Math 131 Final Exam Solutions

1. True or false? Briefly justify your answer. (Correct answers with no justification will not receive full credit.)

   (a) $\dfrac{d}{dx}\displaystyle\int_1^e \ln x\,dx = 1$

   Answer False. The definite integral $\int_1^e \ln x\,dx$ is a constant. Thus its derivative is zero.

   (b) $\displaystyle\sum_{i=3}^{5} i^2 = 50$

   Answer True. $\sum_{i=3}^{5} i^2 = 3^2 + 4^2 + 5^2 = 50$.

   (c) If f is an increasing function on $[a, b]$, then $L_n \leq \displaystyle\int_a^b f(x)\,dx \leq R_n$ for every n.

   Answer True. In any rectangle, the left estimate is less than the actual area, which is less than the right estimate.

   (d) If $f'(1) = 0$ and $f''(1) = 0$, then f must have an inflection point at $x = 1$.

   Answer False. $f(x) = (x-1)^4$ is an example. The function is always concave up.

   (e) The inverse of $f(x) = x^2$ on $(-\infty, \infty)$ is $f^{-1}(x) = \sqrt{x}$.

   Answer False. $f(x) = x^2$ has no inverse since it's not one-to-one.

2. Multiple choice. Briefly justify your answer. (Correct answers with no justification will not receive full credit.)

   (a) In which interval(s) is there a solution of $x^3 + 3x = \sqrt{x^2+1}$?

   (a) $(-1, 0)$ only (b) $(0, 1)$ only (c) both $(-1, 0)$ and $(0, 1)$ (d) neither $(-1, 0)$ nor $(0, 1)$

   Answer Let $f(x) = x^3 + 3x - \sqrt{x^2+1}$. Then $f(-1) = -4 - \sqrt{2} < 0$, $f(0) = -1$, and $f(1) = 4 - \sqrt{2} > 0$. Thus there is only a solution in $(0, 1)$, (b).

   (b) Which value of a would make the function

   $$f(x) = \begin{cases} x^2 + a & x < 2 \\ ax + 1 & x \geq 2 \end{cases}$$

   continuous at $x = 2$?

   (a) $a = 0$ (b) $a = 1$ (c) $a = 2$ (d) $a = 3$

   Answer We have $\lim_{x\to 2^-} f(x) = 2^2 + a = 4 + a$ and $\lim_{x\to 2^+} f(x) = 2a + 1$. To have continuity, the limit should be the same from both sides, $4 + a = 2a + 1$, so $a = 3$. (d).

(c) If $(2,3)$ is a point on the graph of $y = f(x)$, then which of the following points must appear on the graph of $y = 2f(2x) - 1$?

(a) $(4,5)$ (b) $(1,5)$ (c) $(4,4)$ (d) $(1,4)$

Answer We know $f(2) = 3$, and nothing else. Thus $2f(2) - 1 = 5$, so when $x = 1$ and $y = 5$, then $5 = 2f(2 \cdot 1) - 1$. Thus the answer is (b).

(d) If $f(x) = \sqrt{x^2 + 9} - 4$ is being integrated on the interval $[0,4]$, exactly how large must n be so that $R_n - L_n$, the difference between the right-hand sum and the left-hand sum, is less than 0.01?

(a) $n \geq 100$ (b) $n \geq 200$ (c) $n \geq 400$ (d) $n \geq 800$

Answer The general formula is

$$R_n - L_n = \frac{(b-a)[f(b) - f(a)]}{n}.$$

In this case we have

$$R_n - L_n = \frac{4 \cdot 2}{n} \leq 0.01,$$

so that $n \geq 800$. (d).

3. (a) Find the tangent line to the curve

$$xy^3 + \frac{x^3}{y} = 2$$

at the point $(1,1)$.

Answer Use implicit differentiation.

$$3xy^2 \frac{dy}{dx} + y^3 + \frac{3x^2}{y} - \frac{x^3}{y^2} \frac{dy}{dx} = 0,$$

and plugging in $x = 1$, $y = 1$, we get

$$3\frac{dy}{dx} + 1 + 3 - \frac{dy}{dx} = 0,$$

so that $\frac{dy}{dx} = -2$.
Thus the tangent line is $y - 1 = -2(x - 1)$.

