4 Minimal real Kähler surfaces in $\mathbf{R}^6$

In this chapter, we will use the Weierstrass representation for minimal real Kähler surfaces that we described in the last chapter to give a classification and a local construction method in the special case of codimension 2, i.e. when the ambient Euclidean space is $\mathbf{R}^6$. The classification will consist of two non–trivial cases, which will be distinguished by the *rank of the second osculating bundle* F'' of the holomorphic representative F of the minimal real Kähler surface. Here,

$$F'' := \operatorname{span}\left\{ \frac{\partial F}{\partial z_j} \, , \, \frac{\partial^2 F}{\partial z_j \, \partial z_k} \,\middle|\, 1 \le j \, , \, k \le 2 \right\}$$

for any complex chart $(z_1 \, , \, z_2)$ on M. Note that F'' is independent of the choice of the complex chart. In fact, since F'' basically consists of vectors that define the complex Gauss map of f and its "first derivatives", it is also independent of the choice of F, and thus an *invariant* for our minimal real Kähler immersion f. In terms of a Weierstrass representation $((u \, , \, v) \, , \, X \, , \, Y)$ of f (compare page 37 and Proposition 3.2), we obtain

$$F'' = \operatorname{span}\left\{ \begin{pmatrix} \frac{1}{2} \\ \frac{i}{2} \\ X \end{pmatrix} , \begin{pmatrix} -1 \\ i \\ Y \end{pmatrix} , \begin{pmatrix} 0 \\ 0 \\ X_u \end{pmatrix} , \begin{pmatrix} 0 \\ 0 \\ X_v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ Y_u \end{pmatrix} , \begin{pmatrix} 0 \\ 0 \\ Y_v \end{pmatrix} \right\} ,$$

and by Laplace's determinant criterion, it is clear that we have

$$\operatorname{rank} F'' = 2 + \dim \operatorname{span}\{X_u \, , \, X_v = Y_u \, , \, Y_v\} \, . \tag{1}$$

Since F and thus X and Y are holomorphic (and M without loss of generality connected), this rank is, therefore, constant on M, perhaps except for some isolated points. For this reason, it makes sense to talk about "the rank" of the second osculating bundle of F.

The case $\operatorname{rank} F'' = 2$ is trivial, since this means that M is simply a (piece of a) 4-plane in $\mathbf{R}^6$. We will see soon that we must always have $\operatorname{rank} F'' \le 4$, and that the "generic case" is $\operatorname{rank} F'' = 4$, since all these minimal real Kähler surfaces in $\mathbf{R}^6$ will (locally) emerge from the same construction method. The only remaining case, $\operatorname{rank} F'' = 3$, will lead to three classes of minimal real Kähler surfaces, one of which are the ones generated by isotropic cylinders (as in Example 1.6) with *one* fixed isotropic direction

$\mathbf{X}_1$, and another class being all the immersions $f : M \to \mathbf{R}^6$ that are *holomorphic* with respect to some complex structure on $\mathbf{R}^6$. Note that since the maximal dimension of an isotropic subspace of $\mathbf{C}^6$ is 3, the criterion mentioned on page 12 (by [R-T]) implies that the second osculating bundle of such an holomorphic map must necessarily have rank ≤ 3. Thus, our "generic case" will always lead to *non–holomorphic* maps.

To establish that the case $\operatorname{rank} F'' = 4$ is indeed "generic", we need the following proposition, which explores the case that the integrability condition $X_u = Y_v$ is trivially satisfied in detail.

Proposition 4.1: *Let $f : M^4 \to \mathbf{R}^6$ be a minimal real Kähler surface, and let $((u\,,\,v)\,,\,X\,,\,Y)$ be a Weierstrass representation of f that is defined on some open subset U of M (compare page 37 and Proposition 3.2). Furthermore, assume that*

$$X_v = Y_u \equiv 0\;.$$

Then, f is holomorphic with respect to some complex structure on $\mathbf{R}^6$, or f is generated by an isotropic cylinder (see Example 1.6). In the latter case, we have more exactly that for each point p in U, we can find a neighborhood V of p in U, a coordinate system $(z_1\,,\,w)$ on V, a fixed non–zero isotropic vector $\mathbf{X_1}$ in $\mathbf{C}^6$, and a holomorphic map $\mathbf{Z} = \mathbf{Z}(w)$ with values in a complementary subspace to $\mathbf{X_1}$ in the isotropic orthogonal complement of $\mathbf{X_1}$, such that we have on all of V that

$$f(z_1\,,\,w) = \sqrt{2}\;\operatorname{Re}\Big(z_1\,\mathbf{X_1} + \int \mathbf{Z}(w)\,dw\Big) + b_0$$

for some fixed vector $b_0 \in \mathbf{R}^6$.

Proof: If $X = (X_1\,,\ldots,\,X_{N-2})$, then we have by Proposition 3.2 that $\lambda = X_1 - i\,X_2$ and $\xi = \frac{1}{\lambda}\,(X_3\,,\ldots,\,X_{N-2})$. Therefore, our hypothesis implies

$$\lambda_v = 0 \quad \text{and} \quad \xi_v = 0\;;$$

i.e. λ and ξ are functions "in u alone". Similarly, we find that μ and ζ are functions "in v alone"; i.e.

$$\mu_u = 0 \quad \text{and} \quad \zeta_u = 0\;.$$

So, we have $((\xi-\zeta)^2)_u = 2\,(\xi-\zeta)\cdot\xi_u$, and thus $((\xi-\zeta)^2)_{uv} = -2\,\xi_u\cdot\zeta_v$. Using that we always have $\mu = -\frac{2}{\lambda\,(\xi-\zeta)^2}$, we see that

$$\left(\frac{-2}{\lambda\,\mu}\right)_{uv} = -2\,\xi_u\cdot\zeta_v\ ,$$

and noting that $\lambda_v = 0$ and $\mu_u = 0$, we find

$$\left(\frac{1}{\lambda}\right)_u\left(\frac{1}{\mu}\right)_v = \xi_u\cdot\zeta_v\ . \tag{2}$$

Here, we have to distinguish two cases: either we have that λ or μ is constant, and thus that $\xi_u\cdot\zeta_v \equiv 0$; or we have that $\lambda_u \neq 0$, $\mu_v \neq 0$, and $\xi_u\cdot\zeta_v \neq 0$ almost everywhere (except perhaps at some isolated points).

Case 1: λ or μ is constant, and $\xi_u\cdot\zeta_v \equiv 0$. We will show that in this case, f is *generated by an isotropic cylinder.*

First, we will prove that it suffices to show that either $\mathbf{X}$ or $\mathbf{Y}$ is constant, which by our hypothesis $X_v = Y_u = 0$ is equivalent to showing that $X_u = 0$ or $Y_v = 0$. For assume e.g. that $\mathbf{X}$ is constant. Then, by $0 = \mathbf{X}_v = (F_u)_v = (F_v)_u = \mathbf{Y}_u$, $\mathbf{Y}$ is a function "in v alone". And by integrating, we obtain that

$$F(u\,,\,v) = u\,\mathbf{X} + \tilde{\mathbf{Y}}(v) + \mathbf{C}_0$$

for some map $\tilde{\mathbf{Y}} = \tilde{\mathbf{Y}}(v)$ and a constant vector $\mathbf{C}_0 \in \mathbf{C}^6$. But then $F_v = \frac{d}{dv}\tilde{\mathbf{Y}}$, and hence

$$\frac{d}{dv}(\mathbf{X}\cdot\tilde{\mathbf{Y}}) = \mathbf{X}\cdot F_v = F_u\cdot F_v = 0\ .$$

This means that $\mathbf{X}\cdot\tilde{\mathbf{Y}}$ is constant, which implies that $\mathbf{X}\cdot(\tilde{\mathbf{Y}} - \tilde{\mathbf{Y}}(v_0)) = 0$, if we fix v_0 for some point $p_0 \in U$ such that $v(p_0) = v_0$. Since we can write

$$F(u\,,\,v) = u\,\mathbf{X} + (\tilde{\mathbf{Y}}(v) - \tilde{\mathbf{Y}}(v_0)) + (\tilde{\mathbf{Y}}(v_0) + \mathbf{C}_0)\ ,$$

we can thus assume, without loss of generality, that $\tilde{\mathbf{Y}}$ and $\mathbf{C}_0$ are chosen in a way such that

$$\mathbf{X}\cdot\tilde{\mathbf{Y}} = 0 \quad \text{on all of } U\ .$$

This means that $\tilde{\mathbf{Y}}$ is always in the "isotropic orthogonal complement" of $\mathbf{X}$ in $\mathbf{C}^6$, which is 5–dimensional (since "$\cdot$" is non–degenerate). Denote this complement by $\mathbf{X}^{\perp^\bullet}$ (compare page 14).

Now, $\mathbf{X}^{\perp^\bullet}$ obviously always contains $\mathbf{X}$. Thus, by choosing an complementary subspace W to $\mathbf{X}$ in $\mathbf{X}^{\perp^\bullet}$, we can write

$$\tilde{\mathbf{Y}}(v) = \eta(v)\,\mathbf{X} + \tilde{\mathbf{Z}}(v)\ ,$$

where $\eta(v)$ is a holomorphic function and $\tilde{\mathbf{Z}}(v)$ a holomorphic map with values in the *fixed* subspace W. This means that F can be written as

$$F(u\,,\,v) = (u + \eta(v))\,\mathbf{X} + \tilde{\mathbf{Z}}(v) + \mathbf{C}_0\ .$$

Changing coordinates to $z_1 := u + \eta(v)$ and $w := v$, writing $\mathbf{Z} := \frac{d}{dw}\tilde{\mathbf{Z}}$ (which obviously takes values only in W), and adjusting the domain V of this new complex chart for M, if necessary, obviously gives us the desired form of f as in Example 1.6.

