MAT 131 HW solutions (3.2–3.5)

1 Section 3.2

4. $g'(x) = \frac{1}{2}x^{-1/2}e^x + x^{1/2}e^x$.

13. $\dfrac{dy}{dr} = (2r-2)e^r + (r^2-2r)e^r = (r^2-2)e^r$

20. $f'(x) = \dfrac{(cx+d)(a)-(ax+b)(c)}{(cx+d)^2} = \dfrac{ad-bc}{(cx+d)^2}$

21. $\frac{dy}{dx} = 2e^x + 2xe^x$ so $\frac{dy}{dx}\Big|_{x=0} = 2$, thus the tangent line has slope $m_T = 2$ and the normal line has slope $m_N = -\frac{1}{2}$. Thus the tangent line has equation $y = 2x$ and the normal line has equation $y = -\frac{1}{2}x$.

29. $f(x) = \frac{x^2}{1+x}$ so $f'(x) = \frac{(1+x)(2x)-x^2(1)}{(1+x)^2} = \frac{x^2+2x}{x^2+2x+1}$, so

$$f''(x) = \frac{(x^2+2x+1)(2x+2)-(x^2+2x)(2x+2)}{(x^2+2x+1)^2} = \frac{2x+2}{(x^2+2x+1)^2}.$$

Thus $f''(1) = \frac{1}{4}$.

34. $\dfrac{d}{dx}\left(\dfrac{h(x)}{x}\right)\Big|_{x=2} = \dfrac{xh'(x)-h(x)}{x^2}\Big|_{x=2} = \dfrac{2h'(2)-h(2)}{4} = -\dfrac{5}{2}$.

36. (a) $P'(2) = F(2)G'(2) + G(2)F'(2) = 3\cdot\frac{1}{2} + 2\cdot 0 = \frac{3}{2}$.

(b) $Q'(7) = \dfrac{G(7)F'(7)-F(7)G'(7)}{G(7)^2} = \dfrac{1\cdot\frac{1}{4} - 5\cdot(-\frac{2}{3})}{1^2} = \frac{43}{12}$

38. (a) $\dfrac{dy}{dx} = x^2f'(x) + 2xf(x)$

(b) $\dfrac{dy}{dx} = \dfrac{x^2f'(x)-2xf(x)}{x^4}$

(c) $\dfrac{dy}{dx} = \dfrac{2xf(x)-x^2f'(x)}{f(x)^2}$

(d) $\dfrac{dy}{dx} = -\dfrac{1}{2}x^{-3/2} + \frac{1}{2}x^{-1/2}f(x) + x^{1/2}f'(x)$

41. $f(x) = x^3 e^x$ so $f'(x) = (x^3 + 3x^2)e^x$. We know $e^x > 0$ always, so $f'(x) > 0$ when $x^3 + 3x^2 > 0$, i.e., when $x + 3 > 0$. So f is increasing for $x > -3$.

2 Section 3.4

3. $g'(t) = 3t^2 \cos t - t^3 \sin t$.

8. $\dfrac{dy}{dx} = \dfrac{(x+\cos x)(\cos x) - (1+\sin x)(1-\sin x)}{(x+\cos x)^2} = \dfrac{x \cos x}{(x+\cos x)^2}$

11. $f'(x) = \dfrac{\sin x(xe^x + e^x) - xe^x(\cos x)}{\sin^2 x}$

18. $\dfrac{dy}{dx} = e^x \cos x - e^x \sin x$, so at $x = 0$, we have $\frac{dy}{dx} = 1$. Thus the tangent line is $y - 1 = 1 \cdot (x - 1)$.

26. (a) $g'(\pi/3) = f'(\pi/3)\sin(\pi/3) + f(\pi/3)\cos(\pi/3) = -2 \cdot \frac{\sqrt{3}}{2} + 4 \cdot \frac{1}{2} = 2 - \sqrt{3}$.

    (b) $h'(\pi/3) = \dfrac{f(\pi/3)(-\sin(\pi/3)) - \cos(\pi/3)f'(\pi/3)}{f(\pi/3)^2} = \dfrac{4 \cdot (-\frac{\sqrt{3}}{2}) - \frac{1}{2} \cdot (-2)}{4^2} = \dfrac{-2\sqrt{3}+1}{16}$.

