Weierstrass Representations of Minimal Real Kähler Surfaces

Peter Hennes

Abstract

We give a local classification, construction methods, and some new examples for minimal isometric immersions $f : M^4 \to \mathbf{R}^6$, where M is a Kähler manifold. Our main tool is a local characterization of these so-called minimal real Kähler surfaces for general codimension, which is based on a "Weierstrass coordinate system" for the homogeneous space of all isotropic complex planes in a complex vector space.

1. Introduction

Since the nineteenth century, Weierstrass representations have been used to investigate minimal surfaces in Euclidean 3-space. Recall (e.g. from [S], p.395, [O], or [L]) that for every minimal isometric immersion $f : M^2 \to \mathbf{R}^3$, defined on some surface M that is (locally) endowed with isothermal coordinates $(x\,,\, y)$, one can find a holomorphic function $\lambda(z)$ (with $z = x + i\,y$) and a meromorphic function $\xi(z)$ on M such that $f(x\,,\,y) = \operatorname{Re}\left(\int X(z)\,dz\right)$, where

$$X = \lambda \left(\frac{1-\xi^2}{2}\,,\; i\,\frac{1+\xi^2}{2}\,,\; \xi \right) . \tag{1}$$

This parametrization is a *Weierstrass representation* of the minimal surface. Its key feature is that the complex derivative $X = \frac{\partial f}{\partial z} = \frac{\partial f}{\partial x} - i\,\frac{\partial f}{\partial y}$ is always an *isotropic* vector in $\mathbf{C}^3$; this means that with respect to the standard *symmetric* inner product $X \cdot Y = \sum_{j=1}^{3} X_j\,Y_j$ in $\mathbf{C}^3$, we have $X^2 = X \cdot X = 0$. It is not hard to see that all isotropic vectors $X \in \mathbf{C}^3$ can be written in the form (1), at least as long as $X_1 \neq i\,X_2$. Actually, Hoffman and Osserman showed in [H-O] that the above description works for minimal surfaces in *any* Euclidean space $\mathbf{R}^N$: simply replace the squares in (1) by symmetric inner products $\xi^2 = \xi \cdot \xi$ for vectors $\xi \in \mathbf{C}^{N-2}$.

In the last two decades, it emerged that minimal real Kähler submanifolds share many of the features of minimal surfaces ([D-G$_1$], [D-G$_2$]). For instance, if $f : M^{2n} \to \mathbf{R}^N$ is any such minimal isometric immersion from a Kähler manifold M into an Euclidean space, one finds that with respect to any complex coordinate system $(z_1\,, \ldots, z_n)$ on M, the partial derivatives $X_j = \frac{\partial f}{\partial z_j} = \frac{\partial f}{\partial x_j} - i\,\frac{\partial f}{\partial y_j}$ (with $z_j = x_j + i\,y_j$) are *holomorphic* as maps $M \to \mathbf{C}^N$ and span n-dimensional *isotropic subspaces* in $\mathbf{C}^N$, which means that

$$X_j \cdot X_k = 0 \quad \text{for all } 1 \leq j\,,\, k \leq n\,.$$

Thus, if we could find a simple parametrization for isotropic n-spaces in $\mathbf{C}^N$, we should locally be able to explicitly write down "Weierstrass representations" of f, in the form $f = \operatorname{Re}\left(\int f_*^{\mathbf{C}}\right) = \operatorname{Re}\left(\int X_1\,dz_1 + \ldots + X_n\,dz_n\right)$.

Note that the X_j determine the *complex Gauss map* $\varphi = \operatorname{span}\{X_1, \dots, X_n\}$ of f, here considered as a *holomorphic* map[1] $\varphi : M \to \mathrm{Gr}_n(\mathbf{C}^N)$ into the Grassmannian of all complex n-planes in $\mathbf{C}^N$. Of course, for $n \geq 2$ the X_j must in addition satisfy the *integrability conditions*

$$\frac{\partial}{\partial z_j} X_k = \frac{\partial}{\partial z_k} X_j \quad \text{for all } 1 \leq j\,,\, k \leq n$$