To establish our claim for Case 1, it remains to be shown that we indeed have that $X_u = 0$ or $Y_v = 0$. By our case assumption, we know that either λ or μ is constant, and by Lemma 3.5 we can, without loss of generality, assume that μ is constant. By our hypothesis $X_v = Y_u \equiv 0$, we obviously have that

$$\lambda_v = 0\ \ ,\ \ \xi_v = 0\ \ ,\quad \text{and}\quad \zeta_u = 0\ . \tag{3}$$

Since μ is defined by $\mu = \frac{-2}{\lambda\,(\xi-\zeta)^2}$, the first of these equations implies that we also have

$$\frac{\partial}{\partial v}(\xi-\zeta)^2 = \frac{\partial}{\partial v}\,\frac{-2}{\lambda\,\mu} = \frac{2\,\lambda_v}{\lambda^2\mu} = 0\ . \tag{4}$$

If we write $\xi = (\xi_1\,,\,\xi_2)$ and $\zeta = (\zeta_1\,,\,\zeta_2)$, our case assumption furthermore gives that

$$0 = \xi_u\cdot\zeta_v = (\xi_1)_u\,(\zeta_1)_v + (\xi_2)_u\,(\zeta_2)_v\ .$$

Now assume that $(\zeta_2)_v \neq 0$. Then, we can rewrite the last equation as

$$(\xi_2)_u = -\frac{(\zeta_1)_v}{(\zeta_2)_v}\,(\xi_1)_u\ .$$

But (3) means in particular that ξ is a function "in u alone", whereas ζ is a function "in v alone". Thus, the last equation implies that $-\frac{(\zeta_1)_v}{(\zeta_2)_v}$ must be a constant $C \in \mathbf{C}$. It then follows that

$$(\xi_2)_u = C\,(\xi_1)_u \quad\text{and}\quad (\zeta_1)_v = -C\,(\zeta_2)_v\ ,$$

which implies that there are two more constants $A,\, B \in \mathbf{C}$ such that

$$\xi_2 = C\,\xi_1 + A \quad \text{and} \quad \zeta_1 = -C\,\zeta_2 + B\,.$$

Using these equations, we can calculate that

$$\begin{aligned}
(\xi - \zeta)^2 \;&=\; (\xi_1 - \zeta_1)^2 + (\xi_2 - \zeta_2)^2 = (\xi_1 + C\,\zeta_2 - B)^2 + (C\,\xi_1 + A - \zeta_2)^2 \\
&=\; \xi_1^2 + C^2\zeta_2^2 + B^2 + 2\,C\,\xi_1\,\zeta_2 - 2\,B\,\xi_1 - 2\,BC\,\zeta_2 \\
&\qquad + C^2\xi_1^2 + A^2 + \zeta_2^2 + 2\,AC\,\xi_1 - 2\,C\,\xi_1\,\zeta_2 - 2\,A\,\zeta_2 \\
&=\; (1 + C^2)\,(\xi_1^2 + \zeta_2^2) + 2\,(AC - B)\,\xi_1 - 2\,(BC + A)\,\zeta_2 + (A^2 + B^2)\,.
\end{aligned}$$

Inserting this expression into (4) gives (since $\xi_v = 0$, by (3))

$$\begin{aligned}
0 \;&=\; 2\,(1 + C^2)\,\zeta_2\,(\zeta_2)_v + 0 - 2\,(BC + A)\,(\zeta_2)_v + 0 \\
&=\; 2\,(\zeta_2)_v\,\Big((1 + C^2)\,\zeta_2 - (BC + A)\Big)\,.
\end{aligned}$$

But since we assumed that $(\zeta_2)_v \neq 0$, and thus not constant, this implies that the term in parenthesis must be zero. This, in turn, would mean that ζ_2 is constant, unless

$$1 + C^2 = 0 \quad \text{and} \quad BC + A = 0\,.$$

So, we obtain that $(C = i$ and $A = -i\,B)$ or $(C = -i$ and $A = i\,B)$. In either case, we have that

$$A^2 + B^2 = 0 \quad \text{and} \quad AC - B = 0\,.$$

But this would mean that $(\xi - \zeta)^2 = 0$, which is impossible for a Weierstrass representation.

Thus, we must have that $(\zeta_2)_v = 0$, and since ζ_2 is a function "in v alone", this means that ζ_2 is constant. Using this in (4), we obtain

$$0 = \frac{\partial}{\partial v}(\xi - \zeta)^2 = -2\,(\xi_1 - \zeta_1)\,(\zeta_1)_v\,.$$

So, either $(\zeta_1)_v = 0$ or $\xi_1 = \zeta_1$, which after differentiating with respect to v, and by (3), also gives $(\zeta_1)_v = 0$. In any case, ζ_1 is also constant, which makes ζ constant. Since we assumed that μ is constant, this means that Y

and thus also $\mathbf{Y}$ in the Weierstrass representation of f is constant, which is what we claimed in Case 1.

Case 2: $\lambda_u \neq 0$, $\mu_v \neq 0$, and $\xi_u \cdot \zeta_v \neq 0$ almost everywhere. We will show that in this case, f is *generated by an isotropic graph* (see Example 1.4), and is thus, by Proposition 1.5, *holomorphic* with respect to some complex structure on $\mathbf{R}^6$.

To simpify notation, write the components of ξ and ζ as $\xi = (s\,,\, t)$ and $\zeta = (p\,,\, q)$. Also, write

$$a := \frac{1}{(1/\lambda)_u} \quad \text{and} \quad b := \frac{1}{(1/\mu)_v}\ ,$$

so that, after multiplying by a and b (which by case assumption are almost never zero), equation (2) now reads

$$1 = a\,b\,(\xi_u \cdot \zeta_v) = a\,b\,(s_u\,p_v + t_u\,q_v) = (a\,s_u)(b\,p_v) + (a\,t_u)(b\,q_v)\ . \qquad (5)$$

Note that in the last term, all the factors in parenthesis are either functions in u alone or functions in v alone, respectively. Taking the partial derivative with respect to u gives, after dividing by b,

$$0 = (a\,s_u)_u\,p_v + (a\,t_u)_u\,q_v\ .$$

Now *assume* that we have $(a\,t_u)_u \neq 0$ almost everywhere. Then the last equation gives that

$$q_v = -\frac{(a\,s_u)_u}{(a\,t_u)_u}\,p_v\ .$$

But since q_v and p_v are functions in v alone, $\frac{(a\,s_u)_u}{(a\,t_u)_u}$ must be constant. So there is a $C \in \mathbf{C}$ such that

$$q_v = C\,p_v\ .$$

Taking the derivative with respect to v in (5) gives, after dividing by a,

$$0 = s_u\,(b\,p_v)_v + t_u\,(b\,q_v)_v = s_u\,(b\,p_v)_v + t_u\,(b\,C\,p_v)_v = (s_u + C\,t_u)(b\,p_v)_v\ .$$

Again *assuming* that almost everywhere $(b\,p_v)_v \neq 0$, we obtain

$$s_u = -C\,t_u\ .$$

But this would result in

$$\xi_u \cdot \zeta_v = \begin{pmatrix} s_u \\ t_u \end{pmatrix} \cdot \begin{pmatrix} p_v \\ q_v \end{pmatrix} = s_u\, p_v + t_u\, q_v = -C\, t_u\, p_v + C\, t_u\, p_v \equiv 0 \ ,$$

in contradiction to our case assumption.

Thus, we must have that $(a\, t_u)_u \equiv 0$ or that $(b\, p_v)_v \equiv 0$. By (5), this implies in the first case that we also have that $(a\, s_u)_u \equiv 0$ or that $p_v \equiv 0$, and in the second case that we also have that $(b\, q_v)_v \equiv 0$ or that $t_u \equiv 0$. But if e.g. $p_v \equiv 0$, then (5) gives $1 = (a\, t_u)(b\, q_v)$, so that $(b\, q_v)_v \equiv 0$ and, of course, $(b\, p_v)_v \equiv 0$. In any case, we find that

$$a \begin{pmatrix} s_u \\ t_u \end{pmatrix} = \frac{1}{(1/\lambda)_u}\, \xi_u \text{ is constant, or that } b \begin{pmatrix} p_v \\ q_v \end{pmatrix} = \frac{1}{(1/\mu)_v}\, \zeta_v \text{ is constant,}$$

and by Lemma 3.5 we may, without loss of generality, assume that the former is the case. Thus, there is a constant vector $\xi_1 \in \mathbf{C}^2$ such that $\xi_u = (1/\lambda)_u\, \xi_1$, which implies that there is another constant vector $\xi_0 \in \mathbf{C}^2$ such that

$$\xi = \xi(u) = \frac{1}{\lambda}\, \xi_1 + \xi_0 \ .$$

With this form of ξ, (2) gives $(1/\lambda)_u\, (1/\mu)_v = (1/\lambda)_u\, \xi_1 \cdot \zeta_v$, and since $\lambda_u \neq 0$, this means

$$\xi_1 \cdot \zeta_v = \left(\frac{1}{\mu} \right)_v \ .$$

Furthermore, substituting the form of ξ as above into $(\xi - \zeta)^2 = \frac{-2}{\lambda\,\mu}$, we obtain

$$\tfrac{-2}{\lambda\,\mu} = \left(\tfrac{1}{\lambda}\, \xi_1 + \xi_0 - \zeta \right)^2 = \tfrac{1}{\lambda^2}\, \xi_1^2 + \tfrac{2}{\lambda}\, \xi_1 \cdot (\xi_0 - \zeta) + (\xi_0 - \zeta)^2 \ . \qquad (6)$$

Taking the partial derivative with respect to v and then substituting the expression we found for $\xi_1 \cdot \zeta_v$ results in

$$-\frac{2}{\lambda} \left(\frac{1}{\mu} \right)_v = -\frac{2}{\lambda} \underbrace{\xi_1 \cdot \zeta_v}_{=(1/\mu)_v} -2\, (\xi_0 - \zeta) \cdot \zeta_v \ ,$$

or, after simplifying,

$$0 = 2\, (\xi_0 - \zeta) \cdot \zeta_v = \frac{\partial}{\partial v}\, (\xi_0 - \zeta)^2 \ .$$

But this means that $(\xi_0 - \zeta)^2$ is constant.

Now, taking the partial derivative with respect to u in (6) leads to

$$\frac{2}{\mu}\,\frac{\lambda_u}{\lambda^2} = -\frac{2\,\lambda_u}{\lambda^3}\,\xi_1^2 - \frac{2\,\lambda_u}{\lambda^2}\,\xi_1\cdot(\xi_0-\zeta)\ .$$

Since $\lambda_u \neq 0$ almost everywhere, this can be simplified to

$$\frac{1}{\mu} + \xi_1\cdot(\xi_0-\zeta) = -\frac{1}{\lambda}\,\xi_1^2\ . \tag{7}$$

Taking the derivative with respect to u again gives $0 = \frac{\lambda_u}{\lambda^2}\,\xi_1^2$, and since $\lambda_u \neq 0$, we must have that $\xi_1^2 = 0$, and thus that ξ_1 is *isotropic* in $\mathbf{C}^2$. But the only isotropic vectors in $\mathbf{C}^2$ are of the form $\kappa\,(1\,,\pm i)$ for some $\kappa \in \mathbf{C}$. Thus, we have found that ξ must have the form

$$\xi(u) = \frac{\kappa}{\lambda(u)}\begin{pmatrix}1\\ \pm i\end{pmatrix} + \xi_0\ ,$$

where $\kappa \in \mathbf{C}$ and $\xi_0 \in \mathbf{C}^2$ are constant.

For (7), this means $\xi_1\cdot(\xi_0-\zeta) = -\frac{1}{\mu}$, and using all these relations in (6) gives

$$\tfrac{-2}{\lambda\,\mu} = \tfrac{2}{\lambda}\,\xi_1\cdot(\xi_0-\zeta) + (\xi_0-\zeta)^2 = \tfrac{-2}{\lambda\,\mu} + (\xi_0-\zeta)^2\ .$$

Thus, $(\xi_0-\zeta)^2 = 0$; i.e. $\xi_0 - \zeta$ is also isotropic in $\mathbf{C}^2$, and we can find a holomorphic function $\eta = \eta(v)$ such that

$$\zeta(v) = \eta(v)\begin{pmatrix}1\\ \pm i\end{pmatrix} + \xi_0\ .$$

On the other hand, we need

$$-\frac{1}{\mu} = \xi_1\cdot(\xi_0-\zeta) = \xi_1\cdot\left(-\eta(v)\begin{pmatrix}1\\ \pm i\end{pmatrix}\right) = -\kappa\,\eta(v)\begin{pmatrix}1\\ \pm i\end{pmatrix}\cdot\begin{pmatrix}1\\ \pm i\end{pmatrix}\ .$$

Since this expression can never be zero, κ cannot be zero, and we always must have the opposite signs in ξ_1 and ζ, so that we find, without loss of generality, that

$$\xi_1 = \kappa\begin{pmatrix}1\\ \pm i\end{pmatrix} \quad \text{and} \quad \zeta(v) = \eta(v)\begin{pmatrix}1\\ \mp i\end{pmatrix} + \xi_0\ .$$

This means that

$$\frac{1}{\mu} = \kappa\,\eta(v) \begin{pmatrix} 1 \\ \pm i \end{pmatrix} \cdot \begin{pmatrix} 1 \\ \mp i \end{pmatrix} = \kappa\,\eta\,(1 - i^2) = 2\,\kappa\,\eta\;;$$

i.e. $\eta(v) = \frac{1}{2\,\kappa\,\mu(v)}$, and we have found that

$$\zeta(v) = \frac{1}{2\,\kappa\,\mu(v)} \begin{pmatrix} 1 \\ \mp i \end{pmatrix} + \xi_0$$

with the same $\kappa \neq 0$ and ξ_0 as above for ξ .

