MAT 131 HW solutions (3.6–3.7)

1 Section 3.6

1. (a) $xy' + y + 2 + 6x = 0$, so $y' = -\frac{y+2+6x}{x}$.

   (b) $y = 4x^{-1} - 3x - 2$, so $y' = -4x^{-2} - 3$.

   (c) Plugging y from (b) into y' from (a), we get $y' = -x^{-1}(4x^{-1} - 3x - 2 + 2 + 6x) = -x^{-1}(4x^{-1} + 3x) = -4x^{-2} - 3$, which agrees with the answer derived in (b).

10.

$$\begin{aligned} y\sin(x^2) &= x\sin(y^2) \\ 2x\cos(x^2)y + \sin(x^2)\frac{dy}{dx} &= \sin(y^2) + 2xy\cos(y^2)\frac{dy}{dx} \\ 2xy\cos(x^2) - \sin(y^2) &= (2xy\cos(y^2) - \sin(x^2))\frac{dy}{dx} \\ \frac{dy}{dx} &= \frac{2xy\cos(x^2) - \sin(y^2)}{2xy\cos(y^2) - \sin(x^2)} \end{aligned}$$

13. If $f(x)+x^2[f(x)]^3 = 10$ for all x then $f'(x)+3x^2[f(x)]^2f'(x)+2x[f(x)]^3 = 0$ for all x. Plugging in $x = 1$ and $f(1) = 2$, we get the equation $f'(1) + 12f'(1) + 16 = 0$, which implies $f'(1) = -\frac{16}{13}$.

19.

$$\begin{aligned} \frac{d}{dx}\left[2(x^2+y^2)^2\right] &= \frac{d}{dx}\left[25(x^2-y^2)\right] \\ 4(x^2+y^2)\frac{d}{dx}(x^2+y^2) &= 25\left(2x - 2y\frac{dy}{dx}\right) \\ 4(x^2+y^2)\left(2x + 2y\frac{dy}{dx}\right) &= 25\left(2x - 2y\frac{dy}{dx}\right) \end{aligned}$$

At point $(3, 1)$, we get the equation

$$4(10)\left(6 + 2\frac{dy}{dx}\right) = 25\left(6 - 2\frac{dy}{dx}\right),$$

with solution $\frac{dy}{dx} = -\frac{9}{13}$.

Thus the tangent line is

$$y - 1 = -\frac{9}{13}(x - 3)$$

30. If $y = \sqrt{\tan^{-1} x}$, then $y = u^{1/2}$ with $u = \tan^{-1} x$. So

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{1}{2}u^{-1/2}\frac{1}{1+x^2} = \frac{1}{2(1+x^2)\sqrt{\tan^{-1} x}}$$

35. If $y = \arcsin(\tan\theta)$ then $y = \arcsin u$ with $u = \tan\theta$. So

$$\frac{dy}{d\theta} = \frac{dy}{du}\frac{du}{d\theta} = \frac{1}{\sqrt{1-u^2}}\sec^2\theta = \frac{1}{\cos^2\theta\sqrt{1-\tan^2\theta}}$$

52. The slope of the tangent line to the ellipse $x^2 + 4y^2 = 36$ satisfies $2x + 8y\frac{dy}{dx} = 0$, so that $\frac{dy}{dx} = -\frac{x}{4y}$. Thus for any particular point (x_o, y_o) on the ellipse, the slope of the tangent line is $m_o = -\frac{x_o}{4y_o}$ and the tangent line has equation

$$y - y_o = -\frac{x_o}{4y_o}(x - x_o).$$

For this to pass through the point $(12, 3)$, we must have

$$\begin{aligned} 3 - y_o &= -\frac{x_o}{4y_o}(12 - x_o) \\ 12y_o - 4y_o^2 &= -12x_o + x_o^2 \\ 12y_o + 12x_o &= x_o^2 + 4y_o^2 \end{aligned}$$

Now (x_o, y_o) must also satisfy the equation $x_o^2 + 4y_o^2 = 36$, so that we get $12y_o + 12x_o = 36$ or $x_o + y_o = 3$. Now we plug this equation again into $x_o^2 + 4y_o^2 = 36$ and get

$$\begin{aligned} x_o^2 + 4(3 - x_o)^2 &= 36 \\ 5x_o^2 - 24x_o &= 0 \end{aligned}$$

with solutions $x_o = 0$ and $x_o = \frac{24}{5}$. Thus the corresponding y_o solutions are $y_o = 3$ and $y_o = -\frac{9}{5}$. So the two tangent lines are $y = 3$ and $y + \frac{9}{5} = \frac{2}{3}(x - \frac{24}{5})$.

2 Section 3.7

5. $f'(x) = \frac{1}{5x}(\ln x)^{-4/5}$

10. $f'(t) = \dfrac{(1-\ln t)\frac{1}{t} - (1+\ln t)(-\frac{1}{t})}{(1-\ln t)^2} = \dfrac{2}{t(1-\ln t)^2}$

11. First write $F(t) = 3\ln(2t+1) - 4\ln(3t-1)$. Then $F'(t) = \frac{6}{2t+1} - \frac{12}{3t-1}$.

14. First write $y = 4\ln x + 2\ln\sin x$. Then $\frac{dy}{dx} = \frac{4}{x} + \frac{2\cos x}{\sin x}$.

20. $y' = x^{-3} - 2x^{-3}\ln x$ and $y'' = -5x^{-4} + 6x^{-4}\ln x$.

25. (a) If $f(x) = x\ln x$ then $f'(x) = 1 + \ln x$. f is decreasing when $f'(x) < 0$, i.e., when $1 + \ln x < 0$, i.e., when $x < \frac{1}{e}$.

    (b) $f''(x) = \frac{1}{x}$. f is concave upward when $f''(x) > 0$, i.e., when $x > 0$.

27. $y = (2x+1)^5(x^4-3)^6$, so $\ln y = 5\ln(2x+1) + 6\ln(x^4-3)$ and

$$\frac{1}{y}\frac{dy}{dx} = \frac{10}{2x+1} + \frac{24x^3}{x^4-3};$$

thus

$$\frac{dy}{dx} = (2x+1)^5(x^4-3)^6\left[\frac{10}{2x+1} + \frac{24x^3}{x^4-3}\right]$$

31. If $y = x^x$ then $\ln y = \ln(x^x) = x\ln x$. So

$$\frac{1}{y}\frac{dy}{dx} = \ln x + 1$$

and

$$\frac{dy}{dx} = (\ln x + 1)x^x$$

38. If $x^y = y^x$ then $\ln(x^y) = \ln(y^x)$ so that $y\ln x = x\ln y$. Therefore

$$(\ln x)y' + \frac{y}{x} = x\frac{y'}{y} + \ln y$$

$$\left(\ln x - \frac{x}{y}\right)y' = \left(\ln y - \frac{y}{x}\right)$$

$$y' = \frac{\ln y - \frac{y}{x}}{\ln x - \frac{x}{y}}$$