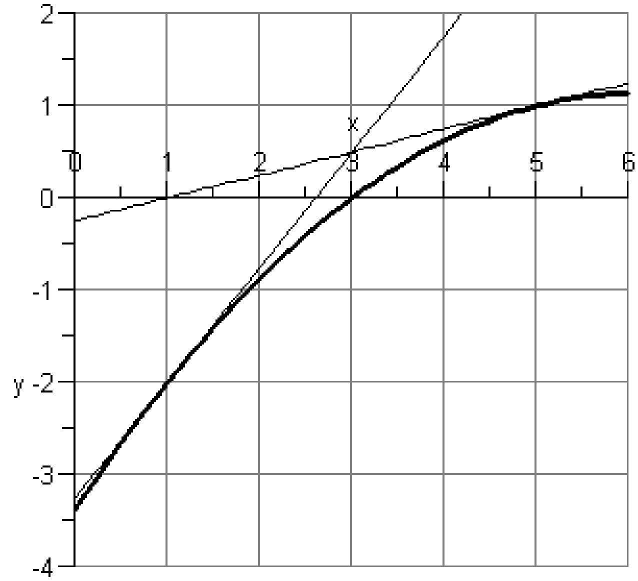
(b) Use the linear approximation from part (a) to estimate the solution of

$$0.9y^3 + \frac{0.9^3}{y} = 2$$

Answer When $x = 0.9$, we have

$$y \approx 1 - 2(x - 1) = 1 - 2(0.9 - 1) = 1 - 2(-0.1) = 1.2$$

4. Consider $y = f(x)$, given by the following graph.



If Newton's method is used to find the solution of $f(x) = 0$, with initial guess $x_1 = 5$, illustrate Newton's method graphically (for two steps), and display on the graph the x_2 and x_3 that you obtain. Your answer may not be exact; some reasonable tolerance will be allowed.

Answer The graph of the two tangent lines are shown. We have $x_1 = 5$, $x_2 = 1$, and $x_3 \approx 2.6$.

5. Consider $f(x) = (x^2 + 1)e^{-x}$.

   (a) Determine when f is increasing and when it is decreasing.

   Answer f is increasing when $f'(x) > 0$. We compute

$$f'(x) = (x^2 + 1)(-e^{-x}) + 2xe^{-x} = -(x^2 - 2x + 1)e^{-x} = -(x - 1)^2 e^{-x}.$$

   So $f'(x) \leq 0$ for every x, and hence $f'(x)$ is always decreasing. The point $x = 1$ is a stationary point.

   (b) Determine when f is concave up and when it is concave down.

   Answer f is concave up when $f''(x) > 0$. We compute

$$f''(x) = -(x^2 - 2x + 1)(-e^{-x}) - (2x - 2)e^{-x} = (x^2 - 4x + 3)e^{-x} = (x - 1)(x - 3)e^{-x}.$$

   So $f''(x) = 0$ when $x = 1$ or $x = 3$. For $x < 1$, $f''(x) > 0$. For $1 < x < 3$, $f''(x) < 0$. For $x > 3$, $f''(x) < 0$. Thus f is concave up for $x < 1$, concave down for $1 < x < 3$, and concave up for $x > 3$.

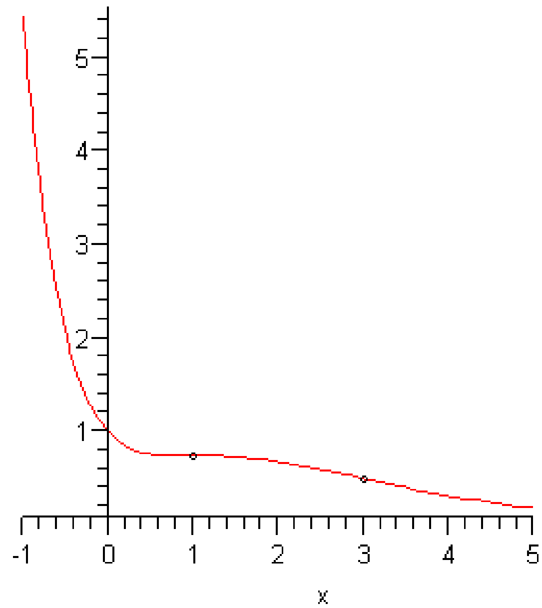
(c) Determine $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$.

