3 Minimal real Kähler surfaces and their Weierstrass representations

One key feature of the Weierstrass representation for minimal surfaces in $\mathbf{R}^N$ is that it uses a simple parametrization for isotropic vectors in $\mathbf{C}^N$ (see [H-O]). If one wants to find a similar representation for isometric immersions of a Kähler manifold M^{2n} into $\mathbf{R}^N$, one could, thus, try to first construct a similar parametrization for isotropic n-dimensional complex subspaces of $\mathbf{C}^N$. We will in fact do this for $n = 2$, i.e. for isotropic complex *planes* in $\mathbf{C}^N$ (for $N \geq 5$), and will then find a local characterization for all minimal (complex) *Kähler surfaces* in $\mathbf{R}^N$ (so: of *real* dimension 4). This characterization corresponds to the Weierstrass representation of a minimal surface (=complex *curve*) described on page 2. It is not only interesting in its own right, but also because all isometric immersions $f : M^{2n} \to \mathbf{R}^{2n+2}$ in codimension 2 that are *not* holomorphic with respect to some complex structure in $\mathbf{R}^{2n+2}$ must locally be affine vector bundles of (real) rank $2n-4$ over a 4-dimensional Kähler manifold $\tilde{M}^4$ (see our remarks on page 6). And the complex Gauss map of such an f restricted to $\tilde{M}$ can be viewed as a map from $\tilde{M}$ into the space of all isotropic complex planes in $\mathbf{C}^{2n+2}$.

In general, denote the space of all isotropic complex n-subspaces in $\mathbf{C}^N$ by $\mathbf{I}_n(\mathbf{C}^N)$, or $\mathbf{I}_n^N$ for short. It is well-known (see e.g. [R-T]) that $\mathbf{I}_n^N$ is a compact complex homogeneous space, namely

$$\mathbf{I}_n(\mathbf{C}^N) \cong {SO(N)}\Big/{\big(U(n) \times SO(N-2n)\big)} \quad .$$

This implies for the (real) dimension of this manifold

$$\left.\begin{aligned} \dim_{\mathbf{R}}(\mathbf{I}_n(\mathbf{C}^N)) &= \tbinom{N}{2} - (n^2 + \tbinom{N-2n}{2}) = 2\,N\,n - 3\,n^2 - n \\ &= n \cdot (2\,N - 3\,n - 1)\ . \end{aligned}\right\} \qquad (1)$$

In fact, as a complex submanifold of the Grassmannian $\mathrm{Gr}_n(\mathbf{C}^N)$, which is a Kähler manifold with its standard (Fubini-Study like) metric (see [K-N], pages 133–134 and pages 160–161), $\mathbf{I}_n^N$ is itself a Kähler manifold. However, we will not use this last fact here.

We will now give a complex coordinate system for $\mathbf{I}_2^N$, whose *complex* dimension is, by (1), $\frac{1}{2} \cdot 2 \cdot (2\,N - 3 \cdot 2 - 1) = 2\,N - 7$. Note that since $\mathbf{I}_2^N$ is homogeneous and since we will be mainly interested in *local* considerations, any "small" coordinate system of $\mathbf{I}_2^N$ suffices for our purposes.

We will write the complex coordinates in $\mathbf{C}^{2N-7}$ as $(\xi\,,\,\zeta\,,\,\lambda)$, where $\xi = (\xi_1\,,\ldots,\,\xi_{N-4})$ and $\zeta = (\zeta_1\,,\ldots,\,\zeta_{N-4})$ are in $\mathbf{C}^{N-4}$ and λ is a complex number. Then set

$$\left.\begin{aligned} &\mathbf{X} := \left(\tfrac{1}{2}\,,\,\tfrac{i}{2}\,,\,X\right), && \text{where } X := \lambda\left(\tfrac{1-\xi^2}{2},\, i\,\tfrac{1+\xi^2}{2},\,\xi\right), \\ &\text{and} && \\ &\mathbf{Y} := \left(-X\cdot Y\,,\,i\,X\cdot Y\,,\,Y\right), && \text{where } Y := \left(\tfrac{1-\zeta^2}{2},\, i\,\tfrac{1+\zeta^2}{2},\,\zeta\right). \end{aligned}\right\} \qquad (2)$$

Here, $X\cdot Y$ is the standard symmetric inner product in $\mathbf{C}^{N-2}$, and $\xi^2 = \xi\cdot\xi$ and $\zeta^2 = \zeta\cdot\zeta$ refer to the analogous inner product in $\mathbf{C}^{N-4}$ (see page 10). It is easy to check that $\mathbf{X}$ and $\mathbf{Y}$ span an *isotropic subspace* of $\mathbf{C}^N$, and that X and Y are isotropic *vectors* in $\mathbf{C}^{N-2}$. However, in general X and Y will *not* span an isotropic sub*space* of $\mathbf{C}^{N-2}$. Now, define the map $\Phi : \mathbf{C}^{2N-7} \to \mathbf{I}_2^N$ by

$$\Phi(\xi\,,\,\zeta\,,\,\lambda) := \operatorname{span}\{\mathbf{X}\,,\mathbf{Y}\}\ ,$$

with $\mathbf{X}$ and $\mathbf{Y}$ defined in terms of $(\xi\,,\,\zeta\,,\,\lambda)$ as in (2). Note that this map is well-defined, i.e. that $\operatorname{span}\{\mathbf{X}\,,\mathbf{Y}\}$ is in fact always 2-dimensional; for let us assume that there were complex numbers α and β such that

$$0 = \alpha\,\mathbf{X} + \beta\,\mathbf{Y} = \begin{pmatrix} \frac{\alpha}{2} - \beta\,X\cdot Y \\ i\,\frac{\alpha}{2} + i\,\beta\,X\cdot Y \\ \alpha\,X + \beta\,Y \end{pmatrix}.$$

Dividing the second component by i and adding it to the first component gives $\alpha = 0$, so that in particular we must have $\beta\,Y = 0$. But looking at the form of Y as in (2), we see that the first and the second component of Y can never be zero simultaneously. Thus, β must also be zero.

We will have to restrict the domain of Φ to obtain the (inverse of the) coordinate system for which we are looking. Thus, let

$$G\Phi := \{\,(\xi\,,\,\zeta\,,\,\lambda) \in \mathbf{C}^{2N-7} \mid \lambda\,(\xi-\zeta)^2 \neq 0\,\}\ .$$

Note that the condition in this definition is equivalent to $X\cdot Y \neq 0$, since

$$\left.\begin{aligned} X\cdot Y &= \lambda\Big(\frac{(1-\xi^2)(1-\zeta^2)}{4} + i^2\frac{(1+\xi^2)(1+\zeta^2)}{4} + \xi\cdot\zeta\Big) \\ &= \lambda\Big(-\tfrac{1}{2}\,\xi^2 - \tfrac{1}{2}\,\zeta^2 + \xi\cdot\zeta\Big) = -\frac{\lambda}{2}\,(\xi-\zeta)^2 \neq 0\ . \end{aligned}\right\} \qquad (3)$$

Thus, span$\{X\,,\,Y\}$ will, in fact, *never* be an isotropic subspace of $\mathbf{C}^{N-2}$ as long as $(\xi\,,\,\zeta\,,\,\lambda)\in G\Phi$. Obviously, $G\Phi$ is an open set in $\mathbf{C}^{2N-7}$, and it is easy to see that $G\Phi$ is connected. Then we have