Let us now insert these expressions for ξ and ζ into X and Y. If $\xi_0 = (\alpha\,,\,\beta)$, then we have

$$\xi^2 = \frac{2\,\kappa}{\lambda}\,(\alpha \pm i\,\beta) + \xi_0^2 \quad \text{and } \zeta^2 = \frac{1}{\kappa\,\mu}\,(\alpha \mp i\,\beta) + \xi_0^2\;.$$

Inserting this into X and Y gives, after some reordering,

$$X = \lambda \begin{pmatrix} \frac{1-\xi_0^2}{2} \\ i\,\frac{1+\xi_0^2}{2} \\ \xi_0 \end{pmatrix} + \begin{pmatrix} -\kappa\,(\alpha \pm i\,\beta) \\ i\,\kappa\,(\alpha \pm i\,\beta) \\ \kappa \\ \pm i\,\kappa \end{pmatrix}$$

and

$$Y = \mu \begin{pmatrix} \frac{1-\xi_0^2}{2} \\ i\,\frac{1+\xi_0^2}{2} \\ \xi_0 \end{pmatrix} + \begin{pmatrix} -\frac{1}{2\,\kappa}\,(\alpha \mp i\,\beta) \\ \frac{i}{2\,\kappa}\,(\alpha \mp i\,\beta) \\ \frac{1}{2\,\kappa} \\ \mp\frac{i}{2\,\kappa} \end{pmatrix}\;.$$

Finally, for F_u and F_v this means

$$F_u = \lambda(u) \underbrace{\begin{pmatrix} 0 \\ 0 \\ \frac{1-\xi_0^2}{2} \\ i\,\frac{1+\xi_0^2}{2} \\ \xi_0 \end{pmatrix}}_{=:\mathbf{X}_3} + \underbrace{\begin{pmatrix} \frac{1}{2} \\ \frac{i}{2} \\ -\kappa\,(\alpha \pm i\,\beta) \\ i\,\kappa\,(\alpha \pm i\,\beta) \\ \kappa \\ \pm i\,\kappa \end{pmatrix}}_{=:\mathbf{X}_1}$$

and

$$F_v = \mu(v)\,\mathbf{X}_3 + \underbrace{\begin{pmatrix} -1 \\ i \\ -\frac{1}{2\,\kappa}\,(\alpha \mp i\,\beta) \\ \frac{i}{2\,\kappa}\,(\alpha \mp i\,\beta) \\ \frac{1}{2\,\kappa} \\ \mp\frac{i}{2\,\kappa} \end{pmatrix}}_{=:\mathbf{X}_2} .$$

Note that $\mathbf{X}_1$, $\mathbf{X}_2$, and $\mathbf{X}_3$ are *constant* in $\mathbf{C}^6$, and a straightforward calculation shows that they indeed span an *isotropic* subspace of $\mathbf{C}^6$. Integrating, we thus obtain

$$F(u\,,\,v) = u\,\mathbf{X}_1 + v\,\mathbf{X}_2 + \left(\int \lambda(u)\,du \;+\; \int \mu(v)\,dv\right)\,\mathbf{X}_3 + \mathbf{C}_0$$

for some constant vector $\mathbf{C}_0 \in \mathbf{C}$, and thus by Example 1.4 that f is generated by an isotropic graph, which is what we had to show in Case 2. This finally finishes the proof of Proposition 4.1.

Note that if $\mathbf{X} = F_u$ or $\mathbf{Y} = F_v$ is constant, we immediately have that $\operatorname{rank} F'' \leq 3$. The same is trivially true when f is holomorphic (see Remarks on page 43). Thus, the last proposition and Lemma 3.5 give us the following corollary, which we will need to establish our main result about the "generic case":

Corollary 4.2: *Let* $F : V \to \mathbf{C}^6$ *be a (local) holomorphic representative of a minimal real Kähler surface* $f : M^4 \to \mathbf{R}^6$ *. If the second osculating space* F'' *of* F *has rank at least* 4 *at a point* $p \in V$ *, then there is a Weierstrass representation* $((u\,,\,v)\,,\,X\,,\,Y)$ *on some neighborhood* U *of* p *in* V *such that the map* $X : V \to \mathbf{C}^4$ *has rank* 2 *; i.e.* X_u *and* X_v *are* linearly independent *everywhere on V.*

We will now start to describe the **generic case**, i.e. that $\operatorname{rank} F'' \geq 4$. Working in a given Weierstrass representation $((u\,,\,v)\,,\,X\,,\,Y)$ of F, we first obtain that, since X and Y are isotropic maps (i.e. $X^2 = Y^2 = 0$),

$$X \cdot X_u = X \cdot X_v = Y \cdot Y_u = Y \cdot Y_v = 0\;.$$

But we have that $X \cdot Y = 1$, and so, since $X_v = Y_u$, the above equations imply

$$X_u \cdot Y = -X \cdot Y_u = -X \cdot X_v = 0 \quad \text{and} \quad X \cdot Y_v = -X_v \cdot Y = -Y_u \cdot Y = 0 \, .$$

Therefore, all these equations together mean that

$$\operatorname{span}\{X_u \, , \, X_v = Y_u \, , \, Y_v\} \perp^{\bullet} \operatorname{span}\{X \, , \, Y\} \, , \tag{8}$$

where $A \perp^{\bullet} B$ for $A \, , \, B \in \mathbf{C}^n$ means by definition that $A \cdot B = 0$ with respect to the standard symmetric inner product in $\mathbf{C}^n$. But since X and Y are isotropic and $X \cdot Y = 1$, X and Y must be everywhere linearly independent, and thus $\operatorname{span}\{X \, , \, Y\}$ always a *non-degenerate* 2-dimensional subspace of $\mathbf{C}^4$. Since the standard symmetric inner product in $\mathbf{C}^4$ is non-degenerate, this means that its orthogonal complement in $\mathbf{C}^4$, $\{X \, , \, Y\}^{\perp^{\bullet}}$, must also be 2-dimensional and non-degenerate. Since (8) says exactly that

$$\operatorname{span}\{X_u \, , \, X_v = Y_u \, , \, Y_v\} \subset \{X \, , \, Y\}^{\perp^{\bullet}} \, ,$$

we have shown by (1) that for the holomorphic representative F of a minimal real Kähler submanifold in $\mathbf{R}^6$, we must indeed have that

$$\operatorname{rank} F'' \leq 4 \, .$$

Having established that the "generic case" is $\operatorname{rank} F'' = 4$, the last corollary tells us that we may assume that we have a Weierstrass representation of F on an open set $U \subset M$ such that

$$X_u \text{ and } X_v \in \mathbf{C}^4 \text{ are linearly independent on all of } U \, .$$

By (8), this means exactly that

$$\operatorname{span}\{X_u \, , \, X_v\} = \operatorname{span}\{X_u \, , \, X_v = Y_u \, , \, Y_v\} = \{X \, , \, Y\}^{\perp^{\bullet}} \, . \tag{9}$$

As we will show presently, this proves to be equivalent to saying that X, X_u, and X_v are linearly independent of all of U. This, in turn, is equivalent to saying that

$$\xi_u \text{ and } \xi_v \in \mathbf{C}^2 \text{ are linearly independent on all of } U \, . \tag{10}$$

The reason for this is the following. Write, as in Proposition 3.2,

$$X = \lambda \left(\frac{1-\xi^2}{2} \, , \, i\,\frac{1+\xi^2}{2} \, , \, \xi \right) ,$$

where ξ is a map from U into $\mathbf{C}^2$. Differentiation with respect to u and v gives

$$X_u = \frac{\lambda_u}{\lambda}\, X + \lambda \Big(-\, \xi \cdot \xi_u \, , \, i\, \xi \cdot \xi_u \, , \, \xi_u \Big) ,$$

and, likewise,

$$X_v = \frac{\lambda_v}{\lambda}\, X + \lambda \Big(-\, \xi \cdot \xi_v \, , \, i\, \xi \cdot \xi_v \, , \, \xi_v \Big) .$$

Now, it is clear that X_u and X_v are linearly independent if X, X_u, and X_v are. Next, assume that X_u and X_v are linearly independent, and that $\beta\, \xi_u + \gamma\, \xi_v = 0$ for some complex numbers β and γ. Then, by the above formulas for X_u and X_v, we have

$$\beta\, X_u + \gamma\, X_v = \left(\beta\, \frac{\lambda_u}{\lambda} + \gamma\, \frac{\lambda_v}{\lambda} \right) X + \lambda \begin{pmatrix} -\xi \cdot (\beta\, \xi_u + \gamma\, \xi_v) \\ i\, \xi \cdot (\beta\, \xi_u + \gamma\, \xi_v) \\ \beta\, \xi_u + \gamma\, \xi_v \end{pmatrix} ,$$

and the last term vanishes by our hypothesis. Taking the symmetric product of the resulting equation with X_u and X_v, respectively, we obtain by (9) that

$$\beta\, X_u^2 + \gamma\, X_u \cdot X_v = 0 \quad \text{and} \quad \beta\, X_u \cdot X_v + \gamma\, X_v^2 = 0 \, ,$$

or equivalently

$$\begin{pmatrix} X_u^2 & X_u \cdot X_v \\ X_u \cdot X_v & X_v^2 \end{pmatrix} \begin{pmatrix} \beta \\ \gamma \end{pmatrix} = 0 \, .$$

But the matrix in this equation is the Gram matrix of the vectors X_u and X_v with respect to the symmetric inner product "$\cdot$", and it is well–known that the determinant of such a matrix is zero *exactly* if $\operatorname{span}\{X_u \, , \, X_v\}$ is not 2–dimensional or is degenerate, neither of which we know to be the case here. Thus, we must have $\beta = \gamma = 0$, and so ξ_u and ξ_v are linearly independent.

Finally, assume that ξ_u and ξ_v are linearly independent, and that for some $\alpha \, , \, \beta \, , \, \gamma \in \mathbf{C}$,

$$0 = \alpha\, X + \beta\, X_u + \gamma\, X_v \, .$$

Using the forms of X, X_u, and X_v as above, this means that

$$0 = \underbrace{(\alpha\,\lambda + \beta\,\lambda_u + \gamma\,\lambda_v)}_{=\alpha'} \begin{pmatrix} \frac{1-\xi^2}{2} \\ i\,\frac{1+\xi^2}{2} \\ \xi \end{pmatrix} + \beta\,\lambda \begin{pmatrix} -\xi\cdot\xi_u \\ i\,\xi\cdot\xi_u \\ \xi_u \end{pmatrix} + \gamma\,\lambda \begin{pmatrix} -\xi\cdot\xi_v \\ i\,\xi\cdot\xi_v \\ \xi_v \end{pmatrix} .$$

But dividing the second components by i and adding the resulting equation to the one given by the first components of these vectors immediately gives $\alpha' = 0$, so that we necessarily must have that $\beta\,\xi_u + \gamma\,\xi_v = 0$, and thus that $\beta = \gamma = 0$. This, in turn, implies that α must also be zero, and we have proven the claimed equivalences.