in order for the above integral to exist. In this case, the map $F = \frac{1}{\sqrt{2}} \int f_*^{\mathbf{C}}$ (defined on some open subset $W \subset M$) is called a *holomorphic representative*[2] of f, and we have

$$f|_W = \sqrt{2}\,\mathrm{Re}(F)\ .$$

It now turns out that these necessary conditions (holomorphic, maximum rank, isotropic, integrable) are also locally sufficient for the existence of minimal real Kähler submanifolds (see e.g. [D-G$_2$], or [A-P-S]). In fact, based on the work [H-O] mentioned above, Dajczer and Gromoll use this approach in [D-G$_2$] to give Weierstrass representations for (generically) all *complete* minimal real Kähler immersions $f : M^{2n} \to \mathbf{R}^{2n+2}$ if $n \geq 3$. Their method allows the explicit construction of examples of such submanifolds.

However, the assumptions of completeness and on the dimension of M are crucial for their method to work. Thus, it is for example not clear how such immersions may look *locally* if M is *not complete*, or what may happen if the (real) dimension of M is four, i.e. in the case of a minimal real Kähler *surface* in $\mathbf{R}^6$. To our knowledge, only a few examples in this case were known up to this point; see [F] and [D-G$_3$]. Recently, Arezzo, Pirola, and Solci were able to give entire series of examples of such surfaces; see [A-P-S]. But a classification of these submanifolds had not been established until this time. Following the ideas outlined above, we will be able to give such a local classification in §3.

This article is a summary of the results of the author's dissertation [H]; readers interested in the proofs and further details can download a complete copy from the author's homepage. I want to thank my advisor, Detlef Gromoll, for his support while I was completing this work.

2. Complex Isotropic Planes and Minimal Real Kähler Surfaces

It is well-known (see [R-T]) that the space of all *isotropic* complex n-subspaces in $\mathbf{C}^N$, here denoted by $\mathbf{I}_n(\mathbf{C}^N)$, is a compact complex homogeneous space, namely

$$\mathbf{I}_n(\mathbf{C}^N) \;\cong\; SO(N)/\big(U(n) \times SO(N-2n)\big)\ .$$

We will now describe a coordinate system for the complex $(2\,N-7)$-dimensional space $\mathbf{I}_2(\mathbf{C}^N)$. Write the complex coordinates in $\mathbf{C}^{2N-7}$ as $(\xi\,,\,\zeta\,,\,\lambda)$, where $\xi = (\xi_1\,, \dots, \xi_{N-4})$ and $\zeta = (\zeta_1\,, \dots, \zeta_{N-4})$ are in $\mathbf{C}^{N-4}$ and λ is a complex

---

[1] One finds the *antiholomorphic* complex Gauss map $\bar{\varphi}$ in the literature more frequently.

[2] The factor $\sqrt{2}$ makes $f \oplus 0$ and F congruent in $\mathbf{R}^{2N}$.

number. Then define the holomorphic map $\Phi : \mathbf{C}^{2N-7} \to \mathbf{I}_2(\mathbf{C}^N)$ by

$$\Phi(\xi\,,\,\zeta\,,\,\lambda) = \text{ span}\left\{ \begin{pmatrix} \frac{1}{2} \\ \frac{i}{2} \\ X \end{pmatrix} , \begin{pmatrix} -X\cdot Y \\ i\,X\cdot Y \\ Y \end{pmatrix} \right\} ,$$

where

$$X = \lambda\left(\frac{1-\xi^2}{2}\,,\, i\,\frac{1+\xi^2}{2}\,,\,\xi \right) \quad\text{and}\quad Y = \left(\frac{1-\zeta^2}{2}\,,\, i\,\frac{1+\zeta^2}{2}\,,\,\zeta \right) \tag{2}$$

are in $\mathbf{C}^{N-2}$, and $X \cdot Y$, $\xi^2 = \xi \cdot \xi$, and $\zeta^2 = \zeta \cdot \zeta$ refer to the standard *symmetric* inner products in $\mathbf{C}^{N-2}$ and $\mathbf{C}^{N-4}$, respectively (compare §1). To obtain a coordinate system, restrict Φ to the open and connected set

$$G\Phi = \left\{ \,(\xi\,,\,\zeta\,,\,\lambda) \in \mathbf{C}^{2N-7} \;\middle|\; X \cdot Y = -\frac{\lambda}{2}\,(\xi-\zeta)^2 \neq 0 \,\right\} .$$

Then it is straightforward (if tedious) to show that $\Phi : G\Phi \to \mathbf{I}_2(\mathbf{C}^N)$ is an injective immersion, and thus gives a coordinate system on $\Phi(G\Phi) \subset \mathbf{I}_2(\mathbf{C}^N)$.