Lemma 3.1: *The map* $\Phi:$

$$\begin{array}{rcl} G\Phi & \longrightarrow & \mathbf{I}_2^N \\ (\xi\,,\,\zeta\,,\,\lambda) & \longmapsto & \mathrm{span}\{\mathbf{X}\,,\mathbf{Y}\}\ , \end{array}$$

where $G\Phi := \{\,(\xi\,,\,\zeta\,,\,\lambda)\in\mathbf{C}^{2N-7}\mid\lambda\,(\xi-\zeta)^2\neq 0\,\}$*, and* $\mathbf{X}$ *and* $\mathbf{Y}$ *are given by* (2) *as*

$$\mathbf{X} := \begin{pmatrix} \frac{1}{2} \\ \frac{i}{2} \\ \lambda\,\frac{1-\xi^2}{2} \\ i\,\lambda\,\frac{1+\xi^2}{2} \\ \lambda\,\xi \end{pmatrix}\ ,\ \textit{and}\quad \mathbf{Y} := \begin{pmatrix} \frac{\lambda}{2}\,(\xi-\zeta)^2 \\ -\frac{i\,\lambda}{2}\,(\xi-\zeta)^2 \\ \frac{1-\zeta^2}{2} \\ i\,\frac{1+\zeta^2}{2} \\ \zeta \end{pmatrix}\ ,$$

is an injective holomorphic immersion between complex manifolds of the same (complex) dimension $2N-7$*, and thus (the inverse of) a coordinate system for* $\mathbf{I}_2^N$*.*

We call the inverse chart Φ as in Lemma 3.1 a **Weierstrass coordinate system** for $\mathbf{I}_2^N$.

Proof: First, we prove that Φ is *injective* on $G\Phi$. Thus, suppose that

$$\mathrm{span}\{\mathbf{X}\,,\mathbf{Y}\} = \mathrm{span}\{\tilde{\mathbf{X}}\,,\tilde{\mathbf{Y}}\}\ ,$$

which means that there are complex numbers α, β, γ, and δ such that

$$\left.\begin{array}{rcl} \tilde{\mathbf{X}} & = & \alpha\,\mathbf{X}+\beta\,\mathbf{Y}\ , \\ \tilde{\mathbf{Y}} & = & \gamma\,\mathbf{X}+\delta\,\mathbf{Y}\ . \end{array}\right\} \tag{4}$$

By (2), the first equation means

$$\begin{pmatrix} \frac{1}{2} \\ \frac{i}{2} \\ \tilde{X} \end{pmatrix} = \begin{pmatrix} \frac{\alpha}{2}-\beta\,X\cdot Y \\ i\,\frac{\alpha}{2}+i\,\beta\,X\cdot Y \\ \alpha\,X+\beta\,Y \end{pmatrix}\ .$$

Dividing the second components by i and adding them to the first components on either side gives $\alpha = 1$, whereas subtracting these components gives $2\,\beta\, X\cdot Y = 0$. Since we are in $G\Phi$, we have by (3) that $X\cdot Y \neq 0$, and thus that $\beta = 0$. So, we have found that $\tilde{\mathbf{X}} = \mathbf{X}$, and thus, in particular, that $\tilde{X} = X$. By the second equation in (4) and by (2), we have

$$\begin{pmatrix} -X\cdot\tilde{Y} \\ i\,X\cdot\tilde{Y} \\ \tilde{Y} \end{pmatrix} = \begin{pmatrix} \frac{\gamma}{2} - \delta\, X\cdot Y \\ i\,\frac{\gamma}{2} + i\,\delta\, X\cdot Y \\ \gamma\, X + \delta\, Y \end{pmatrix} .$$

Again, dividing the second components by i and adding them to the first, we immediately have $\gamma = 0$, and thus $\tilde{Y} = \delta\, Y$. But according to (2), this means that the first two components of $\tilde{Y}$ and $\delta\, Y$ give

$$\frac{1-\tilde{\zeta}^2}{2} = \delta\ \frac{1-\zeta^2}{2} \quad \text{and} \quad \frac{1+\tilde{\zeta}^2}{2} = \delta\ \frac{1+\zeta^2}{2} .$$

Adding these equations gives $\delta = 1$, and thus $\tilde{\mathbf{Y}} = \mathbf{Y}$. By (2), this obviously means that the coordinates $(\xi\,,\,\zeta\,,\,\lambda)$ and $(\tilde{\xi}\,,\,\tilde{\zeta}\,,\,\tilde{\lambda})$ generating the subspaces with which we started have to be the same. Thus, Φ is injective on $G\Phi$.

We will now show that Φ is *holomorphic and immersive* on $G\Phi$. For this, we need a complex chart for $\mathbf{I}_2^N$. Since $\mathbf{I}_2^N$ is a *compact* complex submanifold of the Grassmannian $\mathrm{Gr}_2(\mathbf{C}^N)$, its topology is the subspace topology of $\mathrm{Gr}_2(\mathbf{C}^N)$, and thus it suffices to use a complex chart φ_α of $\mathrm{Gr}_2(\mathbf{C}^N)$ as in the following diagram:

$$\begin{tikzcd} G\Phi \arrow[r, "\Phi"] & \mathbf{I}_2^N \arrow[r, hook] & \mathrm{Gr}_2(\mathbf{C}^N) \arrow[r, "\varphi_\alpha"] & \mathrm{Mat}((N-2)\times 2,\,\mathbf{C}) \\ (\xi\,,\,\zeta\,,\,\lambda) \arrow[r, mapsto] & \underset{\mathbf{X}\,,\,\mathbf{Y}\text{ as in (2)}}{\mathrm{span}\{\mathbf{X}\,,\mathbf{Y}\}\,,} \arrow[r, mapsto] & \mathrm{span}\{\mathbf{X}\,,\mathbf{Y}\} \arrow[r, mapsto] & \varphi_\alpha\big(\mathrm{span}\{\mathbf{X}\,,\mathbf{Y}\}\big)\,. \end{tikzcd}$$

It is clear that if we can show that the map

$$\left.\begin{array}{rcl} \Phi_\alpha : \quad G\Phi & \longrightarrow & \mathrm{Mat}\big((N-2)\times 2,\,\mathbf{C}\big) \\ (\xi\,,\,\zeta\,,\,\lambda) & \longmapsto & \varphi_\alpha(\mathrm{span}\{\mathbf{X}\,,\mathbf{Y}\}) \end{array}\right\} \qquad (5)$$

is a holomorphic immersion, then the same has to be true for Φ. Of course, we need an explicit formula for φ_α. We will use the one described in Example 2.4 in [K-N], page 133 (where $p = 2$ and $p+q = N$). Thus, if $(z_1\,,\ldots,\,z_{_N})$

is the standard complex coordinate system in $\mathbf{C}^N$, consider the z_j as linear maps $\mathbf{C}^N \to \mathbf{C}$, and choose a set $\alpha := \{\alpha_1\,,\,\alpha_2\}$ of integers such that $1 \leq \alpha_1 < \alpha_2 \leq N$. Let U_α be the subset of all complex planes $S \subset \mathbf{C}^N$ such that $z_{\alpha_1}|S$ and $z_{\alpha_2}|S$ are linearly independent; i.e. they form basis of the dual space of S. Write the other coordinates as $z_{\alpha_3}\,,\ldots,\,z_{\alpha_N}$, where $\{\alpha_3\,,\ldots,\,\alpha_N\}$ is the complement of α in $\{1\,,\ldots,\,N\}$, written in increasing order. Then, we can find complex numbers s_{kj} such that

$$z_{\alpha_k}|S = \sum_{j=1}^{2} s_{kj}(z_{\alpha_j}|S) \quad \text{for} \quad k = 3\,,\ldots,\,N\ ,$$

and the complex chart for which we are looking is given by

$$\begin{aligned} \varphi_\alpha :\ U_\alpha &\longrightarrow \operatorname{Mat}\Big((N-2)\times 2,\,\mathbf{C}\Big) \\ S &\longmapsto (s_{kj})_{\substack{k=3,\ldots,N\\ j=1,2}} \end{aligned} \ .$$

