MAT 131 HW solutions (4.3–4.6)

1 Section 4.3

1. The secant line from $(0, f(0))$ to $(8, f(8))$ has slope $m = \frac{1}{4}$. Sliding a piece of paper across the graph, we see tangent lines of slope $\frac{1}{4}$ at a number slightly less than 1, a number slightly greater than 3, a number slightly less than 4.5, and a number slightly greater than 6.

16. $f(x) = \frac{x}{x^2+4}$ so $f'(x) = \frac{(x^2+4)(1)-x(2x)}{(x^2+4)^2} = \frac{4-x^2}{(x^2+4)^2}$. The critical points are $x = \pm 2$.

    We have $f''(x) = \frac{(x^2+4)^2(-2x)-(4-x^2)(2)(x^2+4)(2x)}{(x^2+4)^4}$ so $f''(2) = \frac{-4}{64} < 0$ and $f''(-2) = \frac{4}{64} > 0$. Thus $x = 2$ is a local maximum and $x = -2$ is a local minimum, by the Second Derivative Test.

    We can also compute $f'(-3) = \frac{-5}{169} < 0$, $f'(0) = \frac{1}{4}$, and $f'(3) = \frac{-5}{169}$, so that $x = -2$ is a local minimum and $x = 2$ is a local maximum by the First Derivative Test.

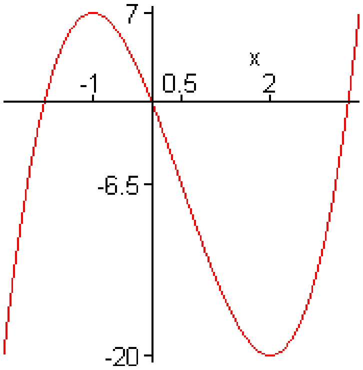
    The First Derivative Test is a little easier, since we have to use several derivative rules to compute f''.

19. $f'(x) = 6x^2 - 6x - 12$ and $f''(x) = 12x - 6$.

(a), (b) The critical points are found from $f'(x) = 6(x-2)(x+1) = 0$, so they are $x = 2$ or $x = -1$. $f''(2) = 18 > 0$ and $f''(-1) = -18$, so that $x = 2$ is a local minimum and $x = -1$ is a local maximum. Therefore, the graph is increasing for $x \leq -1$, decreasing for $-1 \leq x \leq 2$, and increasing again for $x \geq 2$.

(c) The solution of $f''(x) = 0$ is $x = \frac{1}{2}$. For $x < \frac{1}{2}$, we have $f''(x) < 0$, and for $x > \frac{1}{2}$ we have $f''(x) > 0$. Thus f is concave up for $x \geq \frac{1}{2}$ and concave down for $x \leq \frac{1}{2}$, with an inflection point at $x = \frac{1}{2}$.

(d) It is enough to plot the local maximum $(-1, 7)$, the local minimum $(2, -20)$, and the inflection point $(\frac{1}{2}, -\frac{13}{2})$. We get the following graph:

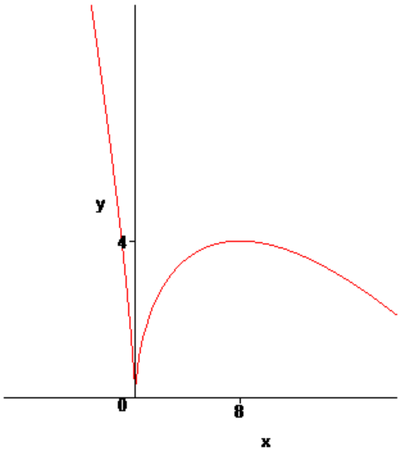


24. $B(x) = 3x^{2/3} - x$, so $B'(x) = 2x^{-1/3} - 1$ and $B''(x) = -\frac{2}{3}x^{-4/3}$.

(a), (b) The critical points are where $B'(x) = 0$, which happens at $x = 8$, and where B' does not exist, which happens at $x = 0$. We have $B''(8) = -\frac{1}{24} < 0$, so that $x = 8$ is a local maximum by the Second Derivative Test. The Second Derivative Test fails at $x = 0$ since $B''(0)$ does not exist. However, we have $B'(x) > 0$ for $0 < x < 8$ and $B'(x) < 0$ for $x < 0$, so that $x = 0$ is a local minimum. We see B is decreasing on $(-\infty, 0]$, then increasing on $[0, 8]$, then decreasing again on $[8, \infty)$.

