) = 0$ and $A(2\sqrt{2}) = 0$ and $A(\sqrt{8/3}) = \frac{64\sqrt{2}}{3\sqrt{3}}$. The dimensions of the rectangle are $\sqrt{\frac{8}{3}} \times \frac{16}{3}$.

32. We modify the diagram somewhat, putting θ as a central angle to make the computations simpler.



Point A is $(-2, 0)$ and point B is $(2\cos\theta, 2\sin\theta)$. So

$$d_{AB} = \sqrt{(2\cos\theta + 2)^2 + (2\sin\theta)^2} = 2\sqrt{\cos^2\theta + 2\cos\theta + 1 + \sin^2\theta}$$
$$= 2\sqrt{2}\sqrt{1+\cos\theta}.$$

In addition the distance from B to C is measured along the circle, so $d_{BC} = 2\theta$.

Thus the total time is

$$T(\theta) = \frac{d_{AB}}{2} + \frac{d_{BC}}{4} = \sqrt{2}\sqrt{1+\cos\theta} + \frac{\theta}{2}.$$

The endpoints are $\theta = 0$ (no walking) and $\theta = \pi$ (no rowing). The critical points are found from $T'(\theta) = \sqrt{2}\frac{1}{2}(1+\cos\theta)^{-1/2}(-\sin\theta) + \frac{1}{2} = 0$, so they happen when

$$\begin{aligned}
\frac{\sqrt{2}\sin\theta}{\sqrt{1+\cos\theta}} &= 1 \\
2\sin^2\theta &= 1+\cos\theta \\
2(1+\cos\theta)(1-\cos\theta) &= 1+\cos\theta \\
2(1-\cos\theta) &= 1 \\
\cos\theta &= \frac{1}{2}.
\end{aligned}$$

The only solution of $\cos\theta = \frac{1}{2}$ for $\theta \in [0, \pi]$ is $\theta = \frac{\pi}{3}$.

We have thus $T(0) = 2$, $T(\pi) = \frac{\pi}{2}$, and $T(\frac{\pi}{3}) = \sqrt{2}\sqrt{\frac{3}{2}} + \frac{\pi}{6} = \sqrt{3} + \frac{\pi}{6}$. Evaluating these numerically, we get $T(0) = 2$, $T(\pi) \approx 1.57$, and $T(\frac{\pi}{3}) \approx 1.73 + 0.52 = 2.25$. Thus the shortest possible time is obtained by walking completely around the lake, without rowing at all.

Tricksy hobbits!

2 Section 4.8

1. (a) According to my estimation, the tangent line at $x_1 = 1$ passes through the x-axis at around $x_2 \approx 2.3$. Then the tangent line at $x_2 \approx 2.3$ passes through the x-axis at around $x_3 \approx 3$.

   (b) $x_1 = 5$ would not have been a better approximation. In fact, the tangent line at $x_1 = 5$ appears to cross the x-axis at around $x_2 = 1$, so we would have ended up at the previous approximation, but one step behind. The problem is that the tangent line is nearly horizontal at $x = 5$, so our next approximation is far from the zero.

4. (a) $x_1 = 0$: the iterates move to $-\infty$ and Newton's Method does not converge.

   (b) $x_1 = 1$: the tangent line does not cross the x-axis, so Newton's Method fails. There is no x_2.

   (c) $x_1 = 3$: the tangent line at $x = 3$ appears to cross the axis at $x_2 = 1$, and then Newton's Method fails.

   (d) $x_1 = 4$: again there is a horizontal tangent, so Newton's Method fails.

   (e) $x_1 = 5$: Newton's Method converges to the zero at $x = 6$.

6. $f(x) = x^5 + 2$, so that $f'(x) = 5x^4$. The sequence of iterates is then

$$x_{n+1} = x_n - \frac{x_n^5 + 2}{5x_n^4} = \frac{4}{5}x_n - \frac{2}{5x_n^4}.$$

Thus if $x_1 = -1$, then $x_2 = -\frac{4}{5} - \frac{2}{5} = -\frac{6}{5} = -1.2000$ and

$$x_3 = \frac{4}{5}\left(-\frac{6}{5}\right) - \frac{2}{5(-\frac{6}{5})^4} = -\frac{24}{25} - \frac{125}{648} = -\frac{-18677}{16200} = -1.1529.$$

For the sake of comparison, $\sqrt[5]{-2} = -1.1487$.