Actually, it is not hard to see how we can write this chart in terms of a basis $v_1\,,\,v_2 \in \mathbf{C}^N$ of the given complex plane S, namely in the following way. Define the two projections $P_\alpha : \operatorname{Mat}(N\times 2,\,\mathbf{C}) \to \operatorname{Mat}(2\times 2,\,\mathbf{C})$ and $Q_\alpha : \operatorname{Mat}(N\times 2,\,\mathbf{C}) \to \operatorname{Mat}\Big((N-2)\times 2,\,\mathbf{C}\Big)$ by

$$P_\alpha\Big((a_{kj})_{\substack{k=1,\ldots,N\\ j=1,2}}\Big) := \begin{pmatrix} a_{\alpha_1,1} & a_{\alpha_1,2} \\ a_{\alpha_2,1} & a_{\alpha_2,2} \end{pmatrix} \text{ and } Q_\alpha\Big((a_{kj})_{\substack{k=1,\ldots,N\\ j=1,2}}\Big) := \begin{pmatrix} a_{\alpha_3,1} & a_{\alpha_3,2} \\ \vdots & \vdots \\ a_{\alpha_N,1} & a_{\alpha_N,2} \end{pmatrix} .$$

Then it is not hard to show that if v_1, v_2 is a basis of $S \in \operatorname{Gr}_2(\mathbf{C}^n)$, S is in U_α exactly if $\det\left(P_\alpha(v_1, v_2)\right) \neq 0$ (where (v_1, v_2) is regarded as the $N\times 2$–matrix whose columns are the components of v_1 and v_2, respectively). Furthermore, one can easily show that we have

$$\varphi_\alpha(S) = Q_\alpha(v_1, v_2)\cdot\Big(P_\alpha(v_1, v_2)\Big)^{-1}\ , \tag{6}$$

which is the form of φ_α with which we will work. Note that, in particular, this formula does not depend on the choice of the basis v_1, v_2 for $S \in U_\alpha$.

We will now use the above formulas in the special case $\alpha = \{1, 2\}$ and for the complex planes that are given by our map Φ. Substituting $v_1 = \mathbf{X}$ and $v_2 = \mathbf{Y}$, we see that

$$\det\Big(P_\alpha(\mathbf{X}\,,\mathbf{Y})\Big) = \det\begin{pmatrix} \frac{1}{2} & -X\cdot Y \\ \frac{i}{2} & i\,X\cdot Y \end{pmatrix} = i\,X\cdot Y\ ,$$

which is different from zero *exactly* if $\mathrm{span}\{\mathbf{X}\,,\mathbf{Y}\}\in G\Phi$ (see (3)). This means that our map $\Phi_\alpha=\Phi_{\{1,2\}}$ in (5) is defined on all of $G\Phi$, and it is clear by (2) and (6) that its components are *rational* functions in the complex variables $(\xi\,,\,\zeta\,,\,\lambda)\in G\Phi$, and thus that Φ_α is a *holomorphic* map.

It remains to be shown that Φ_α is *immersive* on $G\Phi$. To establish this, we have to explicitly calculate how Φ_α looks in terms of $(\xi\,,\,\zeta\,,\,\lambda)$:

$$\begin{aligned}\Phi_\alpha(\xi\,,\,\zeta\,,\,\lambda)\;&=\;Q_\alpha(\mathbf{X}\,,\mathbf{Y})\cdot\Big(P_\alpha(\mathbf{X}\,,\mathbf{Y})\Big)^{-1}\\ &=\;\begin{pmatrix}|&|\\ X&Y\\ |&|\end{pmatrix}\cdot\frac{1}{i\,X\cdot Y}\begin{pmatrix}i\,X\cdot Y & X\cdot Y\\ -\frac{i}{2} & \frac{1}{2}\end{pmatrix}\ ,\end{aligned}$$

since $\alpha=\{1,2\}$. Multiplying these matrices and substituting the expressions for X and Y according to (2) and for $X\cdot Y$ as in (3), we obtain

$$\Phi_\alpha(\xi\,,\,\zeta\,,\,\lambda)=\Big(X-\tfrac{1}{2\,X\cdot Y}\,Y\ \ ,\ \ -i\,X-\tfrac{i}{2\,X\cdot Y}\,Y\Big)$$

$$=\begin{pmatrix}\frac{\lambda}{2}\,(1-\xi^2)+\frac{1}{2\,\lambda\,(\xi-\zeta)^2}\,(1-\zeta^2) & -\frac{i\,\lambda}{2}\,(1-\xi^2)+\frac{i}{2\,\lambda\,(\xi-\zeta)^2}\,(1-\zeta^2)\\ \frac{i\,\lambda}{2}\,(1+\xi^2)+\frac{i}{2\,\lambda\,(\xi-\zeta)^2}\,(1+\zeta^2) & \frac{\lambda}{2}\,(1+\xi^2)-\frac{1}{2\,\lambda\,(\xi-\zeta)^2}\,(1+\zeta^2)\\ \lambda\,\xi+\frac{1}{\lambda\,(\xi-\zeta)^2}\,\zeta & -i\,\lambda\,\xi+\frac{i}{\lambda\,(\xi-\zeta)^2}\,\zeta\end{pmatrix}.$$

We will now determine the partial derivatives of Φ_α in terms of the complex coordinates $(\xi_1\,,\ldots,\,\xi_{N-4}\,,\,\zeta_1\,,\ldots,\,\zeta_{N-4}\,,\,\lambda)$. Note that since

$$\xi^2=\xi_1^2+\ldots+\xi_{N-4}^2\ \ ,\quad \zeta^2=\zeta_1^2+\ldots+\zeta_{N-4}^2\ \ ,$$
$$\text{and}\quad (\xi-\zeta)^2=(\xi_1-\zeta_1)^2+\ldots+(\xi_{N-4}-\zeta_{N-4})^2\ ,$$

we have that, for all $1\le j\,,k\le N-4$,

$$\frac{\partial\,\xi^2}{\partial\,\xi_j}=2\,\xi_j\ \ ,\quad \frac{\partial\,\xi^2}{\partial\,\zeta_k}=0\ \ ,\quad \frac{\partial\,\zeta^2}{\partial\,\xi_j}=0\ \ ,\quad \frac{\partial\,\zeta^2}{\partial\,\zeta_k}=2\,\zeta_k\ \ ,$$

$$\frac{\partial\,(\xi-\zeta)^2}{\partial\,\xi_j}=2\,(\xi_j-\zeta_j)\ \ ,\quad\text{and}\quad \frac{\partial\,(\xi-\zeta)^2}{\partial\,\zeta_k}=-2\,(\xi_k-\zeta_k)\ .$$