Now, ξ is a holomorphic map from the complex 2–dimensional manifold $U \subset M$ into $\mathbf{C}^2$, which implies that our assumption (10) means that ξ is a *local biholomorphism* on U. If we thus make U slightly smaller, if necessary, we may assume that $\xi : U \to \mathbf{C}^2$ is a biholomorphism on all of U, and hence a *complex chart* of M on U. For convenience, we will write the component functions of this chart ξ as

$$s := \xi_1 \quad \text{and} \quad t := \xi_2 \,,$$

and will now write X and Y "in the new coordinates $(s\,,\,t)$"; i.e. we are looking at the maps $X \circ \xi^{-1}$ and $Y \circ \xi^{-1}$, for which we will use the "classical" notation

$$\left.\begin{aligned} &X(s\,,\,t) = \lambda(s\,,\,t)\left(\frac{1-s^2-t^2}{2}\,,\ i\,\frac{1+s^2+t^2}{2}\,,\ s\,,\ t\right) \\ &\text{and} \\ &Y(s\,,\,t) = \mu(s\,,\,t)\left(\frac{1-\zeta^2(s\,,\,t)}{2}\,,\ i\,\frac{1+\zeta^2(s\,,\,t)}{2}\,,\ \zeta_1(s\,,\,t)\,,\ \zeta_2(s\,,\,t)\right). \end{aligned}\right\} \quad (11)$$

Remark: This change of coordinates essentially amounts to the following. Our assumption (9) means that the map $X : U \to \mathbf{C}^4$ is an immersion whose "Gauss map" $\mathrm{span}\{X_u\,,\,X_v\}$ is basically identical to the complex Gauss map of f on U. Since X is also isotropic, we have the following commutative diagram:

$$\begin{array}{ccc} & X & \\ U & \longrightarrow & \mathbf{C}^4 - \{0\} \\ \tilde{X} \downarrow & & \downarrow \pi \\ Q_2 & \hookrightarrow & \mathbf{C}P^3 \end{array}$$

where $Q_2 = \{\pi(Z) \,|\, Z^2 = 0\}$ is the complex quadric in complex projective 3-space $\mathbf{C}P^3$, and $\pi : (\mathbf{C}^4 - \{0\}) \to \mathbf{C}P^3$ the canonical projection. Since our assumption that X_u and X_v are linearly independent is equivalent to X, X_u, and X_v being linearly independent, we have that $\tilde{X}$ is also an immersion, and since the (complex) dimension of Q_2 is 2 (thus the index), this implies that $\tilde{X}$ is a local biholomorphism. It is well–known that our ξ is nothing but a parametrization for Q_2 (see e.g. [H-O]), so that $\tilde{X}$ corresponds exactly to our coordinate system $\xi = (s\,,\,t)$. Thus, $X(s\,,\,t)$ and $Y(s\,,\,t)$ "*represent our minimal real Kähler submanifold in (complex) Gauss map coordinates*". This corresponds to the classical case where one takes the Gauss map of a minimal surface in 3–space to find isothermal coordinates for the minimal surface (see e.g. [S], page 385 and 386).

We see in formula (11) that changing to the new coordinates $(s\,,\,t)$ determines X up to the scaling factor $\lambda(s\,,\,t)$. We will find that Y is also *completely* determined by $\lambda(s\,,\,t)$ alone! Observe that by (11), X_s and X_t simply have the form

$$X_s = \frac{\lambda_s}{\lambda}\,X + \lambda\,(-s\,,\,i\,s\,,\,1\,,\,0\,) \quad \text{and} \quad X_t = \frac{\lambda_t}{\lambda}\,X + \lambda\,(-t\,,\,i\,t\,,\,0\,,\,1\,)\,. \tag{12}$$

But by (9), we still have

$$X_s \cdot Y = \frac{\partial u}{\partial s}\,X_u \cdot Y + \frac{\partial v}{\partial s}\,X_v \cdot Y = 0\;,$$

and likewise $X_t \cdot Y = 0$. Replacing the expressions for X_s and X_t, as above, and the expression for Y in (11) gives (remembering that $X \cdot Y = 1$)

$$\begin{aligned} 0 \;=\; X_s \cdot Y &= \frac{\lambda_s}{\lambda}\,X \cdot Y + \lambda\,\mu\,(-s\,,\,i\,s\,,\,1\,,\,0\,) \cdot \left(\frac{1-\zeta^2}{2}\,,\; i\,\frac{1+\zeta^2}{2}\,,\;\zeta_1\,,\;\zeta_2 \right) \\ &= \frac{\lambda_s}{\lambda} + \lambda\mu \left(\frac{-s + s\,\zeta^2 - s - s\,\zeta^2}{2} + \zeta_1 \right) = \frac{\lambda_s}{\lambda} + \lambda\mu\,(\zeta_1 - s)\;, \end{aligned}$$

and in a completely analogous fashion we find that

$$\frac{\lambda_t}{\lambda} + \lambda\mu\,(\zeta_2 - t) = 0\ .$$

Using that

$$\lambda\,\mu = \frac{-2}{(\xi-\zeta)^2} = \frac{-2}{(s-\zeta_1)^2 + (t-\zeta_2)^2}\ ,$$

we obtain

$$\zeta_1 - s = \frac{\lambda_s}{2\,\lambda}\left((\zeta_1 - s)^2 + (\zeta_2 - t)^2\right) \quad \text{and} \quad \zeta_2 - t = \frac{\lambda_t}{2\,\lambda}\left((\zeta_1 - s)^2 + (\zeta_2 - t)^2\right)\ .$$

Multiplying the first equation by λ_t , the second by λ_s , and then subtracting these equations from each other results in

$$\lambda_t\,(\zeta_1 - s) - \lambda_s\,(\zeta_2 - t) = 0\ . \tag{13}$$

Now, note that λ cannot be constant, since otherwise $\lambda_s = \lambda_t = 0$, and in this case the above equations give that $\zeta = (s\,,\ t) = \xi$, which cannot happen, since for a Weierstrass representation we have that $(\xi - \zeta)^2 \neq 0$. Therefore, we may assume, without loss of generality, that λ_s is not identically equal to zero, and thus, as a holomorphic function, only zero at some isolated points in U. By (13), we then have at all places where $\lambda_s \neq 0$ that

$$\zeta_2 - t = \frac{\lambda_t}{\lambda_s}\,(\zeta_1 - s)\ ,$$

and inserting this expression into the one for $\zeta_1 - s$ gives

$$\zeta_1 - s = \frac{\lambda_s}{2\,\lambda}\,(\zeta_1 - s)^2\left(1 + \frac{\lambda_t^2}{\lambda_s^2}\right) = \frac{\lambda_s^2 + \lambda_t^2}{2\,\lambda\,\lambda_s}\,(\zeta_1 - s)^2\ .$$

But if $\zeta_1 - s = 0$ at some point where $\lambda_s \neq 0$, then by (13) we also would have that $\zeta_2 - t = 0$, which again would lead to $(\xi - \zeta)^2 = 0$, and is thus impossible. Consequently, we can divide by $\zeta_1 - s$ (almost everywhere), which leads to the following result:

λ has no singularities in U, and we have

$$\left.\zeta_1 = s + \frac{2\,\lambda\,\lambda_s}{\lambda_s^2 + \lambda_t^2} \quad \textit{and} \quad \zeta_2 = t + \frac{2\,\lambda\,\lambda_t}{\lambda_s^2 + \lambda_t^2}\ .\right\} \tag{14}$$

Note that these formulas are still true at the isolated points $q \in U$ where $\lambda_s(q) = 0$; by what we showed above, we must then necessarily have that $\lambda_t(q) \neq 0$. These are the promised formulas that give ζ in terms of λ alone. Now we can also calculate μ in terms of λ alone:

$$\mu = \frac{-2}{\lambda\,((s-\zeta_1)^2 + (t-\zeta_2)^2)} = \frac{-2\,(\lambda_s^2+\lambda_t^2)^2}{\lambda\,((2\,\lambda\,\lambda_s)^2 + (2\,\lambda\,\lambda_t)^2)}\,,$$

which gives

$$\mu = -\frac{\lambda_s^2+\lambda_t^2}{2\,\lambda^3}\,. \tag{15}$$

For later reference, we also need to calculate Y_s and Y_t in terms of λ alone. This can be done directly by using the formulas above, but the following approach is much shorter. By our assumption (9), we know that Y_s and Y_t are always in the span of X_s and X_t, so that we can find holomorphic functions a, b, c, and d such that

$$Y_s = a\,X_s + b\,X_t \quad \text{and} \quad Y_t = d\,X_s + c\,X_t\,.$$

Taking the symmetric inner product of either equation with X_s and X_t, and recalling that $Y \cdot X_s = 0$ and $Y \cdot X_t = 0$, we obtain the equations

$$\left.\begin{aligned} Y_s \cdot X_s &= -Y \cdot X_{ss} = a\,X_s^2 + b\,X_s \cdot X_t\,,\\ Y_s \cdot X_t &= -Y \cdot X_{st} = a\,X_s \cdot X_t + b\,X_t^2\,,\\ Y_t \cdot X_s &= -Y \cdot X_{ts} = d\,X_s^2 + c\,X_s \cdot X_t\,,\\ Y_t \cdot X_t &= -Y \cdot X_{tt} = d\,X_s \cdot X_t + c\,X_t^2\,. \end{aligned}\right\} \tag{16}$$

Let us calculate the expressions for X_s^2, X_t^2, and $X_s \cdot X_t$. By (12), and since $X \cdot X_s = 0$, we have

$$\begin{aligned} X_s^2 &= \left(\tfrac{\lambda_s}{\lambda}\,X + \lambda\,(-s\,,\, i\,s\,,\,1\,,\,0\,)\right)\cdot X_s\\ &= \lambda\,(-s\,,\,i\,s\,,\,1\,,\,0\,)\cdot\left[\lambda_s\left(\tfrac{1-s^2-t^2}{2}\,,\, i\,\tfrac{1+s^2+t^2}{2}\,,\; s\,,\; t\right) + \lambda\,(-s\,,\,i\,s\,,\,1\,,\,0\,)\right]\\ &= \lambda\,\lambda_s\left(\tfrac{-s+s^3+s\,t^2-s-s^3-s\,t^2}{2} + s\right) + \lambda^2\,(s^2-s^2+1)\\ &= \lambda^2\,, \end{aligned}$$

and in the same way we find

$$X_t^2 = \lambda^2\,.$$

Further,

$$\begin{aligned} X_s \cdot X_t &= \left(\tfrac{\lambda_s}{\lambda} X + \lambda\,(-s\,,\, i\,s\,,\, 1\,,\, 0\,)\right) \cdot X_t \\ &= \lambda\,(-s\,,\, i\,s\,,\, 1\,,\, 0\,) \cdot \left[\lambda_t \left(\tfrac{1-s^2-t^2}{2}\,,\, i\,\tfrac{1+s^2+t^2}{2}\,,\; s\,,\; t\right) + \lambda\,(-t\,,\, i\,t\,,\, 0\,,\, 1\,)\right] \\ &= \lambda\,\lambda_t \left(\tfrac{-s+s^3+s\,t^2-s-s^3-s\,t^2}{2} + s\right) + \lambda^2\,(s\,t - s\,t) \\ &= 0\,. \end{aligned}$$