Using this "Weierstrass coordinate system" to locally represent the complex Gauss map φ of a minimal real Kähler surface, and in addition the integrability conditions for φ, one can now prove the following proposition, which gives the promised *local characterization* of such submanifolds.

Proposition 1: *Let $f : M^4 \to \mathbf{R}^N$ be a minimal isometric immersion from a (real) 4–dimensional Kähler manifold M into Euclidean N-space, and let $F : W \to \mathbf{C}^N$ be a holomorphic representative of f (as in* §1*). Then we can find a neighborhood U about every point in W, a complex coordinate system $(u\,,\,v)$ of M defined on U, and holomorphic maps $\xi\,,\,\zeta : U \to \mathbf{C}^{N-4}$ and $\lambda : U \to \mathbf{C}$ such that everywhere*

$$\lambda\,(\xi-\zeta)^2 \neq 0\,, \tag{3}$$

and such that, up to isometry in $\mathbf{R}^N$,

$$F_u = \left(\frac{1}{2}\,,\,\frac{i}{2}\,,\,X \right) \quad\textit{and}\quad F_v = \big(-1\,,\,i\,,\,Y \big)\,, \tag{4}$$

with

$$X = \lambda\left(\frac{1-\xi^2}{2}\,,\, i\,\frac{1+\xi^2}{2}\,,\,\xi \right) \quad\textit{and}\quad Y = \mu\left(\frac{1-\zeta^2}{2}\,,\, i\,\frac{1+\zeta^2}{2}\,,\,\zeta \right)\,, \tag{5}$$

where μ is defined by

$$\mu = -\frac{2}{\lambda\,(\xi-\zeta)^2}\,. \tag{6}$$

Conversely, any holomorphic functions ξ, ζ, and λ as above in two complex variables $(u\,,\,v) \in \mathbf{C}^2$ that satisfy (3) *and, in addition,*

$$\lambda_v = \mu_u\,,\;\; (\lambda\,\xi)_v = (\mu\,\zeta)_u\,,\;\; \textit{and}\;\; (\lambda\,\xi^2)_v = (\mu\,\zeta^2)_u \tag{7}$$

(with μ defined as in (6)*), give rise to a minimal real Kähler surface in Euclidean N-space whose holomorphic representative F has partial derivatives with respect to u and v which have the form as in* (4) *and* (5)*.*

In the situation of this proposition, we call the triple $\big((u\,,\,v)\,,\,X\,,\,Y\big)$ a *Weierstrass representation* of the minimal real Kähler surface. Note that (7) is equivalent to the integrability condition $X_v = Y_u$ for the maps X and Y. Also, (4) and (5) give in particular that the complex Gauss map φ of f factors over U as $\varphi|_U = \Phi \circ (\xi\,,\,\zeta\,,\,\lambda)$.

Looking at (7), it seems somewhat questionable how useful Proposition 1 really could be in studying the structure of minimal real Kähler surfaces for general dimension N. However, in the case $N = 6$ (i.e. for real codimension two), Proposition 1 suffices to completely uncover the local structure of these submanifolds, as we will see in the next section.