Also, it will be convenient to write

$$(\xi-\zeta)^4 := \Big((\xi-\zeta)^2\Big)^2 = \Big((\xi_1-\zeta_1)^2+\ldots+(\xi_{N-4}-\zeta_{N-4})^2\Big)^2\,,$$

and to denote the $j^{\,th}$ coordinate vector in $\mathbf{C}^{N-4}$ by $\mathbf{e}_j$. Using these notations, we obtain for all $1 \le j\,, k \le N-4$:

$$\frac{\partial\,\Phi_\alpha}{\partial\,\xi_j} = \lambda \underbrace{\begin{pmatrix} -\xi_j & i\,\xi_j \\ i\,\xi_j & \xi_j \\ \mathbf{e}_j & -i\,\mathbf{e}_j \end{pmatrix}}_{=:\,A_j} - \frac{\xi_j-\zeta_j}{\lambda\,(\xi-\zeta)^4} \underbrace{\begin{pmatrix} 1-\zeta^2 & i\,(1-\zeta^2) \\ i\,(1+\zeta^2) & -(1+\zeta^2) \\ 2\,\zeta & 2\,i\,\zeta \end{pmatrix}}_{=:\,Z}\,,$$

$$\frac{\partial\,\Phi_\alpha}{\partial\,\zeta_k} = \frac{1}{\lambda\,(\xi-\zeta)^2} \underbrace{\begin{pmatrix} -\zeta_k & -i\,\zeta_k \\ i\,\zeta_k & -\zeta_k \\ \mathbf{e}_k & i\,\mathbf{e}_k \end{pmatrix}}_{=:\,B_k} + \frac{\xi_k-\zeta_k}{\lambda\,(\xi-\zeta)^4} \underbrace{\begin{pmatrix} 1-\zeta^2 & i\,(1-\zeta^2) \\ i\,(1+\zeta^2) & -(1+\zeta^2) \\ 2\,\zeta & 2\,i\,\zeta \end{pmatrix}}_{=\,Z}\,,$$

and finally

$$\frac{\partial\,\Phi_\alpha}{\partial\,\lambda} = \frac{1}{2} \underbrace{\begin{pmatrix} 1-\xi^2 & -i\,(1-\xi^2) \\ i\,(1+\xi^2) & 1+\xi^2 \\ 2\,\xi & -2\,i\,\xi \end{pmatrix}}_{=:\,\Xi} - \frac{1}{2\,\lambda^2\,(\xi-\zeta)^2} \underbrace{\begin{pmatrix} 1-\zeta^2 & i\,(1-\zeta^2) \\ i\,(1+\zeta^2) & -(1+\zeta^2) \\ 2\,\zeta & 2\,i\,\zeta \end{pmatrix}}_{=\,Z}\,.$$

Now, if we can show that the $(N-2)\times 2$–matrices A_j $(1 \le j \le N-4)$, B_k $(1 \le k \le N-4)$, Ξ , and Z defined as above are linearly independent in $\mathrm{Mat}\Big((N-2)\times 2\,,\mathbf{C}\Big)$ whenever $(\xi\,,\,\zeta\,,\,\lambda)$ is in $G\Phi$, then the same has to be true for the partial derivatives of Φ_α. For assume that we have complex numbers a_j , b_j $(1 \le j \le N-4)$, and c such that

$$0 = \sum_{j=1}^{N-4} \Big(a_j\,\frac{\partial\,\Phi_\alpha}{\partial\,\xi_j} + b_j\,\frac{\partial\,\Phi_\alpha}{\partial\,\zeta_j}\Big) \;+\; c\,\frac{\partial\,\Phi_\alpha}{\partial\,\lambda}\,.$$

Replacing the partial derivatives of Φ_α as above and ordering the resulting terms, one finds that

$$0 = \sum_{j=1}^{N-4} \lambda\, a_j\, A_j + \sum_{j=1}^{N-4} \frac{b_j}{\lambda\,(\xi-\zeta)^2}\, B_j + \frac{c}{2}\;\Xi \;+\; \begin{pmatrix}\text{some}\\ \text{large sum}\end{pmatrix} Z\ .$$

Since we assume that the A_j , B_j , Ξ, and Z are linearly independent, we have, in particular, that

$$\lambda\, a_j = 0\ ,\quad \frac{b_j}{\lambda\,(\xi-\zeta)^2} = 0\ \ (1 \le j \le N-4)\ ,\quad \text{and}\quad c = 0\ .$$

Since we are in $G\Phi$, where $\lambda\,(\xi-\zeta)^2$ (and thus λ) is never zero, this means that all the a_j , b_j , and c must be zero, and thus that Φ_α is indeed *immersive* on all of $G\Phi$.

To show that the A_j , B_j , Ξ, and Z are linearly independent, we assume that there are complex numbers a_j , b_j $(1 \le j \le N-4)$, c, and d such that

$$0 = \sum_{j=1}^{N-4} \Big(a_j\, A_j + b_j\, B_j\Big) \;+\; c\,\Xi \;+\; d\,Z\ .$$

Combining the a_j and the b_j to the vectors

$$a := (a_1\,, \ldots\,, a_{N-4}) \quad\text{and}\quad b := (b_1\,, \ldots\,, b_{N-4}) \in \mathbf{C}^{N-4}\ ,$$

and using the definitions of our matrices, we can rewrite the above equation in terms of the entries of the matrices:

$(1\,,1)$–entry: $$0 = -a\cdot\xi - b\cdot\zeta + c\,(1-\xi^2) + d\,(1-\zeta^2) \qquad \text{(a)}$$

$(1\,,2)$–entry: $$0 = i\,\Big(a\cdot\xi - b\cdot\zeta - c\,(1-\xi^2) + d\,(1-\zeta^2)\Big) \qquad \text{(b)}$$

$(2\,,1)$–entry: $$0 = i\,\Big(a\cdot\xi + b\cdot\zeta + c\,(1+\xi^2) + d\,(1+\zeta^2)\Big) \qquad \text{(c)}$$

$(2\,,2)$–entry: $$0 = a\cdot\xi - b\cdot\zeta + c\,(1+\xi^2) - d\,(1+\zeta^2) \qquad \text{(d)}$$

rest of 1^{st} column: $$0 = a + b + 2\,c\,\xi + 2\,d\,\zeta \qquad \text{(e)}$$

rest of 2^{nd} column: $$0 = i\,(-a + b - 2\,c\,\xi + 2\,d\,\zeta) \qquad \text{(f)}$$

Dividing (f) by i and adding it to (e) gives, after dividing by 2, $b = -2\,d\,\zeta$, whereas subtracting gives $a = -2\,c\,\xi$. Dividing (b) by i, adding it to (a), and using the formula for b we just obtained gives

$$0 = -2\,(\underbrace{-2\,d\,\zeta}_{=b})\cdot\zeta + 2\,d - 2\,d\,\zeta^2 = 2\,d + 2\,d\,\zeta^2\ ,$$

whereas subtracting and using the formula for a results in

$$0 = -2\,(\underbrace{-2\,c\,\xi}_{=a}) \cdot \xi + 2\,c - 2\,c\,\xi^2 = 2\,c + 2\,c\,\xi^2 \;.$$

Performing similar operations on (c) and (d) gives

$$0 = 2\,c - 2\,c\,\xi^2 \quad \text{and} \quad 0 = 2\,d - 2\,d\,\zeta^2 \;,$$

and adding the corresponding equations for c and d obviously gives $c = d = 0$. Thus, we also have that $a = b = 0$, which is what we set out to prove.