By (16), these equations immediately give that

$$a = -\frac{Y \cdot X_{ss}}{\lambda^2} \quad , \quad b = d = -\frac{Y \cdot X_{st}}{\lambda^2} \quad , \quad \text{and} \quad c = -\frac{Y \cdot X_{tt}}{\lambda^2}\;. \tag{17}$$

Furthermore, we have that

$$X_{ss} = \left(\frac{\lambda_s}{\lambda}\right)_s X + \frac{\lambda_s}{\lambda} X_s + \lambda_s\,(-s\,,\, i\,s\,,\, 1\,,\, 0\,) + \lambda\,(-1\,,\, i\,,\, 0\,,\, 0\,)\;.$$

Taking the symmetric inner product of X_{ss} with Y, and remembering that $X \cdot Y = 1$ and $X_s \cdot Y = 0$, results in

$$\begin{aligned} X_{ss} \cdot Y &= \left(\tfrac{\lambda_s}{\lambda}\right)_s X \cdot Y + \tfrac{\lambda_s}{\lambda} X_s \cdot Y \\ &\qquad + \lambda_s\,(-s\,,\, i\,s\,,\, 1\,,\, 0\,) \cdot \mu \left(\tfrac{1-\zeta^2}{2}\,,\, i\,\tfrac{1+\zeta^2}{2}\,,\; \zeta_1\,,\; \zeta_2\right) \\ &\qquad\quad + \lambda\,(-1\,,\, i\,,\, 0\,,\, 0\,) \cdot \mu \left(\tfrac{1-\zeta^2}{2}\,,\, i\,\tfrac{1+\zeta^2}{2}\,,\; \zeta_1\,,\; \zeta_2\right) \\ &= \left(\tfrac{\lambda_s}{\lambda}\right)_s + \mu\,\lambda_s \left(\tfrac{-s+s\,\zeta^2-s-s\,\zeta^2}{2} + \zeta_1\right) + \lambda\,\mu\,\tfrac{-1+\zeta^2-1-\zeta^2}{2} \\ &= \left(\tfrac{\lambda_s}{\lambda}\right)_s + \mu\,\left(\lambda_s\,(\zeta_1 - s) - \lambda\right)\;. \end{aligned}$$

But by (14) and (15), $\zeta_1 - s$ and μ in the second term can be replaced, to give

$$\begin{aligned} X_{ss} \cdot Y &= \left(\tfrac{\lambda_s}{\lambda}\right)_s - \tfrac{\lambda_s^2+\lambda_t^2}{2\,\lambda^3} \left(\lambda_s\,\tfrac{2\,\lambda\,\lambda_s}{\lambda_s^2+\lambda_t^2} - \lambda\right) = \tfrac{2\,\lambda_{ss}\,\lambda - 2\,\lambda_s^2}{2\,\lambda^2} - \tfrac{2\,\lambda_s^2-(\lambda_s^2+\lambda_t^2)}{2\,\lambda^2} \\ &= \frac{2\,\lambda_{ss}\,\lambda - 3\,\lambda_s^2 + \lambda_t^2}{2\,\lambda^2}\;. \end{aligned}$$

Completely analogously, one will find that

$$X_{tt} \cdot Y = \frac{2\,\lambda_{tt}\,\lambda + \lambda_s^2 - 3\,\lambda_t^2}{2\,\lambda^2} \ .$$

Finally, since

$$X_{st} = \left(\frac{\lambda_s}{\lambda}\right)_t X + \frac{\lambda_s}{\lambda} X_t + \lambda_t\,(-s\,,\; i\,s\,,\; 1\,,\; 0\,)\ ,$$

we find that

$$\begin{aligned} X_{st} \cdot Y \ &= \ \left(\tfrac{\lambda_s}{\lambda}\right)_t X \cdot Y + \tfrac{\lambda_s}{\lambda} X_t \cdot Y \\ &\qquad + \lambda_t\,(-s\,,\; i\,s\,,\; 1\,,\; 0\,) \cdot \mu \left(\tfrac{1-\zeta^2}{2}\,,\; i\,\tfrac{1+\zeta^2}{2}\,,\; \zeta_1\,,\; \zeta_2\right) \\ &= \ \left(\tfrac{\lambda_s}{\lambda}\right)_t + \lambda_t\,\mu\,(\zeta_1 - s) = \frac{\lambda_{st}\,\lambda - \lambda_s\,\lambda_t}{\lambda^2} - \frac{\lambda_t\,(\lambda_s^2 + \lambda_t^2)}{2\,\lambda^3}\,\frac{2\,\lambda\,\lambda_s}{\lambda_s^2 + \lambda_t^2} \\ &= \ \frac{2\,\lambda_{st}\,\lambda - 4\,\lambda_s\,\lambda_t}{2\,\lambda^2} \ . \end{aligned}$$

Using these expressions in (17) gives us a, b, and c, and hence Y_s and Y_t, in terms of λ alone, namely:

$$\left.\begin{aligned} &\textit{We have } \ Y_s = a\,X_s + b\,X_t \ \ \textit{and} \ \ Y_t = b\,X_s + c\,X_t\ ,\ \textit{where} \\ &a = \frac{-2\,\lambda_{ss}\,\lambda + 3\,\lambda_s^2 - \lambda_t^2}{2\,\lambda^4} \quad , \quad b = \frac{-2\,\lambda_{st}\,\lambda + 4\,\lambda_s\,\lambda_t}{2\,\lambda^4}\ , \\ &\textit{and} \quad c = \frac{-2\,\lambda_{tt}\,\lambda - \lambda_s^2 + 3\,\lambda_t^2}{2\,\lambda^4}\ . \end{aligned}\right\} \quad (18)$$

Note that we have just shown that *in "$(s\,,\,t)$-coordinates", X and Y are essentially determined by λ alone.* The question remains whether *every* function λ is possible. We will see shortly that this is basically the case, the only essential restriction being that λ has *no singularities* (cf. (14)).

To this end, note that our integrability condition $X_v = Y_u$ in $(u\,,\,v)$-coordinates looks as follows in $(s\,,\,t)$-coordinates, if we use (18) (and write $s_u = \frac{\partial s}{\partial u}$, etc.):

$$\begin{aligned} X_v \ &= \ \frac{\partial s}{\partial v} X_s + \frac{\partial t}{\partial v} X_t = Y_u = \frac{\partial s}{\partial u} Y_s + \frac{\partial t}{\partial u} Y_t \\ &= \ (a\,s_u + b\,t_u)\,X_s + (b\,s_u + c\,t_u)\,X_t\ . \end{aligned}$$

Since in our generic case X_s and X_t must be everywhere linearly independent, this leads to the following condition, which is equivalent to the integrability condition $X_v = Y_u$:

$$s_v = a\,s_u + b\,t_u \quad \text{and} \quad t_v = b\,s_u + c\,t_u\ . \tag{19}$$

But recall that we have (in "classical notation") $(s\,,\,t) = \xi(u\,,\,v)$, and thus $(u\,,\,v) = \xi^{-1}(s\,,\,t)$. By virtue of the Inverse Mapping Theorem, this gives

$$\begin{pmatrix} u_s & u_t \\ v_s & v_t \end{pmatrix} = d(\xi^{-1}) = (d\,\xi)^{-1} \circ \xi^{-1} = \begin{pmatrix} s_u & s_v \\ t_u & t_v \end{pmatrix}^{-1} \circ \xi^{-1}$$

$$= \left(\frac{1}{\det(d\,\xi)} \begin{pmatrix} t_v & -s_v \\ -t_u & s_u \end{pmatrix} \right) \circ \xi^{-1}\ ,$$

and comparing the entries of these matrices gives

$$\begin{array}{rclcrcl} s_u &=& \det(d\,\xi)\,(v_t \circ \xi) &,& s_v &=& -\det(d\,\xi)\,(u_t \circ \xi)\ , \\ t_u &=& -\det(d\,\xi)\,(v_s \circ \xi) &,& t_v &=& \det(d\,\xi)\,(u_s \circ \xi)\ . \end{array}$$

Replacing these expressions in (19) leads to

$$-(\det(d\,\xi) \circ \xi^{-1})\,u_t = (\det(d\,\xi) \circ \xi^{-1})\,(a\,v_t - b\,v_s)$$

and

$$(\det(d\,\xi) \circ \xi^{-1})\,u_s = (\det(d\,\xi) \circ \xi^{-1})\,(b\,v_t - c\,v_s)\ ,$$

so that canceling the common (non-zero) factor results in the following *integrability condition in terms of* $(s\,,\,t)$*-coordinates* :

$$X_v = Y_u \quad \Longleftrightarrow \quad \left\{ \begin{array}{l} u_s = -c\,v_s + b\,v_t \\ \text{and} \\ u_t = b\,v_s - a\,v_t\ . \end{array} \right\} \tag{20}$$

Note that the coefficients a, b, and c are determined by $\lambda(s\,,\,t)$ alone as in (18), and that the equations in (20) determine u (as a function in $(s\,,\,t)$-coordinates) in terms of v alone, up to a constant. But for the equations in (20) to be (locally) integrable, they in turn need to satisfy the integrability condition $(u_s)_t = (u_t)_s$; i.e. by (20):

$$(u_s)_t = -c_t\,v_s - c\,v_{st} + b_t\,v_t + b\,v_{tt} = (u_t)_s = b_s\,v_s + b\,v_{ss} - a_s\,v_t - a\,v_{st}\ ,$$

or equivalently:

$$b\, v_{ss} + (c-a)\, v_{st} - b\, v_{tt} + (b_s + c_t)\, v_s - (b_t + a_s)\, v_t = 0\ . \tag{21}$$

This is a second order *linear* homogeneous partial differential equation in v (as a function in $(s\,,\,t)$-coordinates) whose coefficients are holomorphic in $(s\,,\,t)$ and completely determined by λ as in (18). By the Cauchy–Kowalewski Theorem (see [Hö], pages 348 to 350), such a partial differential equation always has (local) solutions, where we even have the freedom to choose functions in one complex variable along certain complex curves, the so–called "characteristics" of the equation (see our Example 4.4 below).