3. Local Classification and Construction of Minimal Real Kähler Surfaces in $\mathbf{R}^6$

Our classification consists of two non-trivial cases, which are distinguished by the *rank of the second osculating bundle* F'' of a holomorphic representative F of the minimal real Kähler surface in question. This bundle is essentially the span of the 2-jet of F; for any complex coordinates $(z_1\,,\,z_2)$ on M, we have

$$F'' = \operatorname{span}\left\{ \frac{\partial F}{\partial z_j}\,,\ \frac{\partial^2 F}{\partial z_j\,\partial z_k}\ \middle|\ 1 \le j\,,\,k \le 2 \right\}\,.$$

In terms of a Weierstrass representation $((u\,,\,v)\,,\,X\,,\,Y)$ of f as described in §2, we obtain

$$F'' = \operatorname{span}\left\{ \begin{pmatrix} \frac{1}{2} \\ \frac{i}{2} \\ X \end{pmatrix}, \begin{pmatrix} -1 \\ i \\ Y \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ X_u \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ X_v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ Y_u \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ Y_v \end{pmatrix} \right\},$$

and thus

$$\operatorname{rank} F'' = 2 + \dim \operatorname{span}\{X_u\,,\,X_v = Y_u\,,\,Y_v\}\,.$$

Since F and thus X and Y are holomorphic, this rank is (locally) constant on M, except perhaps for some isolated points.

The case that $\operatorname{rank} F'' = 2$ is trivial, since this means that M is simply a piece of a 4-plane in $\mathbf{R}^6$. Further, using the isotropy of X and Y, the fact that $X \cdot Y \equiv 1$, and the integrability condition $X_v = Y_u$, it is easy to show that we must always have $\operatorname{rank} F'' \le 4$. In the "generic" case $\operatorname{rank} F'' = 4$, the map X (or the map Y) turns out to be locally immersive. The key idea to solve this case is to use the map X, or better the map $\xi = (s\,,\,t)$ as in (5), to parametrize our minimal real Kähler surface; this is somewhat similar to using the (real) Gauss map of a minimal surface in $\mathbf{R}^3$ to find its isothermal coordinates (see e.g. [S], pp.385–386). This approach leads to the crucial observation that in the new coordinates $(s\,,\,t)$, the (complex) Gauss map of the Kähler surface is completely determined by the "scaling function" λ of X as in (5).

The following theorem summarizes how one can construct such surfaces, starting with (almost) any holomorphic function in two variables.

Theorem 2: *Let* $\tilde{\lambda}$ *be any* nowhere zero *holomorphic function* without singularities *in the two complex variables* $(s\,,\,t)$*, defined on some open subset of* $\mathbf{C}^2$*. Define the holomorphic functions* $a\,,\ b\,,$ *and* c *by*

$$a = \frac{-2\,\tilde{\lambda}_{ss}\,\tilde{\lambda} + 3\,\tilde{\lambda}_s^2 - \tilde{\lambda}_t^2}{2\,\tilde{\lambda}^4} \quad , \quad b = \frac{-2\,\tilde{\lambda}_{st}\,\tilde{\lambda} + 4\,\tilde{\lambda}_s\,\tilde{\lambda}_t}{2\,\tilde{\lambda}^4}\ ,$$

$$\text{and} \qquad c = \frac{-2\,\tilde{\lambda}_{tt}\,\tilde{\lambda} - \tilde{\lambda}_s^2 + 3\,\tilde{\lambda}_t^2}{2\,\tilde{\lambda}^4}\ .$$

Next, solve the second order linear homogeneous partial differential equation

$$b\,v_{ss} + (c-a)\,v_{st} - b\,v_{tt} + (b_s + c_t)\,v_s - (b_t + a_s)\,v_t = 0 \tag{8}$$

for v*. Then locally, there is a function* u *such that*

$$u_s = c\,v_s + b\,v_t \quad \text{and} \quad u_t = b\,v_s - a\,v_t\ .$$

On an open subset where the map $(u\,,\,v)$ has (complex) rank two, *calculate the inverse* $\xi = \xi(u\,,\,v)$ *of the map* $\big(u(s\,,\,t)\,,\,v(s\,,\,t)\big)$*, and set* $\lambda = \tilde{\lambda}\circ\xi = \lambda(u\,,\,v)$*. Finally, define the following holomorphic functions:*

$$\zeta_1 = \xi_1 + \left(\frac{2\,\tilde{\lambda}\,\tilde{\lambda}_s}{\tilde{\lambda}_s^2 + \tilde{\lambda}_t^2}\right)\circ\xi \quad , \quad \zeta_2 = \xi_2 + \left(\frac{2\,\tilde{\lambda}\,\tilde{\lambda}_t}{\tilde{\lambda}_s^2 + \tilde{\lambda}_t^2}\right)\circ\xi\ ,$$