This finishes the proof of Lemma 3.1.

Before we proceed, a few more remarks about Weierstrass coordinate systems Φ as in Lemma 3.1 are in order. First, we have that Φ is actually holomorphic as a map from *all of* $\mathbf{C}^{2N-7}$ into $\mathbf{I}_2^n$.

[To see this, note that if $(\xi\,,\,\zeta\,,\,\lambda) \notin G\Phi$, i.e. according to (3) that $X \cdot Y = 0$ (with X and Y as in (2)), then the first two components of $\mathbf{Y}$ as in (2) are zero, i.e. $\mathbf{Y} = (\,0\,,\,0\,,\,Y\,)$. But since the first two components of Y are never zero simultaneously, we find that either

$$\det\begin{pmatrix} \mathbf{X}_1 & \mathbf{Y}_1 \\ \mathbf{X}_3 & \mathbf{Y}_3 \end{pmatrix} = \det\begin{pmatrix} \frac{1}{2} & 0 \\ X_1 & Y_1 \end{pmatrix} = \frac{1}{2}\,Y_1 = \frac{1}{4}\,(1-\zeta^2) \neq 0 \;,$$

or that

$$\det\begin{pmatrix} \mathbf{X}_1 & \mathbf{Y}_1 \\ \mathbf{X}_4 & \mathbf{Y}_4 \end{pmatrix} = \det\begin{pmatrix} \frac{1}{2} & 0 \\ X_2 & Y_2 \end{pmatrix} = \frac{1}{2}\,Y_2 = \frac{i}{4}\,(1+\zeta^2) \neq 0 \;.$$

This means that in a neighborhood of any point in $\mathbf{C}^{2N-7} - G\Phi$, we can repeat the procedure on pages 28 and 29 with $\alpha = \{1\,,3\}$ or $\alpha = \{1\,,4\}$, and we see that, again, we will obtain a map from this neighborhood to $\mathrm{Mat}\big((N-2)\times 2\,,\,\mathbf{C}\big)$, all of whose component functions are rational, and which, thus, is holomorphic. This makes Φ *holomorphic* in this neighborhood.]

On the other hand, we can*not* assume Φ to be *immersive* at points that are not in $G\Phi$.

[To see this, note e.g. that whenever $\lambda = 0$ and ζ is fixed, *any* ξ will give the same isotropic plane in $\mathbf{I}_2^N$ under Φ. Thus, Φ will *not* be locally injective in any neighborhood of such a point, and thus cannot be immersive there.]

The next Proposition gives the promised *local characterization of minimal real Kähler surfaces* (so: of *real* dimension 4) in Euclidean spaces:

Proposition 3.2: *Let $f : M^4 \to \mathbf{R}^N$ be a minimal isometric immersion from a (real) 4–dimensional Kähler manifold M into $\mathbf{R}^N$ (where $N \geq 5$), and let $p \in M$ be any point in M. Further, let $F : W \to \mathbf{C}^N$ be a holomorphic representative of f, defined in a neighborhood W of p in M (see page 5). Then we can find a neighborhood U of p in W, a complex chart $(u\,,\, v)$ of M defined on U, and holomorphic maps $\xi\,,\,\zeta : U \to \mathbf{C}^{N-4}$ and $\lambda : U \to \mathbf{C}$ such that on all of U*

$$\lambda\,(\xi - \zeta)^2 \neq 0\,, \tag{7}$$

and such that, up to isometry in $\mathbf{R}^N$, we have that on all of U

$$\left.\begin{aligned} F_u \;&=\; (\tfrac{1}{2}\,,\,\tfrac{i}{2}\,,\,X)\,, \quad \textit{where}\;\; X := \lambda\left(\tfrac{1-\xi^2}{2},\, i\,\tfrac{1+\xi^2}{2},\,\xi\right), \\ &\textit{and} \\ F_v \;&=\; (-1\,,\,i\,,\,Y)\,, \quad \textit{where}\;\; Y := \mu\left(\tfrac{1-\zeta^2}{2},\, i\,\tfrac{1+\zeta^2}{2},\,\zeta\right), \\ &\qquad\qquad\qquad \textit{with}\;\; \mu := -\tfrac{2}{\lambda\,(\xi-\zeta)^2}\;. \end{aligned}\right\} \tag{8}$$

In particular, we have that the complex Gauss map $\varphi := \mathrm{im}(F_*)$ of f (as defined on page 9) factors on U as

$$\varphi = \Phi \circ (\xi\,,\,\zeta\,,\,\lambda)\;. \tag{9}$$

Proof: Let $(z_1,\, z_2)$ be some complex chart of M, without loss of generality defined on W. By Theorem 1.1, we know that $\varphi(q)$ is an isotropic complex plane in $\mathbf{C}^N$ for every $q \in M$. Let Φ denote the Weierstrass coordinate system for $\mathbf{I}_2^N$ as in Lemma 3.1. If we have that $\varphi(p) \notin \Phi(G\Phi)$ for the chosen point p, we can find an $A \in SO(N)$ such that $A(\varphi(p)) \in \Phi(G\Phi)$, since $\mathbf{I}_2^N$ is homogeneous. Note that by Lemma 1.3, $A \circ F$ is the holomorphic representative of $A \circ f : M \to \mathbf{R}^N$, which is congruent to f in $\mathbf{R}^N$. Thus, we can assume from now on that, without loss of generality, $\varphi(p) \in \Phi(G\Phi)$.

Now, the Gauss map $q \mapsto \varphi(q) = \mathrm{span}\{F_{z_1}(q),\, F_{z_2}(q)\}$ is easily seen to be a holomorphic map $W \to \mathbf{I}_2^N$ (compare the arguments on pages 28 and 29). Let $V \subset W$ be a simply connected neighborhood of p such that, for all $q \in V$, $\varphi(q) \in \Phi(G\Phi)$. Then the map $\Phi^{-1} \circ \varphi : V \to \mathbf{C}^{2N-7}$ is also holomorphic; i.e. with respect to the given chart (z_1, z_2) restricted to V, we have *holomorphic* functions ξ , ζ , and λ as in Proposition 3.2 such that (9) is true on all of V. Also, by the definition of $G\Phi$, we know that (7) is satisfied on all of V.