Since all the necessary conditions for the integrability of the functions involved are locally sufficient, we have, therefore, proven the following

Theorem 4.3: *Let* $\tilde\lambda : V \to \mathbf{C}$ *be any* nowhere zero *holomorphic function in the two complex variables* $(s\,,\,t)$ *on an open subset* V *of* $\mathbf{C}^2$*, and assume that* $\tilde\lambda$ *has* **no singularities** *on* V*. Define the holomorphic functions* a*,* b*,* $c : V \to \mathbf{C}$ *by*

$$a := \frac{-2\,\tilde\lambda_{ss}\,\tilde\lambda + 3\,\tilde\lambda_s^2 - \tilde\lambda_t^2}{2\,\tilde\lambda^4} \quad , \quad b := \frac{-2\,\tilde\lambda_{st}\,\tilde\lambda + 4\,\tilde\lambda_s\,\tilde\lambda_t}{2\,\tilde\lambda^4}\ ,$$

$$\textit{and} \quad c := \frac{-2\,\tilde\lambda_{tt}\,\tilde\lambda - \tilde\lambda_s^2 + 3\,\tilde\lambda_t^2}{2\,\tilde\lambda^4}\ .$$

Next, solve the second order linear homogeneous partial differential equation

$$b\, v_{ss} + (c-a)\, v_{st} - b\, v_{tt} + (b_s + c_t)\, v_s - (b_t + a_s)\, v_t = 0$$

on some simply connected open subset $\tilde V$ *of* V *. Then there is a function* $u : \tilde V \to \mathbf{C}$ *such that*

$$u_s = -c\, v_s + b\, v_t \quad \textit{and} \quad u_t = b\, v_s - a\, v_t\ .$$

At some fixed point in $\tilde V$ **where the map** $(u\,,\,v) : \tilde V \to \mathbf{C}^2$ **has (complex) rank 2***, calculate the inverse map* $\xi = \xi(u\,,\,v)$ *of the map* $(u\,,\,v) = \big(u(s\,,\,t)\,,\,v(s\,,\,t)\big)$*, which is defined on some open, simply connected subset* U *of* $\mathbf{C}^2$*, and view* $\xi = (\xi_1\,,\,\xi_2)$ *as a map in the two complex variables* $(u\,,\,v) \in U \subset \mathbf{C}^2$*. Define* $\lambda := \tilde\lambda \circ \xi$*, and view* λ *as a function in the two complex variables* $(u\,,\,v) \in U \subset \mathbf{C}^2$*.*

Now, define the following maps: $\zeta := (\zeta_1, \zeta_2) : U \to \mathbf{C}^2$ *, where*

$$\zeta_1 := \xi_1 + \left(\frac{2\,\tilde\lambda\,\tilde\lambda_s}{\tilde\lambda_s^2 + \tilde\lambda_t^2} \right) \circ \xi \quad \textit{and} \quad \zeta_2 := \xi_2 + \left(\frac{2\,\tilde\lambda\,\tilde\lambda_t}{\tilde\lambda_s^2 + \tilde\lambda_t^2} \right) \circ \xi \,,$$

$$\mu := \frac{-2}{\lambda\,(\xi - \zeta)^2} = -\left(\frac{\tilde\lambda_s^2 + \tilde\lambda_t^2}{2\,\tilde\lambda^3} \right) \circ \xi : U \to \mathbf{C} \,,$$

and further

$$\mathbf{X} := \left(\tfrac{1}{2}\,,\, \tfrac{i}{2}\,,\, X\right)\,, \quad \textit{where } X := \lambda\left(\tfrac{1-\xi^2}{2},\, i\,\tfrac{1+\xi^2}{2},\, \xi\right),$$

and

$$\mathbf{Y} := (-1\,,\, i\,,\, Y)\,, \quad \textit{where } Y := \mu\left(\tfrac{1-\zeta^2}{2},\, i\,\tfrac{1+\zeta^2}{2},\, \zeta\right).$$

Then the $\mathbf{C}^6$*–valued* 1*-form*

$$\omega := \mathbf{X}\,du + \mathbf{Y}\,dv$$

is exact on U*, and if* $F : U \to \mathbf{C}^6$ *is a holomorphic map such that* $dF = \omega$ *, then* $f := \sqrt{2}\,\mathrm{Re}(F) : M \to \mathbf{R}^6$ *is a minimal isometric immersion of the Kähler manifold* $M := (U\,,\, f^* <,>)$ *into* $\mathbf{R}^6$ *.*

Conversely, if $F : W \to \mathbf{C}^6$ *is a holomorphic representative of a minimal isometric immersion* $f : M^4 \to \mathbf{R}^6$ *from a 4-dimensional Kähler manifold* M *into* $\mathbf{R}^6$*, and if* p *is a point in* M *where* $\operatorname{rank} F'' = 4$*, then we can find a complex coordinate system* $(u\,,\, v)$ *such that in a neighborhood* U *of* p *in* M*,* $\mathbf{X} = F_u$ *and* $\mathbf{Y} = F_v$ *are given, up to isometry, in terms of a function* $\lambda : U \to \mathbf{C}$ *without singularities as described above.*

Example 4.4: Let

$$\tilde\lambda(s\,,\, t) := t\,.$$

Then, we have $\tilde\lambda_s = 0$ and $\tilde\lambda_t = 1$, and all second partial derivatives of $\tilde\lambda$ vanish. By (18), we obtain

$$a(s\,,\, t) = \frac{0 + 3\cdot 0 - 1^2}{2\,t^4} = -\frac{1}{2\,t^4}\,,\quad b(s\,,\, t) = \frac{0 + 4\cdot 0\cdot 1}{2\,t^4} = 0\,,$$

$$\text{and} \quad c(s\,,\, t) = \frac{0 - 0 + 3\cdot 1^2}{2\,t^4} = \frac{3}{2\,t^4}\,.$$

Thus, (21) collapses to

$$0 \cdot v_{ss} + \left(\frac{3}{2\,t^4} + \frac{1}{2\,t^4}\right) v_{st} - 0 \cdot v_{tt} + \left(0 + \left(\frac{3}{2\,t^4}\right)_t\right) v_s - \left(0 + \left(-\frac{1}{2\,t^4}\right)_s\right) v_t$$

$$= \frac{2}{t^4}\, v_{st} - \frac{6}{t^5}\, v_s = 0 \; ,$$

which is equivalent to the partial differential equation

$$(v_s)_t = \frac{3}{t}\, v_s \; .$$

If $z := v_s$ is not constantly zero, we can separate this equation to $\dfrac{z_t}{z} = \dfrac{3}{t}$, or equivalently

$$(\log z)_t = (3\, \log t)_t = (\log t^3)_t \; .$$

Integrating this gives that for every function $g = g(s)$ in s alone,

$$z = g(s)\, t^3$$

is a solution for the partial differential equation given above. Choose

$$g(s) := e^{s+C} \; ,$$

where C is a constant that will be determined later. Since then $z = v_s = e^{s+C}\, t^3$, integrating with respect to s (and setting the integration constant equal to zero) gives

$$v(s\, ,\, t) = e^{s+C}\, t^3 \; .$$

Using this function in (20) leads to the following "gradient" of u:

$$u_s \;=\; -c\, v_s + 0 \cdot v_t \;=\; -\frac{3}{2\,t^4}\, e^{s+C}\, t^3 = -\frac{3}{2\,t}\, e^{s+C} \; ,$$

$$u_t \;=\; 0 \cdot v_s - a\, v_t \;=\; \frac{1}{2\,t^4}\, 3\, e^{s+C}\, t^2 = \frac{3}{2\,t^2}\, e^{s+C} \; .$$

Integrating the first equation gives $u(s\, ,\, t) = -\frac{3}{2\,t}\, e^{s+C} + h(t)$ with a function h in t alone. Inserting this into the second equation results in

$$\frac{3}{2\,t^2}\, e^{s+C} + h'(t) = \frac{3}{2\,t^2}\, e^{s+C} \; ,$$

which means that $h(t) = 0$; i.e. h is constant. Choosing this constant to be zero, we thus find

$$u(s\,,\,t) = -\frac{3}{2\,t}\,e^{s+C}\ .$$

One can easily check that the Jacobian of $(u\,,\,v)$ as a holomorphic map in $(s\,,\,t)$ is never zero (since we must have that $t \neq 0$).

We now have to find the inverse of this map; i.e. we have to express s and t as functions of u and v. To this end,

$$\frac{v}{u} = \frac{t^3\,e^{s+C}}{-\frac{3}{2\,t}\,e^{s+C}} = -\frac{2\,t^4}{3}\ .$$

Using the function $z^{\frac{1}{4}} = \exp(\frac{1}{4}\,\log z)$ on some branch of the complex logarithm gives

$$t = \xi_2(u\,,\,v) = \left(\frac{-3\,v}{2\,u}\right)^{\frac{1}{4}}\ .$$

Thus, we have that

$$e^{s+C} = -\frac{2\,t}{3}\,u = -\frac{2}{3}\left(\frac{-3\,v}{2\,u}\right)^{\frac{1}{4}}\,u\ ,$$

or, taking the logarithm on both sides,

$$s = -C + \log\left(-\frac{2}{3}\right) + \frac{1}{4}\,\left[\log\left(-\frac{3}{2}\right) + \log v - \log u\right] + \log u\ .$$

Choosing C in a way such that all constant terms in this expression cancel out, we finally obtain

$$s = \xi_1(u\,,\,v) = \frac{1}{4}\,\log(v\,u^3)\ .$$

Note also that we have

$$\lambda = \tilde{\lambda} \circ (s(u\,,\,v)\,,\,t(u\,,\,v)) = t(u\,,\,v) = \left(\frac{-3\,v}{2\,u}\right)^{\frac{1}{4}}\ ,$$

and by (14) and (15), we obtain

$$\zeta_1 = s + \frac{2\,t\cdot 0}{0^2+1^2} = s = \frac{1}{4}\,\log(v\,u^3)\ ,$$

$$\zeta_2 = t + \frac{2\,t \cdot 1}{0^2 + 1^2} = 3\,t = 3\,\left(\frac{-3\,v}{2\,u}\right)^{\frac{1}{4}} \;,$$

and

$$\mu = -\frac{0^2 + 1^2}{2\,t^3} = -\frac{1}{2}\,\left(\frac{-3\,v}{2\,u}\right)^{-\frac{3}{4}} \;.$$

Inserting these maps $\xi = (\xi_1\,,\,\xi_2)\,,\ \zeta = (\zeta_1\,,\,\zeta_2)\,,\ \lambda$, and μ into our equations for X and Y gives, according to Theorem 4.3, the (complex Gauss map of an) isometric immersion from a minimal real Kähler manifold with respect to the complex chart $(u\,,\,v)$ into $\mathbf{R}^6$.

We will now investigate the **remaining case** that $\operatorname{rank} F'' = 3$. If we exclude the case that f is holomorphic or generated by an isotropic cylinder (as in Proposition 4.1), then we can assume that for all $p \in U$ (staying away from some isolated points),

$$Z(p) := X_v(p) = Y_u(p) \neq 0 \;.$$

Since by (1) X_u and Y_v have to point in the same complex direction as Z, we can find holomorphic functions α and $\beta : U \to \mathbf{C}$ such that

$$X_u = \alpha\,Z \quad \text{ and } \quad Y_v = \beta\,Z \;.$$

Using the integrability conditions for X and Y, we must have

$$(X_u)_v = (\alpha\,Z)_v = \alpha_v\,Z + \alpha\,Z_v = (X_v)_u = Z_u$$

and

$$(Y_v)_u = (\beta\,Z)_u = \beta_u\,Z + \beta\,Z_u = (Y_u)_v = Z_v \;.$$

Inserting Z_u as in the first equation into the second equation, and vice versa for Z_v, results in

$$\alpha_v\,Z + \alpha\,(\beta_u\,Z + \beta\,Z_u) = Z_u \quad \text{ and } \quad \beta_u\,Z + \beta\,(\alpha_v\,Z + \alpha\,Z_v) = Z_v \;,$$

or equivalently

$$(\alpha_v + \alpha\,\beta_u)\,Z = (1 - \alpha\,\beta)\,Z_u \quad \text{ and } \quad (\beta_u + \beta\,\alpha_v)\,Z = (1 - \alpha\,\beta)\,Z_v \;. \qquad (22)$$

At this point, we have to distinguish two cases: either that the factor $1 - \alpha\,\beta$ is identically zero, or that it is different from zero almost everywhere (recall that α and β are holomorphic).

