Spring 2006 early exam solutions

1. The equation of the straight line through the points $(-1,3)$ and $(2,-3)$ is

   (a) $y-1=2x$;

   (b) $y+1=2x$;

   (c) $y=5-2x$;

   (d) $y=1-2x$;

   (e) none of the above.

   Answer The line has slope $m=\frac{\Delta y}{\Delta x}=\frac{3-(-3)}{-1-2}=-2$. Plug point $(2,-3)$ into the general equation $y=-2x+b$ and obtain $b=1$. So the line is $y=1-2x$, choice (d).

2. The equation of the straight line through point $(1,1)$ parallel to line $y=3x-1$ is

   (a) $y=x$

   (b) $y=3x$

   (c) $y=-\frac{1}{3}x+1$

   (d) $y=-\frac{1}{3}x-1$

   (e) none of the above.

   Answer The parallel line must have the same slope, $m=3$. Plug point $(1,1)$ into the general formula $y=3x+b$ and get $b=-2$. So the desired line is $y=3x-2$. This is not an option, so the answer is (e).

3. The set of all solutions to the inequality $x^2-4x+3>0$ is

   (a) $(-1,3)$

   (b) $(1,3)$

   (c) $(-\infty,1)\cup(3,\infty)$

   (d) $(-\infty,-1)\cup(3,\infty)$

   (e) none of the above.

Answer The inequality factors as $(x-1)(x-3) > 0$. The product of two numbers is positive if either both numbers are positive or both are negative. So either $x-1>0$ *and* $x-3>0$, or $x-1<0$ *and* $x-3<0$. If $x>1$ and $x>3$, then $x>3$ ($x>1$ is automatically satisfied when $x>3$). Similarly, if $x<1$ and $x<3$, then $x<1$. So the inequality will be satisfied when either $x>3$ or $x<1$. The answer is thus (c).

4. $\dfrac{2^{2x}3^{x+1}}{6^{x+1}}$ is equal to

   (a) 2^{x+1}
   (b) 2^{x-1}
   (c) $\left(\frac{2}{3}\right)^{x-1}$
   (d) 2^{2x-1}
   (e) none of the above.

   Answer The formula simplifies as

   $$\frac{2^{2x}3^{x+1}}{6^{x+1}} = \frac{2^{2x}3^{x+1}}{2^{x+1}3^{x+1}} = 2^{2x-(x+1)}3^{(x+1)-(x+1)} = 2^{x-1},$$

   so the answer is (b).

5. After simplification, expression $\dfrac{(ab)^3a^{-2}}{\sqrt{ab}}$ is equal to

   (a) $\sqrt{ab^3}$
   (b) $a^{1/2}b^{5/2}$
   (c) $\dfrac{ab^2}{\sqrt{a}}$
   (d) $(\sqrt{a})^3b^3$
   (e) none of the above.

   Answer The formula simplifies as

   $$\frac{(ab)^3a^{-2}}{\sqrt{ab}} = \frac{a^3b^3a^{-2}}{a^{1/2}b^{1/2}} = a^{3-2-1/2}b^{3-1/2} = a^{1/2}b^{5/2},$$

   so the answer is (b).

6. If $a = \frac{2}{x-1}$, $b = \frac{2}{x+1}$, then $\dfrac{1}{a} - \dfrac{1}{b}$ is equal to

(a) $\frac{2}{(x-1)(x+1)}$

(b) $\frac{-4}{(x-1)(x+1)}$

(c) -1

(d) $\frac{x}{2}$

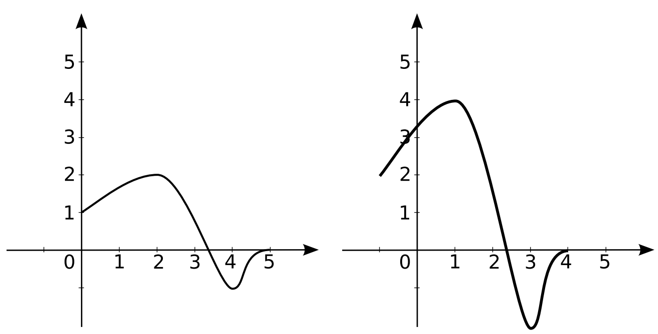
(e) none of the above.

Answer The fractions can be written as

$$\frac{1}{a} - \frac{1}{b} = \frac{1}{\frac{2}{x-1}} - \frac{1}{\frac{2}{x+1}} = \frac{x-1}{2} - \frac{x+1}{2} = \frac{-1-1}{2} = -1.$$

So the answer is (c).

7.



(a) (b)

Figure (a) shows the graph of function $f(x)$. Then the graph in Figure (b) is the graph of

(a) $f(2(x+1))$

(b) $2f(x+1)$

(c) $2f(x-1)$

(d) $f(\frac{1}{2}(x-1))$

(e) none of the above.

Answer The graph is twice as high and twice as low as the original graph, so it is being stretched vertically by a factor of 2. Thus we must multiply f on the outside by 2. In addition, the graph has been moved to the left by 1 unit. So we must add inside the parentheses one unit, i.e., replace (x) with $(x+1)$. Thus the graph is of $y = 2f(x+1)$, choice (b).

8. Function $f(x) = \cos(2x) + 2x^2$ is

   (a) even but not odd

   (b) odd but not even

   (c) both even and odd

   (d) neither even nor odd

   Answer Both cosine and the squaring function are even functions, and therefore the sum is also an even function. No even function can be an odd function (except for the zero function), and thus the answer is (a).