(c) Since $x^{-4/3} > 0$ for all $x \neq 0$, $B''(x) < 0$ for all $x \neq 0$. Thus B is always concave down and has no inflection points.

(d) We need to plot a curve with vertical tangent at $(0, 0)$, concave down everywhere, with a local maximum at $(8, 4)$. We get the following graph:



29. (a) The vertical asymptotes happen at $x = 1$ and $x = -1$. We have

$$\lim_{x\to 1^-} \frac{x^2}{x^2-1} = -\infty, \quad \lim_{x\to 1^+} \frac{x^2}{x^2-1} = +\infty,$$
$$\lim_{x\to -1^-} \frac{x^2}{x^2-1} = +\infty, \quad \lim_{x\to -1^+} \frac{x^2}{x^2-1} = -\infty.$$

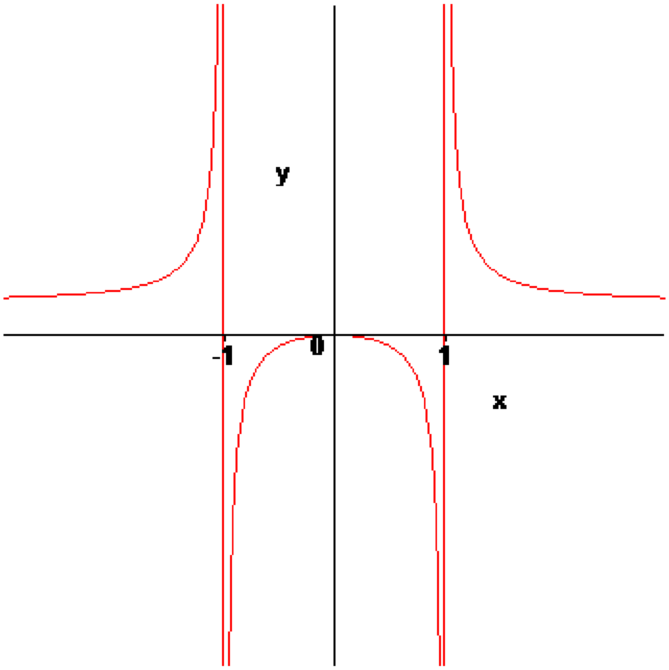
The horizontal asymptote happens at $y = 1$.

(b) $f'(x) = \frac{(x^2-1)(2x)-(x^2)(2x)}{(x^2-1)^2} = \frac{-2x}{(x^2-1)^2}$, so that $f'(x) > 0$ for $x < 0$ and $f'(x) < 0$ for $x > 0$. f is increasing for $x < -1$ and also for $-1 < x < 0$, then decreasing for $0 < x < 1$ and for $1 < x < \infty$.

(c) The only local extreme value is at $x = 0$, where we have a local maximum.

(d) $f''(x) = \frac{-(x^2-1)^2(2)-(-2x)(2)(x^2-1)(2x)}{(x^2-1)^4} = \frac{-2(x^2-1)+8x^2}{(x^2-1)^3} = \frac{2(3x^2+1)}{(x^2-1)^3}$. Thus f is concave down for $-1 < x < 1$, concave up for $x > 1$, and concave up for $x < -1$. There are no inflection points; the concavity only changes at the asymptotes.

(e) Here is the graph of the function.