Case 1: $\alpha\,\beta \neq 1$ almost everywhere. In this case, (22) gives that Z_u and Z_v are always linearly dependent, and thus that the map Z has rank ≤ 1 everywhere. But then the Rank Theorem (see e.g. [B-J], pages 45–48[1]) gives us that, away from isolated singularities, Z can be factored as

$$Z(u\,,\,v) = \tilde{Z} \circ \gamma(u\,,\,v)\ ,$$

where $\tilde{Z} = \tilde{Z}(w)$ is a holomorphic map in one complex variable w, and $\gamma : U \to \mathbf{C}$ is some holomorphic function on U. Using this more specific form of Z in our integrability conditions for X and Y results in (writing $\tilde{Z}' = \frac{d}{dw}\,\tilde{Z}$)

$$(X_v)_u = Z_u = \gamma_u\,(\tilde{Z}' \circ \gamma) = (X_u)_v = (\alpha\,Z)_v = \alpha_v\,(\tilde{Z} \circ \gamma) + \alpha\,\gamma_v\,(\tilde{Z}' \circ \gamma)$$

and

$$(Y_u)_v = Z_v = \gamma_v\,(\tilde{Z}' \circ \gamma) = (Y_v)_u = (\beta\,Z)_u = \beta_u\,(\tilde{Z} \circ \gamma) + \beta\,\gamma_u\,(\tilde{Z}' \circ \gamma)\ ,$$

so that, after simplifying,

$$\alpha_v\,(\tilde{Z} \circ \gamma) = (\gamma_u - \alpha\,\gamma_v)\,(\tilde{Z}' \circ \gamma) \quad \text{and} \quad \beta_u\,(\tilde{Z} \circ \gamma) = (\gamma_v - \beta\,\gamma_u)\,(\tilde{Z}' \circ \gamma)\,. \tag{23}$$

Now, *assume* that $\tilde{Z}$ and $\tilde{Z}'$ would be linearly independent on some open set in $\mathbf{C}$. Then, by (23), we would clearly have

$$\alpha_v = 0 \ \ ,\ \ \beta_u = 0 \ \ ,\ \ \gamma_u = \alpha\,\gamma_v \ \ ,\ \ \text{and} \ \ \gamma_v = \beta\,\gamma_u\ .$$

The last two equations give that $\gamma_u = \alpha\,\beta\,\gamma_u$ and $\gamma_v = \alpha\,\beta\,\gamma_v$. But we cannot have that γ_u and γ_v are both zero everywhere, since in this case γ would be constant, and we could replace $\tilde{Z}$ by a constant vector, in contradiction to $\tilde{Z}$ and $\tilde{Z}'$ being linearly independent almost everywhere. But then we would find that $\alpha\,\beta = 1$ everywhere, in contradiction to our case assumption.

Thus, $\tilde{Z}$ and $\tilde{Z}'$ are *everywhere linearly dependent*. But that means that we must have a *constant*, non–zero vector $Z_0 \in \mathbf{C}^4$ and a function $\delta = \delta(w)$ in w such that

$$\tilde{Z}(w) = \delta(w)\,Z_0$$

(and δ is a solution of a certain differential equation; but we will not need to know δ in more detail here).

[1]There, one can find a proof for *real* manifolds. However, the only tool that is necessary is the Inverse Mapping Theorem, and this theorem is also true in the holomorphic case.

Since $X_v = Y_u = Z = \tilde{Z} \circ \gamma = (\delta \circ \gamma)\, Z_0$, we can integrate to obtain

$$X = g(u\,,\,v)\, Z_0 + X_0 \quad \text{and} \quad Y = h(u\,,\,v)\, Z_0 + Y_0 \,,$$

for some holomorphic functions $g\,,\,h : U \to \mathbf{C}$, and some constant vectors $X_0\,,\,Y_0 \in \mathbf{C}^4$. Using our integrability condition once more, we find

$$X_v = g_v\, Z_0 = Y_u = h_u\, Z_0 \,, \tag{24}$$

so since $Z_0 \neq 0$, we must have $g_v = h_u \neq 0$. Thus (after making U smaller if necessary), there is a holomorphic function $\eta : U \to \mathbf{C}$ without singularities such that

$$g = \eta_u \quad \text{and} \quad h = \eta_v \,.$$

Now, inserting X and Y into the partial derivatives for F results in

$$F_u = \begin{pmatrix} \frac{1}{2} \\ \frac{i}{2} \\ \eta_u\, Z_0 + X_0 \end{pmatrix} \quad \text{and} \quad F_v = \begin{pmatrix} -1 \\ i \\ \eta_v\, Z_0 + Y_0 \end{pmatrix} \,,$$

and integrating these equations, we see that the holomorphic representative F of the minimal real Kähler immersion we are looking for, in this case, must have the form

$$F(u\,,\,v) = u \begin{pmatrix} \frac{1}{2} \\ \frac{i}{2} \\ X_0 \end{pmatrix} + v \begin{pmatrix} -1 \\ i \\ Y_0 \end{pmatrix} + \eta(u\,,\,v) \begin{pmatrix} 0 \\ 0 \\ Z_0 \end{pmatrix} + C_0$$

for some constant vector $C_0 \in \mathbf{C}^6$. Since $g_v = h_u \neq 0$ and thus g and h non–constant, (8) and (24) immediately give that

$$X_0 \cdot Z_0 = Y_0 \cdot Z_0 = Z_0^2 = 0 \,.$$

And since we need $X \cdot Y = 1$, the equations above also give $X_0 \cdot Y_0 = 1$. This means that the vectors in the formula for F indeed span an isotropic subspace of $\mathbf{C}^6$, and thus that $f = \sqrt{2}\,\mathrm{Re}(F)$ *is generated by the graph of the holomorphic function* $\eta(u\,,\,v)$. Proposition 1.5 then says that **such an** f **must be holomorphic with respect to some complex structure on** $\mathbf{R}^6$. So, Case 1 leads to our "trivial examples" for minimal real Kähler submanifolds in $\mathbf{R}^6$.

Case 2: $\alpha\,\beta = 1$ everywhere. This means that $\beta = \dfrac{1}{\alpha}$, where α can never be zero, and since Z is (almost) never zero either, (22) implies that

$$\alpha_v = -\alpha\,\beta_u = -\alpha\,\left(\frac{1}{\alpha}\right)_u = \alpha\,\frac{\alpha_u}{\alpha^2}\ ,$$

and hence that $\alpha = \alpha(u\,,\,v)$ must be a solution of the (non–linear) partial differential equation

$$\alpha_u = \alpha\,\alpha_v = \frac{1}{2}\,(\alpha^2)_v\ . \tag{25}$$

This of course restricts the choice of α greatly. Furthermore, we also have that

$$X_u = \alpha\,Z = \alpha\,X_v \quad \text{and} \quad Y_u = Z = \alpha\,\left(\frac{1}{\alpha}\,Z\right) = \alpha\,(\beta\,Z) = \alpha\,Y_v\ ;$$

i.e. α and the component functions of X and Y satisfy the same *linear* partial differential equation

$$\phi_u = \alpha\,\phi_v\ .$$

But from the theory of linear partial differential equations (see the Proposition in the Appendix), we know that, in this case, there must be holomorphic maps $\tilde{X} = \tilde{X}(w)$ and $\tilde{Y} = \tilde{Y}(w)$ in one complex variable w such that

$$X = \tilde{X} \circ \alpha \quad \text{and} \quad Y = \tilde{Y} \circ \alpha\ .$$

Note that α is a "fundamental system of solutions", since it is never zero and non–constant (see below). Using these relations in our integrability condition, and noting (25), we find that the former is equivalent to

$$X_v = (\tilde{X} \circ \alpha)_v = \alpha_v\,(\tilde{X}' \circ \alpha) = Y_u = (\tilde{Y} \circ \alpha)_u = \alpha_u\,(\tilde{Y}' \circ \alpha) = \alpha_v\,(\alpha\,(\tilde{Y}' \circ \alpha))\ ,$$

where $\tilde{X}'$ again means $\frac{d}{dw}\,\tilde{X}$, and where α_v is not constantly zero since we assumed that $X_v = Y_u$ is almost nowhere zero. Therefore, we must have that

$$0 = \tilde{X}' \circ \alpha - \alpha\,(\tilde{Y}' \circ \alpha) = (\tilde{X}' - w\,\tilde{Y}') \circ \alpha\ ,$$

and since $\tilde{X}$ and $\tilde{Y}$ are holomorphic, this implies that we always have

$$\tilde{X}' = w\,\tilde{Y}'\ .$$

Using our knowledge about the form of X and Y in a Weierstrass representation, we can write this *ordinary* differential equation in w as

$$\begin{pmatrix} \left(\tilde\lambda\,\frac{1-\tilde\xi^2}{2}\right)' \\ \left(i\,\tilde\lambda\,\frac{1+\tilde\xi^2}{2}\right)' \\ (\tilde\lambda\,\tilde\xi)' \end{pmatrix} \;=\; w \begin{pmatrix} \left(\tilde\mu\,\frac{1-\tilde\zeta^2}{2}\right)' \\ \left(i\,\tilde\mu\,\frac{1+\tilde\zeta^2}{2}\right)' \\ (\tilde\mu\,\tilde\zeta)' \end{pmatrix} ,$$

where here $\tilde\lambda$, $\tilde\mu$, $\tilde\xi$, and $\tilde\zeta$ are all holomorphic functions in the one complex variable w. Performing by now familiar operations, the last equation can be simplified to the system

$$\left.\begin{aligned} \tilde\lambda' &= w\,\tilde\mu' , \\ (\tilde\lambda\,\tilde\xi^2)' &= w\,(\tilde\mu\,\tilde\zeta^2)' , \\ \text{and} \quad (\tilde\lambda\,\tilde\xi)' &= w\,(\tilde\mu\,\tilde\zeta)' . \end{aligned}\right\} \tag{26}$$

Note that the first of these equations *determines* $\tilde\mu$ *in terms of* $\tilde\lambda$, up to a constant.