Let now $\mathbf{X}$, $\mathbf{Y}$, X and $\tilde{Y}$ (instead of "Y") be defined in terms of ξ , ζ , and λ as in (2); in particular, they are all holomorphic on V. Since we thus have

$$\Phi \circ (\xi\,,\,\zeta\,,\,\lambda) = \operatorname{span}\{\mathbf{X}\,,\mathbf{Y}\} = \operatorname{span}\left\{ \begin{pmatrix} \frac{1}{2} \\ \frac{i}{2} \\ X \end{pmatrix} , \begin{pmatrix} -X\cdot\tilde{Y} \\ i\,X\cdot\tilde{Y} \\ \tilde{Y} \end{pmatrix} \right\} , \tag{10}$$

by (9) there must be holomorphic functions α , β , γ , $\delta : V \to \mathbf{C}$ such that

$$\left.\begin{aligned} F_{z_1} &= \alpha\,\mathbf{X} + \beta\,\mathbf{Y} \\ F_{z_2} &= \gamma\,\mathbf{X} + \delta\,\mathbf{Y} \end{aligned}\right\} \quad \text{and} \quad \det\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \neq 0 \,. \tag{11}$$

Using the *integrability condition* $(F_{z_1})_{z_2} = (F_{z_2})_{z_1}$ in (11), we obtain

$$\begin{aligned} (F_{z_1})_{z_2} &= \Big((\tfrac{\alpha}{2} - \beta\,X\cdot\tilde{Y})_{z_2}\,,\, i\,(\tfrac{\alpha}{2} + \beta\,X\cdot\tilde{Y})_{z_2}\,,\,(\alpha\,X + \beta\,\tilde{Y})_{z_2} \Big) \\ &\,\| \\ (F_{z_2})_{z_1} &= \Big((\tfrac{\gamma}{2} - \delta\,X\cdot\tilde{Y})_{z_1}\,,\, i\,(\tfrac{\gamma}{2} + \delta\,X\cdot\tilde{Y})_{z_1}\,,\,(\gamma\,X + \delta\,\tilde{Y})_{z_1} \Big) \,. \end{aligned}$$

Dividing the second components on each side by i and then adding or subtracting them from the first ones, respectively, gives

$$\alpha_{z_2} = \gamma_{z_1} \quad \text{and} \quad (\beta\,(X\cdot\tilde{Y}))_{z_2} = (\delta\,(X\cdot\tilde{Y}))_{z_1} \,.$$

Since we chose V to be simply connected, the complex version of Poincaré's Lemma (see e.g. [W], page 49) tells us that we can find two holomorphic functions $u\,,\,v : V \to \mathbf{C}$ such that

$$du = \alpha\,dz_1 + \gamma\,dz_2 \quad \text{and} \quad dv = (\beta\,(X\cdot\tilde{Y}))\,dz_1 + (\delta\,(X\cdot\tilde{Y}))\,dz_2 \,.$$

This means, in particular, that

$$\alpha = \frac{\partial\,u}{\partial\,z_1}\,,\ \gamma = \frac{\partial\,u}{\partial\,z_2}\,,\ \beta\,(X\cdot\tilde{Y}) = \frac{\partial\,v}{\partial\,z_1}\,,\ \text{ and }\ \delta\,(X\cdot\tilde{Y}) = \frac{\partial\,v}{\partial\,z_2}\,.$$

Thus, we obtain that

$$\det\begin{pmatrix} \frac{\partial u}{\partial z_1} & \frac{\partial u}{\partial z_2} \\ \frac{\partial v}{\partial z_1} & \frac{\partial v}{\partial z_2} \end{pmatrix} = \det\begin{pmatrix} \alpha & \gamma \\ \beta\,(X\cdot\tilde{Y}) & \delta\,(X\cdot\tilde{Y}) \end{pmatrix} = (X\cdot\tilde{Y})\,\det\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}^{t} \neq 0\,,$$

by (3), (7), and (11). This means that in some neighborhood $U \subset V$ of p in M, the map $(u\,,\,v)$ is a *local biholomorphism* from U onto an open set in $\mathbf{C}^2$. Thus, $(u\,,\,v)$ is a complex chart of M on $U \ni p$. Changing to these new coordinates u and v, we find

$$F_{z_1} = \frac{\partial\,u}{\partial\,z_1}\,F_u + \frac{\partial\,v}{\partial\,z_1}\,F_v = \alpha\,F_u + \beta\,(X\cdot\tilde{Y})\,F_v\ ,$$

and

$$F_{z_2} = \frac{\partial\,u}{\partial\,z_2}\,F_u + \frac{\partial\,v}{\partial\,z_2}\,F_v = \gamma\,F_u + \delta\,(X\cdot\tilde{Y})\,F_v\ .$$

By (11), this means that if we express F_u and $(X\cdot\tilde{Y})\,F_v$ on one hand, or $\mathbf{X}$ and $\mathbf{Y}$ on the other hand, in components of the basis F_{z_1} and F_{z_2} of $\mathrm{im}F_*$, then the corresponding components will be the same, namely the entries of the inverse of the matrix in (11). Thus, we have that

$$F_u = \mathbf{X} \quad\text{and}\quad F_v = \tfrac{1}{X\cdot\tilde{Y}}\,\mathbf{Y}\ .$$

Together with (10), this means that we have proved (8), by setting

$$Y := \tfrac{1}{X\cdot\tilde{Y}}\,\tilde{Y} = -\tfrac{2}{\lambda\,(\xi-\zeta)^2}\left(\tfrac{1-\zeta^2}{2}\,,\,i\,\tfrac{1+\zeta^2}{2}\,,\,\zeta\right) = \mu\left(\tfrac{1-\zeta^2}{2}\,,\,i\,\tfrac{1+\zeta^2}{2}\,,\,\zeta\right),$$

where we used the expression for $\tilde{Y}$ according to (2) (recall that we used “$\tilde{Y}$” here in place of “Y” in (2)). This completes the proof of Proposition 3.2.

Remark: Note that in the $(u\,,\,v)$–coordinates, Proposition 3.2 gives

$$F(u\,,\,v) = \left(\tfrac{u}{2} - v + C_1\,,\, i\,(\tfrac{u}{2} + v + C_2)\,,\, F_3(u\,,\,v)\,,\ldots,\, F_N(u\,,\,v)\right)$$

for some constants C_1 and C_2. Thus, after a translation in the first two coordinates, Proposition 3.2 essentially describes F locally as the *graph* of a map over the fixed “parameter plane”

$$\mathrm{span}\{\,\tfrac{u}{2}\,(1\,,\,i\,,\,0\,,\ldots,\,0) + v\,(-1\,,\,i\,,\,0\,,\ldots,\,0)\mid u\,,\,v \in \mathbf{C}\} \subset \mathbf{C}^N\,.$$

The next step is to give a procedure for how one can utilize Proposition 3.2 to construct local examples of minimal real Kähler surfaces. Since the maps $(\,\frac{1}{2}\,,\,\frac{i}{2}\,,\,X)$ and $(-1\,,\,i\,,\,Y)$ as in (8) always span an isotropic plane in $\mathbf{C}^2$, by Theorem 1.1 one only needs to find suitable holomorphic maps $\xi\,,\ \zeta\,,$ and λ as in (8) such that, in addition, it is guaranteed that $(\,\frac{1}{2}\,,\,\frac{i}{2}\,,\,X)$ and $(-1\,,\,i\,,\,Y)$ form the "gradient" of a map F with respect to some complex chart $(u\,,\,v)$. By the complex Poincaré Lemma, this is obviously equivalent to requiring that, with respect to these coordinates, we have

$$X_v = Y_u\ . \tag{12}$$

Let us write this relation in terms of $\xi\,,\ \zeta\,,$ and $\lambda\,$. Since by (8)

$$X = \lambda\left(\tfrac{1-\xi^2}{2},\, i\,\tfrac{1+\xi^2}{2},\, \xi\right) \quad \text{and} \quad Y = \mu\left(\tfrac{1-\zeta^2}{2},\, i\,\tfrac{1+\zeta^2}{2},\, \zeta\right),$$

the integrability condition (12) can be rewritten as

$$\left((\lambda\,\tfrac{1-\xi^2}{2})_v\,,\, i\,(\lambda\,\tfrac{1+\xi^2}{2})_v\,,\, (\lambda\,\xi)_v\right) = \left((\mu\,\tfrac{1-\zeta^2}{2})_u\,,\, i\,(\mu\,\tfrac{1+\zeta^2}{2})_u\,,\, (\mu\,\zeta)_u\right)\ .$$