34. (a) There are no vertical asymptotes, since $1+e^x=0$ has no solution. The horizontal asymptotes are

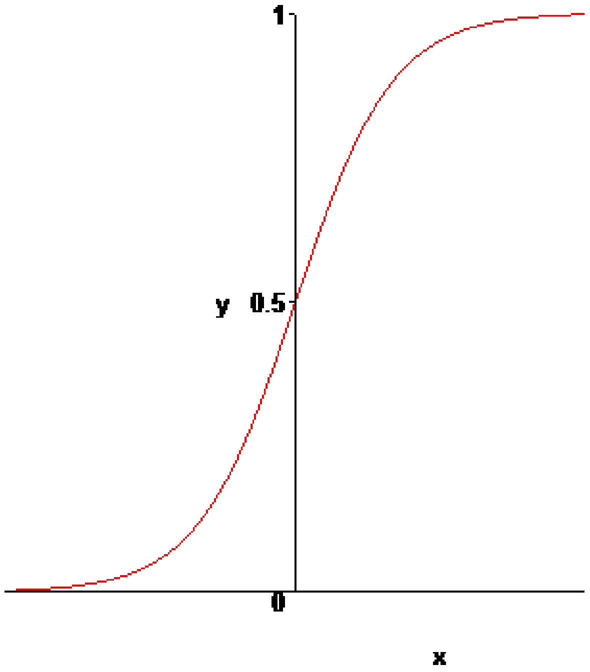
$$\lim_{x\to+\infty}\frac{e^x}{1+e^x}=1 \qquad \text{and} \qquad \lim_{x\to-\infty}\frac{e^x}{1+e^x}=0.$$

(b) Since $f'(x)=\frac{(1+e^x)(e^x)-(e^x)(e^x)}{(1+e^x)^2}=\frac{e^x}{(1+e^x)^2}$, we have $f'(x)>0$ for all x, so that f is always increasing.

(c) There are no local minima or maxima.

(d) Since $f''(x)=\frac{(1+e^x)^2(e^x)-(e^x)(2)(1+e^x)(e^x)}{(1+e^x)^4}=\frac{e^x(1-e^x)}{(1+e^x)^3}$, we have $f''(x)=0$ if and only if $x=0$. For $x>0$, $f''(x)<0$, while for $x<0$, $f''(x)>0$. Thus f is concave up for $x<0$ and concave down for $x>0$. The inflection point occurs at $(0,\frac{1}{2})$.

(e) The graph of $f(x)$ is shown.



37. Suppose the derivative is $f'(x)=(x+1)^2(x-3)^5(x-6)^4$. We know f is increasing when $f'(x)>0$. Since $(x+1)^2\geq 0$ always, and $(x-6)^4\geq 0$ always, we only need to worry about $(x-3)^5$ being greater than 0. This occurs when $x>3$. So f is increasing on $[3,\infty)$.

2 Section 4.5

$$\lim_{x\to a} f(x) = 0 \qquad \lim_{x\to a} g(x) = 0 \qquad \lim_{x\to a} h(x) = 1 \qquad \lim_{x\to a} p(x) = \infty \qquad \lim_{x\to a} q(x) = \infty$$

1. (a) $\displaystyle\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{0}{0}$ is indeterminate.
   (b) $\displaystyle\lim_{x\to a} \frac{f(x)}{p(x)} = \frac{0}{\infty} = 0.$
   (c) $\displaystyle\lim_{x\to a} \frac{h(x)}{p(x)} = \frac{1}{\infty} = 0.$
   (d) $\displaystyle\lim_{x\to a} \frac{p(x)}{f(x)} = \frac{\infty}{0} = \pm\infty.$
   (e) $\displaystyle\lim_{x\to a} \frac{p(x)}{q(x)} = \frac{\infty}{\infty}$ is indeterminate.

2. (a) $\displaystyle\lim_{x\to a} \big[f(x)p(x)\big] = 0 \cdot \infty$ is indeterminate.
   (b) $\displaystyle\lim_{x\to a} \big[h(x)p(x)\big] = 1 \cdot \infty = \infty.$
   (c) $\displaystyle\lim_{x\to a} \big[p(x)q(x)\big] = \infty \cdot \infty = \infty.$

3. (a) $\displaystyle\lim_{x\to a} \big[f(x) - p(x)\big] = 0 - \infty = -\infty.$
   (b) $\displaystyle\lim_{x\to a} \big[p(x) - q(x)\big] = \infty - \infty$ is indeterminate.
   (c) $\displaystyle\lim_{x\to a} \big[p(x) + q(x)\big] = \infty + \infty = \infty.$

4. (a) $\big[f(x)\big]^{g(x)} = 0^0$ is indeterminate.
   (b) $\big[f(x)\big]^{p(x)} = 0^\infty = 0.$
   (c) $\big[h(x)\big]^{p(x)} = 1^\infty$ is indeterminate.
   (d) $\big[p(x)\big]^{f(x)} = \infty^0$ is indeterminate.
   (e) $\big[p(x)\big]^{q(x)} = \infty^\infty = \infty.$
   (f) $\sqrt[q(x)]{p(x)} = \infty^0$ is indeterminate.