and

$$\mu = \frac{-2}{\lambda\,(\xi-\zeta)^2} = -\left(\frac{\tilde{\lambda}_s^2 + \tilde{\lambda}_t^2}{2\,\tilde{\lambda}^3}\right)\circ\xi\ ,$$

and set $\zeta = (\zeta_1,\,\zeta_2)$*. Then the triple* $\big((u\,,\,v)\,,\,X\,,\,Y\big)$*, with* X *and* Y *defined in terms of* ξ*,* ζ*,* λ*, and* μ *as in* (5) *in Proposition* 1*, is the Weierstrass representation of a minimal real Kähler surface in* $\mathbf{R}^6$*.*

Conversely, the holomorphic representative F *of any minimal real Kähler surface in* $\mathbf{R}^6$ *with* $\operatorname{rank} F'' = 4$ *can locally and up to isometry in* $\mathbf{R}^6$ *be constructed in this way.*

Note that (8) is nothing but the integrability condition for the coordinate function u, and that (8) can always be solved locally, by virtue of the Cauchy–Kowalewski Theorem.

Example: Let $\tilde{\lambda}(s\,,\,t) = t$. Then (8) collapses to the separable equation

$$\frac{2}{t^4}\,v_{st} - \frac{6}{t^5}\,v_s = 0\ ,$$

so that $v_s(s\,,\,t) = g(s)\,t^3$ for some function g. Choosing $g(s) = e^{s+C}$, where C is a constant that will be determined later, we obtain $v(s\,,\,t) = e^{s+C}\,t^3$ (up to a constant). This gives the following "gradient" for u:

$$u_s = -\frac{3}{2\,t}\,e^{s+C} \quad \text{and} \quad u_t = \frac{3}{2\,t^2}\,e^{s+C}\ ,$$

which can be integrated to $u(s\,,\,t) = -\frac{3}{2\,t}\,e^{s+C}$. One can easily check that the Jacobian of $(u\,,\,v)$ as a holomorphic map in $(s\,,\,t)$ is never zero, since we must have that $t \neq 0$. To find the inverse of this map, first note that $\dfrac{v}{u} = -\dfrac{2\,t^4}{3}$. Using the function $z^{\frac{1}{4}} = \exp(\frac{1}{4}\,\log z)$ on some branch of the complex logarithm, we obtain

$$t = \xi_2(u\,,\,v) = \left(\frac{-3\,v}{2\,u}\right)^{\frac{1}{4}}\,,$$

so that we have $e^{s+C} = -\frac{2\,t}{3}\,u = -\frac{2}{3}\left(\frac{-3\,v}{2\,u}\right)^{\frac{1}{4}}\,u$. Taking the logarithm on both sides, and then choosing C in such a way that all constant terms add up to zero, we finally obtain

$$s = \xi_1(u\,,\,v) = \frac{1}{4}\,\log(v\,u^3)\,,$$

which then leads to

$$\lambda = t = \left(\frac{-3\,v}{2\,u}\right)^{\frac{1}{4}}\,,\quad \mu = -\frac{1}{2}\left(\frac{-3\,v}{2\,u}\right)^{-\frac{3}{4}}\,.$$

$$\zeta_1 = s = \frac{1}{4}\,\log(v\,u^3)\,,\quad \text{and}\quad \zeta_2 = 3\,t = 3\left(\frac{-3\,v}{2\,u}\right)^{\frac{1}{4}}\,.$$

Inserting these functions into our equations for X and Y as in (5) gives us the complex Gauss map of a minimal real Kähler surface in $\mathbf{R}^6$.

Let us now turn to the "non-generic" case, where $\operatorname{rank} F'' = 3$. This condition leads to three classes of minimal real Kähler surfaces. The first of these stems from all the immersions $f : M \to \mathbf{R}^6$ that are *holomorphic* with respect to some complex structure on $\mathbf{R}^6$. It is a well-known criterion that this is the case if and only if the image of the complex Gauss map of f is contained in a *fixed* isotropic subspace of $\mathbf{C}^N$ (see e.g. [R-T] p.517, or [L]). Since the maximal dimension of an isotropic subspace of $\mathbf{C}^6$ is three, this implies that the second osculating bundle of such an immersion must necessarily have rank ≤ 3. In particular, this shows that our generic case $\operatorname{rank} F'' = 4$ will always lead to *non–holomorphic* maps $f : M \to \mathbf{R}^6$.