By comparing components, we obtain

$$\begin{aligned} \lambda_v - (\lambda\,\xi^2)_v &= \mu_u - (\mu\,\zeta^2)_u\ , \\ \lambda_v + (\lambda\,\xi^2)_v &= \mu_u + (\mu\,\zeta^2)_u\ , \\ \text{and}\quad (\lambda\,\xi)_v &= (\mu\,\zeta)_u\ . \end{aligned}$$

By adding or subtracting the first two equations, respectively, we derive the following three equations, which are equivalent to the **integrability conditions** for F:

$$\left.\begin{aligned} (\lambda\,\xi^2)_v &= (\mu\,\zeta^2)_u\ , \\ \lambda_v &= \mu_u\ , \\ \text{and}\ \ (\lambda\,\xi)_v &= (\mu\,\zeta)_u\ . \end{aligned}\right\} \tag{13}$$

Summing up the results we have obtained so far, we have the following

Proposition 3.3: *Let $f : M^4 \to \mathbf{R}^N$ be a minimal isometric immersion from a (real) 4–dimensional Kähler manifold M into $\mathbf{R}^N$ (where $N \geq 5$). Then every point in M has a neighborhood U with a complex chart $(u\,,\,v)$ defined on U such that the holomorphic representative of f on U is determined by* (8)*, where the holomorphic maps X and Y satisfy $X_v = Y_u\,$, or equivalently, where $\xi\,,\,\zeta\,,\,\lambda\,,$ and μ as defined in* (8) *satisfy* (13)*.*

Conversely, if the holomorphic maps $\xi\,,\zeta : U \to \mathbf{C}^{N-4}$ *and* $\lambda : U \to \mathbf{C}$ *are defined on a simply connected open subset* U *of* $\mathbf{C}^2$*, and if they satisfy* (7) *and* (13) *on all of* U *(with* μ *defined as in* (8)*), then the* $\mathbf{C}^N$*–valued* 1*-form*

$$\omega := \begin{pmatrix} \frac{1}{2} \\ \frac{i}{2} \\ X \end{pmatrix} du + \begin{pmatrix} -1 \\ i \\ Y \end{pmatrix} dv$$

with X *and* Y *as in* (8) *is exact on* U*, and if* $F : U \to \mathbf{C}^N$ *is a holomorphic map such that* $dF = \omega$ *, then* $f := \sqrt{2}\,\mathrm{Re}(F) : M \to \mathbf{R}^N$ *is a minimal isometric immersion from the Kähler manifold* $M := (\,U\,,\, f^* <\,,>)$ *into* $\mathbf{R}^N$ *.*

We will call the triple of maps $(\,(u\,,\,v)\,,\,X\,,\,Y\,)$ that describes the local representation of a minimal real Kähler surface as given in this proposition a **Weierstrass representation** of the complex surface.

In the next chapter, we will use this local characterization of minimal real Kähler surfaces to give an explicit local construction method for such manifolds in codimension 2, i.e. in $\mathbf{R}^6$. But before we conclude the present chapter, let us first give two lemmata that work in all codimensions. The first one asserts that the "scaling function" λ in the Weierstrass representation is actually uniquely determined, up to a constant multiple, by the maps ξ and ζ . The second lemma shows that one can switch the roles played by the maps X and Y in 3.3 (at least up to a factor 2) by reparametrizing and reflecting the minimal real Kähler surface at a hyperplane.

Lemma 3.4: *Let* $\xi\,,\,\zeta : U \to \mathbf{C}^n$ *and* $\lambda : U \to \mathbf{C}$ *be holomorphic maps, defined on some simply connected open set* $U \subset \mathbf{C}^2$ *; write the complex coordinates in* $\mathbf{C}^2$ *as* $(u\,,\,v)$ *. Assume that, on all of* U*,* $\lambda\,(\xi-\zeta)^2 \neq 0$ *, and thus that the holomorphic function*

$$\mu := -\frac{2}{\lambda\,(\xi-\zeta)^2}$$

is well defined on all of U*. Furthermore, assume that these maps satisfy the following partial differential equations:*

$$\begin{aligned} (\lambda\,\xi)_v &= (\mu\,\zeta)_u\ , && (13.a)\\ \lambda_v &= \mu_u\ , && (13.b)\\ \textit{and}\ \ (\lambda\,\xi^2)_v &= (\mu\,\zeta^2)_u\ . && (13.c)\end{aligned}$$

Then we have that, on all of U (which is simply connected!),

$$\begin{aligned} d(\log\,\lambda) \quad &= -\frac{2}{(\xi-\zeta)^2}\,(\xi-\zeta)\cdot d\xi\\ &\left(= -\frac{2}{\sum_{j=1}^{n}(\xi_j-\zeta_j)^2}\ \ \sum_{j=1}^{n}(\xi_j-\zeta_j)\,d\xi_j\right)\ .\end{aligned}$$

Proof: Expanding (13.a) and using (13.b) gives

$$\lambda_v\,\xi + \lambda\,\xi_v = \mu_u\,\zeta + \mu\,\zeta_u = \lambda_v\,\zeta + \mu\,\zeta_u\ ,$$

which is equivalent to

$$\lambda_v\,(\xi-\zeta) = \mu\,\zeta_u - \lambda\,\xi_v\ .$$

Taking the symmetric inner product of the last equation with $(\xi+\zeta)$ and $(\xi-\zeta)$, respectively, gives

$$\lambda_v\,(\xi^2-\zeta^2) = \mu\,\xi\cdot\zeta_u + \mu\,\zeta\cdot\zeta_u - \lambda\,\xi\cdot\xi_v - \lambda\,\zeta\cdot\xi_v \qquad (14)$$

and

$$\lambda_v\,(\xi-\zeta)^2 = \mu\,(\xi-\zeta)\cdot\zeta_u - \lambda\,(\xi-\zeta)\cdot\xi_v\ . \qquad (15)$$

Expanding (13.c) and using (13.b) gives

$$\lambda_v\,\xi^2 + \lambda\,(2\,\xi\cdot\xi_v) = \lambda_v\,\zeta^2 + \mu\,(2\,\zeta\cdot\zeta_u)\ ,$$

which is equivalent to

$$\lambda_v\,(\xi^2-\zeta^2) = 2\,\mu\,\zeta\cdot\zeta_u - 2\,\lambda\,\xi\cdot\xi_v\ .$$

Subtracting the last equation from (14) and reordering terms results in

$$\mu\,(\xi-\zeta)\cdot\zeta_u = -\lambda\,(\xi-\zeta)\cdot\xi_v\ . \qquad (16)$$