30. $f(x) = x - \sin x$, so $f'(x) = 1 - \cos x$ and $f''(x) = \sin x$. f is concave upward when $f'' > 0$, and on $[0, 2\pi]$, this happens when $0 < x < \pi$.

35. We note that $\frac{d}{dx}(\sin x) = \cos x$, $\frac{d^2}{dx^2}(\sin x) = -\sin x$, $\frac{d^3}{dx^3}(\sin x) = -\cos x$, and $\frac{d^4}{dx^4}(\sin x) = \sin x$. So taking 4 derivatives brings us back where we started. Therefore taking another 4 derivatives must also take us back where we started, and in general any 4-multiple derivative of $\sin x$ will be $\sin x$ again. So since $96 = 4 \cdot 24$, we have $\frac{d^{96}}{dx^{96}}(\sin x) = \sin x$, and thus $\frac{d^{99}}{dx^{99}}(\sin x) = \frac{d^3}{dx^3}(\sin x) = -\cos x$.

3 Section 3.5

3. $u = 1 - x^2$ and $y = u^{10}$, so

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = 10u^9(-2x) = -20x(1-x^2)^9$$

6. $u = e^x$ and $y = \sin u$, so

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = (\cos u)(e^x) = e^x \cos e^x$$

11. $u = a^3 + x^3$ and $y = \cos u$, so

$$\frac{dy}{dx} = -(\sin u)(3x^2) = -3x^2 \sin{(a^3 + x^3)}$$

16. The derivative can be written using the product rule as

$$\frac{dy}{dx} = e^{-5x}\frac{d}{dx}(\cos 3x) + \cos 3x \frac{d}{dx}(e^{-5x}).$$

Using the chain rule on each term, we get $\frac{d}{dx}(\cos 3x) = -3\sin 3x$ and $\frac{d}{dx}(e^{-5x}) = -5e^{-5x}$. Therefore,

$$\frac{dy}{dx} = -3e^{-5x}\sin 3x - 5e^{-5x}\cos 3x$$

19. The chain rule, then the product rule, gives

$$\frac{d}{dx}(e^{x\cos x}) = e^{x\cos x}\,\frac{d}{dx}(x\cos x) = e^{x\cos x}(\cos x - x\sin x)$$

32. $y = \sqrt{x + \sqrt{x + \sqrt{x}}}$. We have $y = w^{1/2}$ where $w = x + \sqrt{x + \sqrt{x}}$, so $\frac{dy}{dx} = \frac{1}{2}w^{-1/2}\,\frac{dw}{dx} = \frac{1}{2\sqrt{w}}(1 + \frac{d}{dx}(\sqrt{x + \sqrt{x}}))$. To compute this derivative, we let $v = \sqrt{u}$ and $u = x + \sqrt{x}$, and obtain

$$\frac{d}{dx}(\sqrt{x + \sqrt{x}}) = \frac{dv}{dx} = \frac{dv}{du}\,\frac{du}{dx} = \frac{1}{2}v^{-1/2}(1 + \frac{1}{2}x^{-1/2});$$

therefore

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x + \sqrt{x + \sqrt{x}}}}\left(1 + \frac{1}{2}\,\frac{1 + \frac{1}{2\sqrt{x}}}{\sqrt{x + \sqrt{x}}}\right)$$

43. (a) $h'(1) = f'(g(1))g'(1) = f'(2) \cdot 6 = 30$.

(b) $H'(1) = g'(f(1))f'(1) = g'(3) \cdot 4 = 36$.

46. (a) $h'(2) = f'(f(2))f'(2)$. From the graph, $f(2) = 1$, $f'(2) \approx -1$, and $f'(1) \approx -\frac{1}{2}$, so $h'(2) = f'(1)f'(2) \approx \frac{1}{2}$.

(b) $g'(x) = 2xf'(x^2)$, so $g'(2) = 4f'(4)$. From the graph, $f'(4) \approx 2$, and thus $g'(2) \approx 8$.

54. $y = e^{-x^2}$, so $y' = -2xe^{-x^2}$ and $y'' = -2e^{-x^2} + 4x^2e^{-x^2} = 2(2x^2 - 1)e^{-x^2}$. The graph of $y = e^{-x^2}$ is concave downward whenever $y'' < 0$, which happens when $2x^2 - 1 < 0$, i.e. when $|x| < \frac{1}{\sqrt{2}}$.