The second class of such surfaces consists of the ones that are "*generated by an isotropic cylinder*" with one fixed isotropic direction; i.e., we can locally and up to isometry in $\mathbf{R}^6$ write $f(z\,,\,w) = \sqrt{2}\,\operatorname{Re}\Big(z\,\mathbf{X} + \int \mathbf{Z}(w)\,dw\Big)$, where $(z\,,\,w)$ is a suitable complex coordinate system on M, $\mathbf{X}$ a *fixed* non–zero isotropic vector in $\mathbf{C}^6$, and $\mathbf{Z} = \mathbf{Z}(w)$ a holomorphic map such that $\mathbf{X} \cdot \mathbf{Z} = 0$ and $\mathbf{X}$ and $\mathbf{Z}$ are linearly independent everywhere.

It is easy to see that such an f will indeed have $\operatorname{rank} F'' = 3$, but somewhat tedious to show that they (and certain holomorphic immersions) are *exactly* the ones that satisfy the integrability condition (7) trivially (i.e., $X_v = Y_u \equiv 0$). This means that for the surfaces in the third class, we will have $X_v = Y_u \neq 0$

almost everywhere. Analyzing this condition leads to the construction method described in the following theorem, which summarizes the results in the case $\operatorname{rank} F'' = 3$.

Theorem 3: *Let* $F : W \to \mathbf{C}^6$ *be a holomorphic representative of a minimal real Kähler surface in* $\mathbf{R}^6$*, and assume that* the rank of the second osculating bundle F'' of F equals three *on all of* W*. Then* f *is holomorphic with respect to some complex structure on* $\mathbf{R}^6$*, or* f *is generated by an isotropic cylinder (as described above), or we can represent* f *locally in the following way:*
Let $((u\,,\,v)\,,\,X\,,\,Y)$ *be a Weierstrass representation of* F *as described in* §2. *Then in the neighborhood of any point in* W*, we can find a solution* α *of the non–linear partial differential equation*

$$\alpha_u = \alpha\,\alpha_v = \frac{1}{2}\,(\alpha^2)_v \tag{9}$$

and two holomorphic functions $\tilde\lambda(w)$ *and* $h(w)$ *in one complex variable* w *such that we have*

$$X = \left[\tilde\lambda\,\left(\frac{1-\tilde\xi^2}{2}\,,\,i\,\frac{1+\tilde\xi^2}{2}\,,\,\tilde\xi\right)\right]\circ\alpha \quad \textit{and} \quad Y = \left[\tilde\mu\,\left(\frac{1-\tilde\zeta^2}{2}\,,\,i\,\frac{1+\tilde\zeta^2}{2}\,,\,\tilde\zeta\right)\right]\circ\alpha\,,$$

where $\tilde\mu = \int \frac{\tilde\lambda'}{w}\,dw\,,$ g *is a function such that* $g^2 = \frac{-2}{\tilde\lambda\,\tilde\mu}\,,$

$$\tilde\zeta = (\tilde\zeta_1\,,\,\tilde\zeta_2) = \left(\int \frac{(\tilde\lambda\,g\,\cos h)'}{w\,\tilde\mu-\tilde\lambda}\,dw\,,\,\int \frac{(\tilde\lambda\,g\,\sin h)'}{w\,\tilde\mu-\tilde\lambda}\,dw\right)\,,$$

and

$$\tilde\xi = (\tilde\xi_1\,,\,\tilde\xi_2) = \tilde\zeta + g\,(\cos h\,,\,\sin h)\,.$$

Conversely, for any choice of a non–constant solution $\alpha(u\,,\,v)$ *of* (9)*, a nowhere zero function* $\tilde\lambda(w)$*, and an arbitrary function* $h(w)$*, the formulas above give the Weierstrass representation of a minimal real Kähler surface in* $\mathbf{R}^6$ *with* $\operatorname{rank} F'' = 3$.