Answer

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x^2+1}{e^x} = \frac{\infty}{\infty} = \lim_{x\to\infty} \frac{2x}{e^x} = \lim_{x\to\infty} \frac{\infty}{\infty} = \lim_{x\to\infty} \frac{2}{e^x} = 0.$$

$$\lim_{x\to-\infty} f(x) = \infty \cdot \infty = \infty.$$

(d) Use the information from parts (a)–(c) to sketch a rough graph of f. Label any interesting points.



6. (a) Prove, using only the definition of the derivative, that

$$\frac{d}{dx}(x^3 - 2x^2) = 3x^2 - 4x$$

for every x.

Answer

$$\begin{aligned}
\frac{d}{dx}(x^3 - 2x^2) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{(x+h)^3 - 2(x+h)^2 - x^3 + 2x^2}{h} \\
&= \lim_{h\to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - 2x^2 - 4xh - 2h^2 - x^3 + 2x^2}{h} \\
&= \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3 - 4xh - 2h^2}{h} \\
&= \lim_{h\to 0}(3x^2 + 3xh + h^2 - 4x - 2h) \\
&= 3x^2 - 4x
\end{aligned}$$

(b) Prove, using only the definition of the integral, that

$$\int_0^b (3x^2 - 4x)\, dx = b^3 - 2b^2$$

for every b.

Answer We have $\Delta x = \frac{b}{n}$ and $x_i = \frac{bi}{n}$. Thus $f(x_i) = 3\left(\frac{bi}{n}\right)^2 - 4\left(\frac{bi}{n}\right)$, so the integral from the definition is

$$\begin{aligned}
\int_0^b (3x^2 - 4x)\, dx &= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x \\
&= \lim_{n\to\infty} \frac{b}{n} \sum_{i=1}^{n} 3\left(\frac{bi}{n}\right)^2 - 4\left(\frac{bi}{n}\right) \\
&= \lim_{n\to\infty} \frac{b}{n} \left(\frac{3b^2}{n^2} \sum_{i=1}^{n} i^2 - \frac{4b}{n} \sum_{i=1}^{n} i \right) \\
&= \lim_{n\to\infty} \left(\frac{3b^3}{n^3} \frac{n(n+1)(2n+1)}{6} - \frac{4b^2}{n^2} \frac{n(n+1)}{2} \right) \\
&= b^3 - 2b^2
\end{aligned}$$

7. (a) Evaluate $\displaystyle\int \frac{x^5}{x^3+1}\, dx$

Answer Let $u = x^3 + 1$. Then $du = 3x^2\, dx$, so $x^2\, dx = \frac{1}{3}\, du$. We also have $x^3 = u - 1$. Thus the integral is

$$\int \frac{x^5}{x^3+1}\, dx = \int \frac{1}{3} \frac{u-1}{u}\, du = \frac{1}{3} \int 1 - \frac{1}{u}\, du$$
$$\frac{1}{3}(u - \ln u) + C = \frac{1}{3}(x^3 + 1 - \ln(x^3+1)) + C$$

(b) Evaluate $\displaystyle\int_{1/2}^{2} \frac{(\ln x)^2}{x}\, dx$

Answer Let $u = \ln x$. Then $du = \frac{dx}{x}$, and the integral is

$$\int \frac{(\ln x)^2}{x}\, dx = \int u^2\, du = \frac{1}{3}u^3 = \frac{1}{3}(\ln x)^3.$$

Thus the definite integral is

$$\int_{1/2}^{2} \frac{(\ln x)^2}{x}\, dx = \frac{1}{3}(\ln x)^3\Big|_{1/2}^{2} = \frac{1}{3}(\ln 2)^3 - \frac{1}{3}(-\ln 2)^3 = \frac{2}{3}(\ln 2)^3.$$