And using this last equation in (15) and dividing by $(\xi-\zeta)^2$ (which is never 0 by hypothesis) gives

$$\lambda_v = -\frac{2\,\lambda}{(\xi-\zeta)^2}\,(\xi-\zeta)\cdot\xi_v\ . \tag{17}$$

Now, by defintion of μ, we have that $\lambda\,\mu\,(\xi-\zeta)^2=-2$, so that differentiating this equation with respect to u results in

$$0=\lambda_u\,\mu\,(\xi-\zeta)^2+\lambda\,\mu_u\,(\xi-\zeta)^2+2\,\lambda\,\mu\,(\xi-\zeta)\cdot(\xi_u-\zeta_u)\ .$$

Utilizing (13.b) again to replace μ_u by λ_v, using formula (17) for λ_v, and replacing $\mu\,(\xi-\zeta)\cdot\zeta_u$ by the right–hand side of (16), the last equation gives us that

$$0=\lambda_u\,\mu\,(\xi-\zeta)^2-\lambda\,\frac{2\,\lambda}{(\xi-\zeta)^2}\,((\xi-\zeta)\cdot\xi_v)\,(\xi-\zeta)^2+2\,\lambda\,\mu\,(\xi-\zeta)\cdot\xi_u+2\,\lambda^2\,(\xi-\zeta)\cdot\xi_v\ .$$

As one sees, the second and fourth term on the right hand side cancel each other, so that after reorganizing and dividing by $\mu\,(\xi-\zeta)^2$ (which by hypothesis is never zero) we obtain

$$\lambda_u = -\frac{2\,\lambda}{(\xi-\zeta)^2}\,(\xi-\zeta)\cdot\xi_u\ .$$

Since λ is never zero and U simply connected, we have that

$$(\log\lambda)_u=\frac{\lambda_u}{\lambda}\quad\text{and}\quad(\log\lambda)_v=\frac{\lambda_v}{\lambda}\ ,$$

which together with (17) and the above equation for λ_u proves Lemma 3.4.

Lemma 3.5: *Let $f: M^4\to\mathbf{R}^N$ be a minimal real Kähler immersion, and let $((u\,,\,v)\,,\,X\,,\,Y)$ be a Weierstrass representation of f that is defined on some open subset U of M. Furthermore, let $A:\mathbf{R}^N\to\mathbf{R}^N$ denote the reflection at the hyperplane given by all but the first coordinate in $\mathbf{R}^N$; i.e. the matrix of A is*

$$\begin{pmatrix}-1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1\end{pmatrix}\in O(N)\ .$$

Let $\tilde{f}$ be the to f congruent isometric immersion $\tilde{f} := A \circ f : M \to \mathbf{R}^N$. Then the functions $\tilde{u} := 2\,v$ and $\tilde{v} := \frac{1}{2}\,u$ form a complex coordinate system on U (as an open submanifold of M), and the Weierstrass representation of $\tilde{f}$ with respect to the chart $(\tilde{u}\,,\,\tilde{v})$ on U is given by the maps

$$\tilde{X} = \tfrac{1}{2}\,Y \quad \textit{and} \quad \tilde{Y} = 2\,X \;;$$

i.e., $\tilde{X}$ and $\tilde{Y}$ can be written in terms of suitably chosen holomorphic maps $\tilde{\xi}\,,\,\tilde{\zeta} : U \to \mathbf{C}^{N-4}$ and $\tilde{\lambda} : U \to \mathbf{C}$ as in (8).

This is the precise sense in which we may "switch X and Y in the Weierstrass representation of a minimal real Kähler surface", if we wish.

Proof: Let F be the holomorphic representative of f on U that is given by $((u\,,\,v)\,,\,X\,,\,Y)$. Then by Lemma 1.3, $\tilde{F} := A \circ F$ is the holomorphic representative of $\tilde{f}$ on U. Furthermore, we have by (8) that

$$\begin{pmatrix} \frac{1}{2} \\ \frac{i}{2} \\ \tilde{X} \end{pmatrix} = \tilde{F}_{\tilde{u}} = A \circ \frac{\partial F}{\partial \tilde{u}} = A\Big(\underbrace{\frac{\partial u}{\partial \tilde{u}}}_{=0}\, \frac{\partial F}{\partial u} + \underbrace{\frac{\partial v}{\partial \tilde{u}}}_{=1/2}\, \frac{\partial F}{\partial v} \Big) = \frac{1}{2}\,A \circ F_v$$

$$= \frac{1}{2} \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \begin{pmatrix} -1 \\ i \\ Y \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{i}{2} \\ \frac{1}{2}\,Y \end{pmatrix} .$$

Comparing components, we see that we have

$$\tilde{X} = \frac{1}{2}\,Y \;. \tag{18}$$

Similarly, we obtain that

$$\begin{pmatrix} -1 \\ i \\ \tilde{Y} \end{pmatrix} = \tilde{F}_{\tilde{v}} = A \circ \frac{\partial F}{\partial \tilde{v}} = 2\,A \circ F_u = \begin{pmatrix} -1 \\ i \\ 2\,X \end{pmatrix} ,$$

and thus

$$\tilde{Y} = 2\,X \;. \tag{19}$$

It remains to show that we can indeed express $\tilde{X}$ and $\tilde{Y}$ in terms of $\tilde{\xi}$, $\tilde{\zeta}$, and $\tilde{\lambda}$ as in (8) to establish that they do represent the Weierstrass representation of $\tilde{F}$ in the complex chart $(\tilde{u}\,,\,\tilde{v})$. To this end, set

$$\tilde{\xi} := \zeta \quad , \quad \tilde{\zeta} := \xi \quad , \quad \text{and} \quad \tilde{\lambda} := \frac{\mu}{2} = -\frac{1}{\lambda\,(\xi-\zeta)^2} \ .$$

With these definitions, we obtain

$$\tilde{\lambda} \begin{pmatrix} \frac{1-\tilde{\xi}^2}{2} \\ i\,\frac{1+\tilde{\xi}^2}{2} \\ \tilde{\xi} \end{pmatrix} = \frac{\mu}{2} \begin{pmatrix} \frac{1-\zeta^2}{2} \\ i\,\frac{1+\zeta^2}{2} \\ \zeta \end{pmatrix} = \frac{1}{2}\,Y = \tilde{X} \ ,$$

where we used (18) in the last step. Next, according to Proposition 3.2, we have to set

$$\tilde{\mu} := -\frac{2}{\tilde{\lambda}\,(\tilde{\xi}-\tilde{\zeta})^2} \ ,$$

which by definition of $\tilde{\xi}$, $\tilde{\zeta}$, and $\tilde{\lambda}$ leads to

$$\tilde{\mu} = +\frac{2\,\lambda\,(\xi-\zeta)^2}{(\zeta-\xi)^2} = 2\,\lambda \ .$$

Using this last equation and (19), we obtain, in a similar fashion as above, that

$$\tilde{\mu} \begin{pmatrix} \frac{1-\tilde{\zeta}^2}{2} \\ i\,\frac{1+\tilde{\zeta}^2}{2} \\ \tilde{\zeta} \end{pmatrix} = 2\,\lambda \begin{pmatrix} \frac{1-\xi^2}{2} \\ i\,\frac{1+\xi^2}{2} \\ \xi \end{pmatrix} = 2\,X = \tilde{Y} \ .$$

This finishes the proof of Lemma 3.5.