22. (a) If $f(x) = \frac{1}{x} - a$, then $f'(x) = -\frac{1}{x^2}$. Therefore Newton's Method is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n + x_n^2(\frac{1}{x_n} - a) = 2x_n - ax_n^2.$$

   (b) We want to compute the reciprocal of 1.6984. This is fairly close to $1.\overline{66}$, which is $\frac{5}{3}$, whose reciprocal is $\frac{3}{5} = 0.6$. So let's use $a = 1.6984$ and $x_1 = 0.6$.
   We get $x_2 = 2x_1 - 1.6984x_1^2 = 0.588576$, $x_3 = 2x_2 - 1.6984x_2^2 = 0.5887893715$, $x_4 = 2x_3 - 1.6984x_3^2 = 0.5887894489$, and we notice that the method is converging, to six decimal places, to $x = 0.588789$.

23. If $f(x) = x^3 - 3x + 6$, then $f'(x) = 3x^2 - 3$, so Newton's Method is

$$x_{n+1} = x_n - \frac{x_n^3 - 3x_n + 6}{3x_n^2 - 3},$$

and if $x_1 = 1$, then $x_2 = 1 - \frac{4}{0}$ is undefined. Thus Newton's Method fails.

3 Section 4.9

2. $F(x) = x - \frac{1}{4}x^4 + 2x^6 + C$.

5. $F(x) = 2x^{3/2} - x^{-5} + C$.

8. $G(x) = -x^{-5} + 2x^{-2} + 2x + C$.

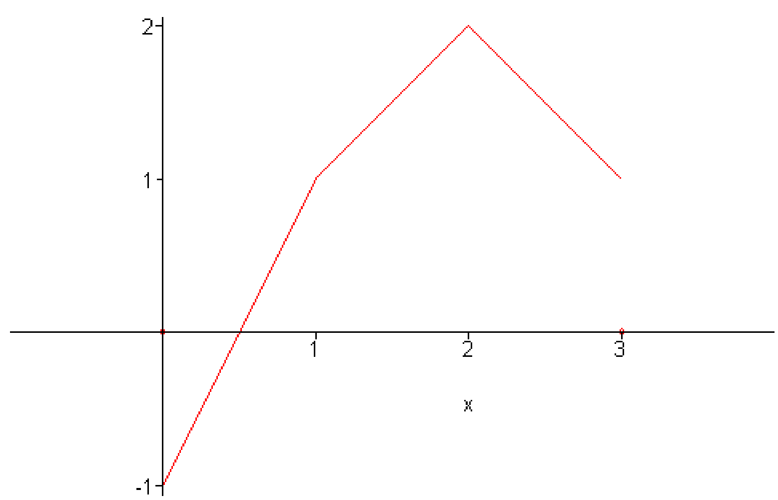
13. $F(x) = x^5 - \frac{1}{3}x^6 + 4$.

16. $f'(x) = 2x + \frac{1}{4}x^4 + \frac{1}{7}x^7 + C$ and $f(x) = x^2 + \frac{1}{20}x^5 + \frac{1}{56}x^8 + Cx + D$.

19. $f(x) = 4x^{3/2} + 2x^{5/2} + C$, and $f(1) = 4 + 2 + C = 6 + C = 10$, so $C = 4$. Thus $f(x) = 4x^{3/2} + 2x^{5/2} + 4$.

24. $f'(x) = 4x - 3x^2 - 10x^4 + 1$ and $f(x) = 2x^2 - x^3 - 2x^5 + x + 2$.

31. The graph of f must be a straight line in each segment; slope 2 in $(0,1)$, slope 1 in $(1,2)$, and slope -1 in $(2,3)$. Since it must be continuous, we know exactly what must happen at the endpoints. Here is the graph:



39. If $v(t) = \sin t - \cos t$ then $s(t) = -\cos t - \sin t + C$. Since $s(0) = 0$, $C - 1 = 0$, so $C = 1$. Thus $s(t) = 1 - \cos t - \sin t$.

50. The deceleration was 16 ft/s^2, so we have $s''(t) = -16$. Thus $s'(t) = -16t + C$ and $s(t) = -8t^2 + Ct + D$. If the initial position was $s(0) = 0$, then $s(t) = -8t^2 + Ct$. The car stops when $s'(t) = 0$, so $t = \frac{C}{16}$. If the car has gone 200 feet, then $s(\frac{C}{16}) = -8\frac{C^2}{16^2} + C\frac{C}{16} = \frac{C^2}{32} = 200$. So $C^2 = 6400$ and thus $C = 80$. Now $s'(0) = C$, so the initial speed was 80 feet per second, which translates into 55 miles per hour.