6. $\displaystyle\lim_{x\to 1} \frac{x^a - 1}{x^b - 1} \left(= \frac{0}{0} = \right) = \lim_{x\to 1} \frac{a x^{a-1}}{b x^{b-1}} = \frac{a}{b}.$

9. $\displaystyle\lim_{t\to 0}\frac{e^t-1}{t^3}\left(=\frac{0}{0}=\right)=\lim_{t\to 0}\frac{e^t}{3t^2}=\infty.$

14. $\displaystyle\lim_{x\to\infty}\frac{\ln\ln x}{x}\left(=\frac{\infty}{\infty}\right)=\lim_{x\to\infty}\frac{\frac{1}{x\ln x}}{1}=\frac{1}{\infty}=0.$

17. $\displaystyle\lim_{x\to 0}\frac{e^x-1-x}{x^2}\left(=\frac{0}{0}=\right)=\lim_{x\to 0}\frac{e^x-1}{2x}\left(=\frac{0}{0}\right)=\lim_{x\to 0}\frac{e^x}{2}=\frac{1}{2}.$

20. $\displaystyle\lim_{x\to 0}\frac{\cos mx-\cos nx}{x^2}\left(=\frac{0}{0}=\right)=\lim_{x\to 0}\frac{-m\sin mx+n\sin nx}{2x}\left(=\frac{0}{0}=\right)=$
    $\displaystyle\lim_{x\to 0}\frac{-m^2\cos mx+n^2\cos nx}{2}=\frac{n^2-m^2}{2}.$

25. $\displaystyle\lim_{x\to 0^+}\sqrt{x}\ln x=\lim_{x\to 0^+}\frac{\ln x}{x^{-1/2}}=\lim_{x\to 0^+}\frac{\frac{1}{x}}{-\frac{1}{2}x^{-3/2}}=\lim_{x\to 0^+}(-2\sqrt{x})=0.$

29. $\lim_{x\to\infty}x^3e^{-x^2}=\lim_{x\to\infty}\frac{x^3}{e^{x^2}}=\lim_{x\to\infty}\frac{3x^2}{2xe^{x^2}}=\lim_{x\to\infty}\frac{3x}{2e^{x^2}}=\lim_{x\to\infty}\frac{3}{4xe^{x^2}}=$
    $0.$

34. $\lim_{x\to 1}\left(\frac{1}{\ln x}-\frac{1}{x-1}\right)=\lim_{x\to 1}\frac{x-1-\ln x}{(x-1)\ln x}=\lim_{x\to 1}\frac{1-\frac{1}{x}}{\ln x+\frac{x-1}{x}}=\lim_{x\to 1}\frac{x-1}{x\ln x+x-1}=$
    $\lim_{x\to 1}\frac{1}{1+\ln x+1}=\frac{1}{2}.$

37. $\lim_{x\to 0}(1-2x)^{1/x}=\exp\left(\lim_{x\to 0}\frac{\ln(1-2x)}{x}\right)=\exp\left(\lim_{x\to 0}\frac{\frac{-2}{1-2x}}{1}\right)=e^{-2}.$

40. $\lim_{x\to\infty}x^{(\ln 2)/(1+\ln x)}=\exp\left(\lim_{x\to\infty}\frac{\ln 2\ln x}{1+\ln x}\right)=\exp\left(\lim_{x\to\infty}\frac{\ln 2}{\frac{1}{\ln x}+1}\right)=$
    $2.$

55. Using L'Hôpital's Rule several times, we get

$$\lim_{x\to\infty}\frac{e^x}{x^n}=\lim_{x\to\infty}\frac{e^x}{nx^{n-1}}=\lim_{x\to\infty}\frac{e^x}{n(n-1)x^{n-2}}=\cdots=\lim_{x\to\infty}\frac{e^x}{n(n-1)(n-2)\cdots 2\cdot 1}=\infty.$$