Example: By a separation ansatz, one finds that for all constants $A\,,\,B \in \mathbf{C}$,

$$\alpha(u\,,\,v) = -\frac{v+A}{u+B}$$

is a solution of (9). Setting $\tilde\lambda = -1$ implies that $\tilde\mu$ also has to be constant, and choosing $\tilde\mu = 2$ (so that $g = 1$) leads to

$$\tilde\zeta_1' = \frac{h'(w)\,\sin(h(w))}{2\,w+1} \quad \text{and} \quad \tilde\zeta_2' = \frac{-h'(w)\,\cos(h(w))}{2\,w+1}\,.$$

Choosing $h(w) = w^2 + w$ then implies

$$\tilde\zeta = \begin{pmatrix} \int \sin(w^2+w)\,dw \\ -\int \cos(w^2+w)\,dw \end{pmatrix} \quad \text{and} \quad \tilde\xi = \tilde\zeta + \begin{pmatrix} \cos(w^2+w) \\ \sin(w^2+w) \end{pmatrix}\,.$$

Remark: Note that we can use Theorem 3 to construct minimal real Kähler *hypersurfaces* in $\mathbf{R}^5$: setting $h = 0$ will give a map f into Euclidean 5–space.

However, there is an easier way to obtain such hypersurfaces. As shown in [H], the Dajczer–Gromoll construction method for the *complete* codimension–two case (see §1) can easily be modified to allow the construction of minimal real Kähler hypersurfaces of *any* dimension. Their structure was first clarified in [D-G$_1$], where Dajczer and Gromoll showed that such hypersurfaces are intimately related to *superminimal* surfaces in Euclidean spheres. The latter were first studied by Calabi in 1968 (see [C]), and are still a topic of research in algebraic geometry. Based on the idea outlined above, the author was able to give two explicit examples of superminimal surfaces in the 4–sphere in [H].

References

[A-P-S] C. Arezzo, G. Pirola, and M. Solci: *The Weierstrass Representation for Pluriminimal Submanifolds.* Preprint

[C] E. Calabi: *Quelques applications de l'analyse complexe aux surfaces d'aire minima.* Topics is Complex Manifolds, University of Montreal, Canada (1968), 59–81

[D-G$_1$] M. Dajczer and D. Gromoll: *Real Kaehler Submanifolds and Uniqueness of the Gauss Map.* J. Diff. Geom. **22** (1985), 13–28

[D-G$_2$] M. Dajczer and D. Gromoll: *The Weierstrass Representation for Complete Minimal Real Kaehler Submanifolds of Codimension Two.* Invent. Math. **119** (1995), 235–242

[D-G$_3$] M. Dajczer and D. Gromoll: *Complete Minimal Kähler Surfaces in* $\mathbf{R}^6$*.* Ann. Glob. Anal. Geom. **15** (1997), 539–541

[F] H. Furuhata: *Construction and Classification of Isometric Minimal Immersions of Kähler Manifolds into Euclidean Spaces.* Bull. London Math. Soc. **26** (1994), 487–496

[H] P. Hennes: *Weierstrass Representations of Minimal Real Kähler Submanifolds.* Ph.D. Thesis, State University of New York at Stony Brook (2001)

[H-O] D. A. Hoffman and R. Osserman: *The Geometry of the Generalized Gauss Map.* Memoirs Am. Math. Soc. **28**, No. 236 (1980)

[L] H. B. Lawson: *Lectures on Minimal Surfaces.* Publish or Perish, Inc., Houston, Texas (1980)

[O] R. Osserman: *A Survey of Minimal Surfaces.* Dover Publications, Inc., New York (1986)

[R-T] M. Rigoli and R. Tribuzy: *The Gauss Map for Kählerian Submanifolds of* $\mathbf{R}^n$*.* Trans. Am. Math. Soc. **332** (1992), 515–528

[S] M. Spivak: *A Comprehensive Introduction to Differential Geometry,* Vol. IV, 2nd edition. Publish or Perish, Inc., Houston, Texas (1979)

Peter Hennes
Alumnus of the State University of New York at Stony Brook
phennes@math.sunysb.edu
http://www.math.sunysb.edu/∼phennes