8. Suppose the acceleration of a helicopter is $a(t) = 10$ ft/sec^2 for $0 \le t \le 8$ seconds. If the helicopter is initially falling from a height of 200 feet with a velocity of -40 feet per second, find

(a) the velocity $v(t)$,

Answer $v(t) = v(0) + \int_0^t a(t)\, dt = -40 + 10t.$

(b) the speed $s(t)$,

Answer $s(t) = |10t - 40|.$

(c) the position $h(t)$, and

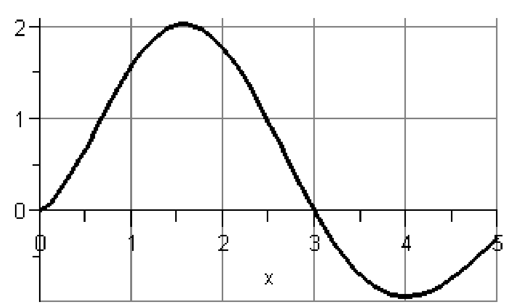
Answer $h(t) = h(0) + \int_0^t h(t)\, dt = 200 + \int_0^t (10t - 40)\, dt = 200 + (5t^2 - 40t).$

(d) the distance travelled between $t = 3$ and $t = 6$ seconds.

Answer The distance is

$$d = \int_3^6 s(t)\, dt = \int_3^6 |10t - 40|\, dt = \int_3^4 (40 - 10t)\, dt + \int_4^6 (10t - 40)\, dt$$
$$= (40t - 5t^2)\Big|_3^4 + (5t^2 - 40t)\Big|_4^6 = (160 - 80) - (120 - 45) + (180 - 240) - (80 - 160) = 25 \text{ ft}$$

9. The graph of $y = f'(x)$ is shown. Answer the following questions about $f(x)$.



(a) Assume that $f(0) = -1$. Estimate $f(3)$ and $f(5)$.

Answer We have

$$f(3) - f(0) = \int_0^3 f'(x)\, dx \approx 3.5,$$

so $f(3) \approx 2.5$.
Also we have

$$f(5) - f(3) = \int_3^5 f'(x)\, dx \approx -1.75,$$

so $f(5) \approx 1.25$.

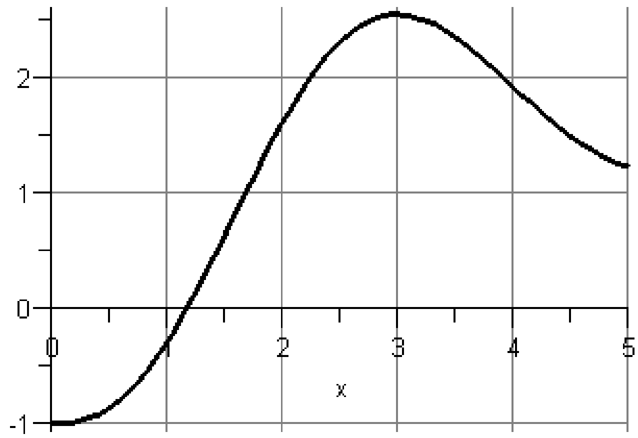
(b) Where is the absolute maximum of $f(x)$? Where is the absolute minimum?

Answer The critical point is $x = 3$, while the endpoints are $x = 0$ and $x = 5$. We already know $f(0) = -1$, $f(3) \approx 2.5$, and $f(5) \approx 1.25$. Thus the absolute maximum occurs at $x = 3$, and the absolute minimum occurs at $x = 0$.

(c) Where are the inflection points of $f(x)$?

Answer Inflection points of $f(x)$ are local extrema of $f'(x)$. These occur at $x = 1.5$ and $x = 4$.

(d) Sketch a graph of $y = f(x)$ using the information from parts (a)–(c).