56. Using L'Hôpital's Rule, we get

$$\lim_{x\to\infty}\frac{\ln x}{x^p}=\lim_{x\to\infty}\frac{\frac{1}{x}}{px^{p-1}}=\lim_{x\to\infty}\frac{1}{px^p}=0.$$

61.

$$\lim_{x\to a}\frac{\sqrt{2a^3x-x^4}-a\sqrt[3]{a^2x}}{a-\sqrt[4]{ax^3}}=\lim_{x\to a}\frac{\frac{1}{2}(2a^3x-x^4)^{-1/2}(2a^3-4x^3)-\frac{1}{3}a^{5/3}x^{-2/3}}{-\frac{3}{4}a^{1/4}x^{-1/4}}$$

$$=\frac{(-a^3)/(a^2)-\frac{1}{3}a}{-\frac{3}{4}}=-\frac{16a}{9}.$$

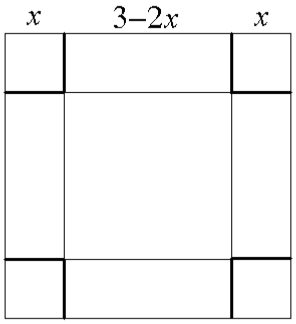
3 Section 4.6

3. Suppose the two numbers are x and y. Then we have $xy = 100$ as the constraint and $P = x + y$ as the function to be minimized. Solve the constraint for y in terms of x: $y = \frac{100}{x}$. Then

$$P(x) = x + \frac{100}{x}.$$

The critical points occur when $P'(x) = 1 - 100x^{-2} = 0$, so $x = \pm 10$. Since x must be positive, we use $x = 10$. Then $y = 10$ also, and the minimum sum is 20.

8. (b) Here is the general situation, with x representing the side of a cut square.



(c-e) The volume is $V = x(3-2x)^2 = 9x - 12x^2 + 4x^3$.

(f) $V'(x) = 9 - 24x + 12x^2 = 3(4x^2 - 8x + 3) = 3(2x-1)(2x-3)$. So the critical points are $x = \frac{1}{2}$ and $x = \frac{3}{2}$. The endpoints of the interval are $x = 0$ and $x = \frac{3}{2}$. The volume at each of these points is $V(0) = 0$, $V(\frac{1}{2}) = 2$, and $V(\frac{3}{2}) = 0$. So the maximum volume occurs when $x = \frac{1}{2}$.

13. The distance of a point (x, y) from the point $(1, 0)$ is $D = \sqrt{(x-1)^2 + y^2}$. Since (x, y) must be on the ellipse $4x^2 + y^2 = 4$, we can solve for y^2 to get $y^2 = 4 - 4x^2$. Thus the distance is

$$D = \sqrt{(x-1)^2 + 4 - 4x^2} = \sqrt{5 - 2x - 3x^2}.$$

The critical points are found from $D'(x) = 0$, and since $D'(x) = \frac{1}{2}(5 - 2x - 3x^2)^{-1/2}(-2 - 6x) = -(1 + 3x)(5 - 2x - 3x^2)^{-1/2}$, we have one critical point at $x = -\frac{1}{3}$. We also have endpoints at $x = \pm 1$ since these are the smallest and largest x-values that appear on the ellipse.

So $D(1) = \sqrt{5 - 2 - 3} = 0$, and $D(-1) = \sqrt{5 + 2 - 3} = 2$, and $D(-\frac{1}{3}) = \sqrt{5 + \frac{2}{3} - \frac{1}{3}} = \frac{4}{\sqrt{3}}$¿ Since $4/\sqrt{3} > 2$, the maximum occurs when $x = -\frac{1}{3}$. At this x, there are two y-values: $y = \pm\sqrt{4 - \frac{4}{9}} = \frac{\sqrt{5}}{3}$. Thus the two points on the ellipse are $(-\frac{1}{3}, \frac{\sqrt{5}}{3})$ and $(-\frac{1}{3}, -\frac{\sqrt{5}}{3})$