9. The function $f(x) = x^2 - 2x$ has

   (a) minimum at $x = 1$

   (b) maximum at $x = 1$

   (c) minimum at $x = 0$

   (d) maximum at $x = -1$

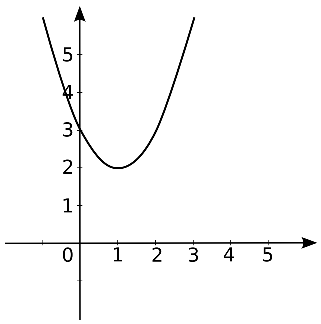
   (e) none of the above.

   Answer Complete the square:

$$x^2 - 2x = (x^2 - 2x + 1) - 1 = (x - 1)^2 - 1.$$

   The squared term is always positive, so we can make it as large as we want. (Thus there is no maximum.) On the other hand, the smallest we can make the squared term is 0, which occurs when $x = 1$. Thus the answer is (a).

10. The figure below is the graph of the function



(a) $f(x) = (x-1)^2 + 2$

(b) $f(x) = (x+1)^2 + 2$

(c) $f(x) = (x+2)^2 + 1$

(d) $f(x) = 2(x+1)^2 + 1$

Answer The vertex of the parabola is $(1, 2)$, so the parabola must be of the form $f(x) = k(x-1)^2 + 2$ (using the standard vertex form of a parabola). Since $f(0) = 3$, the constant k must be 1. Thus the answer is choice (a).

11. If $0 < \theta < \pi/2$ and $\sin\theta = \frac{\sqrt{3}}{2}$, then $\tan\theta$ is equal to

    (a) $\frac{1}{2}$

    (b) $\frac{1}{3}$

    (c) $\frac{1}{\sqrt{3}}$

    (d) $\sqrt{3}$

    (e) none of the above.

Answer If $\sin\theta = \frac{\sqrt{3}}{2}$, then

$$\cos\theta = \pm\sqrt{1 - \sin^2\theta} = \pm\sqrt{1 - \frac{3}{4}} = \pm\sqrt{\frac{1}{4}} = \pm\frac{1}{2}.$$

Since we know θ is in the first quadrant, we are certain that $\cos\theta = \frac{1}{2}$. Thus

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3}.$$

The answer is (d).

12. The set of solutions of inequality $2^{-x} < 4$ is

    (a) $x < 2$
    (b) $x > 2$
    (c) $x > -2$
    (d) $x < -2$
    (e) none of the above.

    Answer The logarithm is an increasing function, so if $2^{-x} < 4$, then $\ln(2^{-x}) < \ln 4$. Thus $-x \ln 2 < 2 \ln 2$, and since $\ln 2$ is positive, we can divide without changing the inequality. So we get $-x < 2$. Now multiply the inequality by -1 to solve for x—we have to switch the inequality when we multiply by a negative number. We get $x > -2$. So the answer is (c).

13. Solution of the equation $9^x \cdot 27 = \left(\frac{1}{3}\right)^x$ is

    (a) $x = 0$
    (b) $x = 1$
    (c) $x = -1$
    (d) $x = \frac{1}{2}$
    (e) none of the above.

    Answer Write everything using the same power, 3: $9^x \cdot 27 = \left(\frac{1}{3}\right)^x$ becomes $3^{2x}3^3 = 3^{-x}$, so that $3^{2x+3} = 3^{-x}$. The only way two powers of 3 can be equal is if the exponents are equal, so we must have $2x + 3 = -1$. Thus $x = -1$, and the answer is (c).

14. If $x = \log_3 5$, then $9^{(1-x)}$ is equal to

    (a) $3^2 - 5^2$
    (b) $3^2 \cdot 5^2$
    (c) $\frac{9}{25}$
    (d) $3^{(2-x)}$

(e) none of the above.

Answer Use laws of exponents:

$$9^{1-x} = 9 \cdot 9^{-x} = 9 \cdot \frac{1}{3^{2x}} = \frac{9}{3^{2\log_3 5}} = \frac{9}{3^{\log_3 5^2}} = \frac{9}{5^2} = \frac{9}{25}.$$

Thus the answer is (c).

15. If $f(x) = 2x + 1$, $g(x) = 2 - \sin(x)$, then $f \circ g(x) =$

(a) $2 - \sin(2x + 1)$

(b) $3 - 2\sin(x)$

(c) $3 - \sin(2x)$

(d) $6 - 2\sin(x))$

(e) none of the above.

Answer To compute $f \circ g(x)$, replace x in the formula for f with $g(x)$. So

$$f \circ g(x) = f\big(g(x)\big) = f\big(2 - \sin x\big) = 2(2 - \sin x) + 1 = 4 - 2\sin x + 1 = 5 - 2\sin x.$$

This is not one of the options, so the answer is (e).

16. If $f(g(x)) = \sin^2(e^x)$ and $f(x) = x^2$, then $g(x)$ is

(a) $\sin^2(e^{\sqrt{x}})$

(b) $\sin(x^2)$

(c) $e^{\sin^2(x)}$

(d) $\sin(e^x)$

(e) none of the above.

Answer Rewrite $f(g(x)) = (\sin(e^x))^2 = f(\sin(e^x))$ to see that $g(x)$ must be $g(x) = \sin(e^x)$. Thus the answer is (d).