Let us now write $\tilde\xi = (s\,,\,t)$ and $\tilde\zeta = (p\,,\,q)$, where s, t, p, and q are holomorphic functions in w. Since $\tilde\mu$ is also given in terms of $\tilde\lambda$, $\tilde\xi$, and $\tilde\zeta$ as $\tilde\mu = \frac{-2}{\tilde\lambda\,(\tilde\xi-\tilde\zeta)^2}$, we can write

$$(\tilde\xi - \tilde\zeta)^2 = (s-p)^2 + (t-q)^2 = \frac{-2}{\tilde\lambda\,\tilde\mu} =: \tilde g(w) .$$

Note that by the last paragraph, the so–defined holomorphic function $\tilde g$ is actually almost entirely determined by $\tilde\lambda$ alone. But we know that $(\tilde\xi - \tilde\zeta)^2$ is never zero. Thus, we can *locally* find a well–defined, holomorphic "square-root" of $\tilde g$, i.e. a holomorphic function $g(w)$ such that

$$(s-p)^2 + (t-q)^2 = \tilde g(w) = g^2(w) .$$

Since the complex trigonometric functions are surjective (see [Ah], page 47), we can therefore conclude that, locally, there is a holomorphic function $h(w)$ such that we can write

$$s(w) = p(w) + g(w)\,\cos(h(w)) \quad \text{and} \quad t(w) = q(w) + g(w)\,\sin(h(w)) . \tag{27}$$

Let us now break the last equation in (26) into first and second components. The first components give

$$(\tilde{\lambda}\, s)' = \tilde{\lambda}'\, s + \tilde{\lambda}\, s' = w\,(\tilde{\mu}\, p)' = w\,\tilde{\mu}'\, p + w\,\tilde{\mu}\, p' \;.$$

Using the first equations of (26) and (27), this can be expanded to

$$\tilde{\lambda}'\, p + \tilde{\lambda}'\, g\, \cos h + \tilde{\lambda}\, p' + \tilde{\lambda}\,(g\, \cos h)' = \tilde{\lambda}'\, p + w\,\tilde{\mu}\, p' \;.$$

Cancelling the first term on both sides and reordering, we obtain

$$(w\,\tilde{\mu} - \tilde{\lambda})\, p' = (\tilde{\lambda}\, g\, \cos h)' \;.$$

But $w\,\tilde{\mu} - \tilde{\lambda}$ can only be zero at isolated points, since otherwise, we would have that $\tilde{\lambda} \equiv w\,\tilde{\mu}$. Differentiating this and then using the first equation in (26) would give

$$\tilde{\lambda}' = \tilde{\mu} + w\,\tilde{\mu}' = \tilde{\mu} + \tilde{\lambda}' \;,$$

and thus $\tilde{\mu} = 0$, which cannot happen. Therefore, we can solve the equation above for p' almost everywhere; and in completely analogous fashion, we can obtain for q':

$$p' = \frac{(\tilde{\lambda}\, g\, \cos h)'}{w\,\tilde{\mu} - \tilde{\lambda}} \quad \text{and} \quad q' = \frac{(\tilde{\lambda}\, g\, \sin h)'}{w\,\tilde{\mu} - \tilde{\lambda}} \;. \tag{28}$$

Note that this means that p and q, and thus by (27) also s and t, are (almost) completely determined by $\tilde{\lambda}$ and h alone.

Finally, let us examine the second equation in (26). We claim that this equation must necessarily be satisfied. To see this, we will replace $\tilde{\xi}$ and $\tilde{\zeta}$ by their component functions, and use the relations we worked out above in the following way. By (27), we find that

$$\begin{aligned}
\tilde{\xi}^2 \;&=\; s^2 + t^2 = (p + g\, \cos h)^2 + (q + g\, \sin h)^2 \\
&=\; p^2 + 2\, p\, g\, \cos h + g^2\, \cos^2 h + q^2 + 2\, q\, g\, \sin h + g^2\, \sin^2 h \\
&=\; \underbrace{p^2 + q^2}_{=\tilde{\zeta}^2} + 2\, g\,(p\, \cos h + q\, \sin h) + g^2 \;.
\end{aligned}$$

Thus, we have

$$(\tilde{\lambda}\,\tilde{\xi}^2)' = \underbrace{\tilde{\lambda}'}_{=w\,\tilde{\mu}'}\, \tilde{\zeta}^2 + 2\,\tilde{\lambda}\,(\tilde{\zeta}\cdot\tilde{\zeta}') + (2\,\tilde{\lambda}\, g\,(p\, \cos h + q\, \sin h) + \tilde{\lambda}\, g^2)' \;. \tag{29}$$

Let us work on the last term on the right–hand side. Recalling (28) and the fact that $g^2 = \tilde g = \frac{-2}{\tilde\lambda\,\tilde\mu}$, we find that this term equals

$$\begin{aligned}
2\,(\tilde\lambda\,g\,\cos h)'\,p + 2\,(\tilde\lambda\,g\,\sin h)'\,q + 2\,\tilde\lambda\,g(p'\,\cos h + q'\,\sin h) - \left(\frac{2}{\tilde\mu}\right)' \\
= \ 2\,(w\,\tilde\mu - \tilde\lambda)\,p'\,p + 2\,(w\,\tilde\mu - \tilde\lambda)\,q'\,q \\
+ \ 2\,\frac{(\tilde\lambda\,g\,\cos h)(\tilde\lambda\,g\,\cos h)' + (\tilde\lambda\,g\,\sin h)(\tilde\lambda\,g\,\sin h)'}{w\,\tilde\mu - \tilde\lambda} - \left(\frac{2}{\tilde\mu}\right)' \\
= \ 2\,(w\,\tilde\mu - \tilde\lambda)\underbrace{(p\,p' + q\,q')}_{=\tilde\zeta\cdot\tilde\zeta'} + \frac{(\tilde\lambda^2\,g^2)'}{w\,\tilde\mu - \tilde\lambda} - \left(\frac{2}{\tilde\mu}\right)' \\
= \ -2\,\tilde\lambda\,(\tilde\zeta\cdot\tilde\zeta') + w\,\tilde\mu\,(\tilde\zeta^2)' + \frac{-2\left(\frac{\tilde\lambda}{\tilde\mu}\right)'}{w\,\tilde\mu - \tilde\lambda} - 2\left(\frac{1}{\tilde\mu}\right)' \ .
\end{aligned}$$

But the last two terms on the last line cancel each other, since, using the first equation of (26) once more, we find that

$$\frac{\tilde\lambda'\,\tilde\mu - \tilde\lambda\,\tilde\mu'}{\tilde\mu^2\,(w\,\tilde\mu - \tilde\lambda)} = \frac{w\,\tilde\mu'\,\tilde\mu - \tilde\lambda\,\tilde\mu'}{\tilde\mu^2\,(w\,\tilde\mu - \tilde\lambda)} = \frac{\tilde\mu'}{\tilde\mu^2} = -\left(\frac{1}{\tilde\mu}\right)' \ .$$

This means that (29) takes the form

$$(\tilde\lambda\,\tilde\xi^2)' = w\,\tilde\mu'\tilde\zeta^2 + 2\,\tilde\lambda\,(\tilde\zeta\cdot\tilde\zeta') - \ 2\,\tilde\lambda\,(\tilde\zeta\cdot\tilde\zeta') + w\,\tilde\mu\,(\tilde\zeta^2)' = w\,(\tilde\mu\,\tilde\zeta^2)' \ ,$$

which is exactly the second equation in (26).

This finishes the investigation of the case that $\mathrm{rank} F'' = 3$. Summarizing our results (including Proposition 4.1), we thus have proven the following theorem, which together with Theorem 4.3 gives the promised complete local classification of all minimal real Kähler surfaces in Euclidean 6–space, away from the isolated “singularities” where the rank of the second osculating bundle F'' is smaller than on the rest of the manifold.

Theorem 4.5: *Let $F : W \to \mathbf{C}^6$ be a holomorphic representative of a minimal isometric immersion $f : M^4 \to \mathbf{R}^6$ from a 4-dimensional Kähler manifold M into $\mathbf{R}^6$, and assume that* **the rank of the second osculating bundle F'' of F equals three** *on all of W. Then f is holomorphic with respect to some complex structure on $\mathbf{R}^6$, or f is generated by an isotropic cylinder (see Example* 1.6*), or we can describe f locally in the following way:*

Let $((u, v), X, Y)$ be a Weierstrass representation of F in the neighborhood of a point $p \in W$ (see page 37 and Proposition 3.2). Then in a (perhaps smaller) neighborhood U of p in M, we can find a solution $\alpha : U \to \mathbf{C}$ of the non–linear partial differential equation

$$\alpha_u = \alpha\, \alpha_v = \frac{1}{2}\, (\alpha^2)_v$$

and two holomorphic functions $\tilde{\lambda}(w)$ and $h(w)$ in one complex variable w such that we have

$$X = (\tilde{\lambda} \circ \alpha) \left(\frac{1 - \tilde{\xi}^2}{2} \,,\, i\, \frac{1 + \tilde{\xi}^2}{2} \,,\, \tilde{\xi} \right) \circ \alpha$$

and

$$Y = (\tilde{\mu} \circ \alpha) \left(\frac{1 - \tilde{\zeta}^2}{2} \,,\, i\, \frac{1 + \tilde{\zeta}^2}{2} \,,\, \tilde{\zeta} \right) \circ \alpha \,,$$

where $\tilde{\mu} := \int \frac{\tilde{\lambda}'}{w}\, dw$, g *is a function such that* $g^2 = \frac{-2}{\tilde{\lambda}\, \tilde{\mu}}$,

$$\tilde{\zeta} := \left(\int \frac{(\tilde{\lambda}\, g\, \cos h)'}{w\, \tilde{\mu} - \tilde{\lambda}}\, dw \,,\, \int \frac{(\tilde{\lambda}\, g\, \sin h)'}{w\, \tilde{\mu} - \tilde{\lambda}}\, dw \right) \,,$$

and

$$\tilde{\xi} := \tilde{\zeta} + g\, (\cos h \,,\, \sin h) \,.$$

Conversely, for any choice of a non–constant α, a nowhere zero $\tilde{\lambda}$, and arbitrary h as above, the map $f := \sqrt{2}\, \mathrm{Re}(F)$ that we obtain through a Weierstrass representation $((u, v), X, Y)$ as defined above gives a minimal real Kähler immersion that is defined on some small neighborhood of $\mathbf{C}^2$.

Remark 4.6: Note that we have found another way to construct minimal real Kähler hypersurfaces in $\mathbf{R}^5$: Simply set $h := 0$, and (28) gives that q has to be a constant. Choosing this constant to be zero, (27) gives that t also has to be zero. Hence, the last components of X and Y and consequentially also the ones of F_u and F_v will be zero, which means that $f = \sqrt{2}\, \mathrm{Re}(F)$ can be considered as a map into $\mathbf{R}^5$.

Example 4.7: By a separation ansatz, one finds (and can easily check) that for all constants A, $B \in \mathbf{C}$,

$$\alpha(u\,,\,v) := -\frac{v+A}{u+B}$$

is a solution of (25). To keep things simple, let us set $\tilde{\lambda} := -1$, and let us determine h a bit later in such a way that the example stays simple. Since $\tilde{\lambda}' = w\,\tilde{\mu}'$, this means that $\tilde{\mu}$ also has to be constant. Let us choose $\tilde{\mu} = 2$, since then $\tilde{g} = \frac{-2}{(-1)\cdot 2} = 1$, and we can choose $g = 1$. Then, the equations for p' and q' have the form

$$p' = \frac{(-\cos h(w))'}{2\,w+1} = \frac{h'(w)\,\sin(h(w))}{2\,w+1}$$

and

$$q' = \frac{(-\sin h(w))'}{2\,w+1} = \frac{-h'(w)\,\cos(h(w))}{2\,w+1}\ .$$

So, if we set $h(w) := w^2 + w$, then

$$p' = \sin(w^2+w) \quad \text{and} \quad q' = -\cos(w^2+w)\ .$$

Setting all integration constants equal to zero and using (27), we finally obtain

$$\tilde{\zeta} = \begin{pmatrix} \displaystyle\int \sin(w^2+w)\,dw \\ \displaystyle -\int \cos(w^2+w)\,dw \end{pmatrix} \qquad \text{and} \qquad \tilde{\xi} = \tilde{\zeta} + \begin{pmatrix} \cos(w^2+w) \\ \sin(w^2+w) \end{pmatrix}\ .$$

Inserting these α, $\tilde{\lambda}$, $\tilde{\mu}$, $\tilde{\xi}$, and $\tilde{\zeta}$ into the Weierstrass representation formulas gives the complex Gauss map of a minimal real Kähler surface whose holomorphic representative has a second osculating bundle of rank three everywhere.