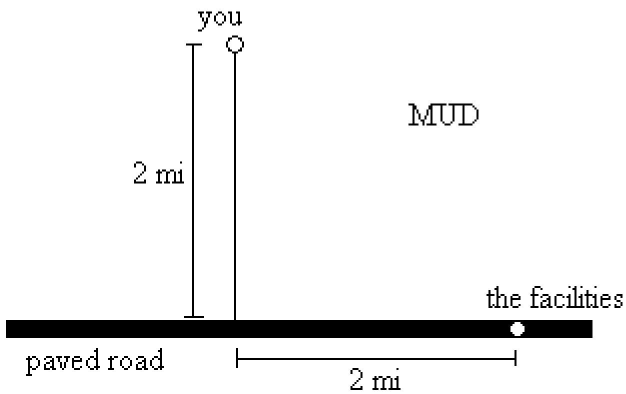


10. Summer! You have saved enough money to take what seems a well-earned vacation in sunny sandy Tattooine. One afternoon you set up a deck chair and an umbrella in a little secluded spot by the rocks, **2 miles** away from the paved road. You pull out your notebook and begin to solve miscellaneous optimization word problems—they were always your favorite kind.

    As you are computing the largest ellipsoid that can be inscribed in a cone, an uncharacteristically heavy rainstorm begins. The sand turns to deep thick mud as you shrink under your umbrella. But soon the food you ate in the cantina catches up with you, and you urgently need to use the nearest facilities, **2 miles** down the road.

    "Crap!" you exclaim, as you realize you'll have to run through the downpour as fast as you can. Your top speed is **10 miles per hour**, but you can only run that fast on the paved road. Through the mud, it will be slower; you estimate it will be about **8 miles per hour**. You decide to **calculate the shortest path** under the umbrella, while you're dry, then make a run for it. As your notebook grows damp, you draw the diagram below. Furiously you assign variables; desperately you determine the natural endpoints; frantically you find the total time required and minimize it.

    What do you get?



Answer Let x be the point on the road (measured from your horizontal location) that you run to, through the mud. Then you run distance $\sqrt{4+x^2}$ through the mud (at 8 miles per hour) and distance $2-x$ on the road (at 10 miles per hour).

Your total time is

$$T(x) = \frac{\sqrt{4+x^2}}{8} + \frac{2-x}{10}.$$

This formula only makes sense if $0 \leq x \leq 2$, so these are the natural boundaries.

We find the critical points of $T(x)$:

$$T'(x) = \frac{x}{8\sqrt{4+x^2}} - \frac{1}{10} = 0,$$

so $10x = 8\sqrt{4+x^2}$ and $100x^2 = 64(4+x^2)$, which implies $36x^2 = 256$, or $x = \frac{8}{3}$. But $\frac{8}{3} > 2$, so this would imply running *past* the facilities and then back on the road, which cannot be right. The problem is that the actual distance on the road is $|2-x|$, not $2-x$.

So the function has *no* critical points in $[0,2]$, and hence the minimum must occur at an endpoint $x=0$ or $x=2$. When $x=0$, we have $T(0) = \frac{1}{2} + \frac{1}{5} = \frac{7}{10}$, while when $x=2$, we have $T(2) = \frac{\sqrt{8}}{8} = \frac{\sqrt{2}4}{\approx}0.4$. Thus the quickest route is to run through the mud diagonally, directly to the facilities, and skip the road entirely.

Bonus question: (10 points)

Suppose $f(x)$ is a differentiable function satisfying

$$\int_0^x tf(t)\,dt = f(x)^2$$

for every x.

(a) Explain why $f(0) = 0$.

Answer

$$0 = \int_0^0 tf(t)\,dt = f(0)^2,$$

so $f(0) = 0$.

(b) One solution of the equation is $f(x) = 0$. Determine what the other solution is. Hint: differentiate both sides with respect to x to get an equation for $f(x)$.

Answer Differentiating both sides, and using the Fundamental Theorem of Calculus, we get

$$\begin{aligned} xf(x) &= 2f(x)f'(x) \\ \frac{x}{2} &= f'(x) \\ \frac{x^2}{4} &= f(x) + C \end{aligned}$$

and $C = 0$ since $f(0) = 0$. So $f(x) = \frac{x^2}{4}$.

(c) Check that your answer satisfies the original equation.

Answer

$$\int_0^x \frac{t^3}{4}\,dt = \frac{1}{16}t^4\Big|_0^x = \frac{x^4}{16} = \left(\frac{x^2}{4}\right)^2.$$