2 Minimal real Kähler hypersurfaces

Locally, any real Kähler hypersurface in $\mathbf{R}^{2n+1}$ which is nowhere flat can be described through its *Gauss parametrization* $\psi : \Lambda \to \mathbf{R}^{2n+1}$, given by

$$\psi(z\,,\,w) = (h \cdot g + \nabla^V h)(z) + w\;, \tag{1}$$

where Λ is the normal bundle along a pseudoholomorphic isometric immersion $g : V^2 \to S^{2n}$ from a surface V into the $2n$–sphere, $h : V \to \mathbf{R}$ some support function on V, and $\nabla^V h$ its gradient in V (see [D-G$_1$], formula (2.4) and Theorem 2.5). The reason that we can find such a parametrization is that for every such hypersurface which is nowhere flat, the index of relative nullity must be constant and equal to $2n-2$ (as we mentioned on page 6), or – which is the same – that its Gauss map must have constant rank 2. In fact, V is nothing but the "Gauss image" of our hypersurface. If the hypersurface is to be *minimal*, we must require that, in addition, h is an eigenfunction of the Laplacian of V for the eigenvalue -2 ([D-G$_1$], Corollary 2.6):

$$\psi \text{ minimal } \iff \triangle^V h + 2\,h = 0 \text{ in } V.$$

Of course, it is neither very easy to find pseudoholomorphic surfaces in S^{2n} – which were first studied by Calabi [C] – nor to determine the eigenfunctions of the Laplacian of V. So the question arose if one could use techniques analogous to those in [D-G$_2$] to find examples of minimal real Kähler hypersurfaces via the Weierstrass representation, similar to the classical case of minimal surfaces in $\mathbf{R}^3$. In fact, this is easily possible by only slightly modifying the methods in [D-G$_2$], as we will now show.

Let $f : M^{2n} \to \mathbf{R}^{2n+1}$ be a minimal isometric immersion of a Kähler manifold M into $\mathbf{R}^{2n+1}$, which is also assumed to be nowhere flat. As mentioned above, the relative nullity bundle $\triangle$ of f must have rank $2n-2$ everywhere. Since this bundle is the kernel of the Jacobian of the (for us) holomorphic complex Gauss map of f (see page 9), it is in fact a holomorphic subbundle of the tangent bundle TM of M (compare [D-G$_2$], page 240). As mentioned on page 6, each leaf of $\triangle$ in M is totally geodesic, and furthermore, f maps any such leaf into an affine subspace of $\mathbf{R}^{2n+1}$.

Now, let $F : M \to \mathbf{C}^{2n+1}$ be a holomorphic representative of f as in formula (3) in Chapter 1 (so we either assume M to be simply connected, or we restrict ourselves to local considerations). The fact that f maps leaves of $\triangle$ into affine subspaces of $\mathbf{R}^{2n+1}$ means that F will map these leaves into complex $(n-1)$–dimensional subspaces of $\mathbf{C}^{2n+1}$ and will, thus, be *holomorphically ruled*. This implies that locally we can find complex coordinates $(z\,,\,w_1\,,\ldots,\,w_{n-1})$ on some open subset $U\times W\subset\mathbf{C}\times\mathbf{C}^{n-1}$ such that

$$\frac{\partial^2 F}{\partial w_j\,\partial w_k}=0\,,\quad 1\le j\,,\,k\le n-1$$

(recall that, by (4), $\frac{\partial F}{\partial z_j}=\sqrt{2}\,\frac{\partial f}{\partial z_j}$ for each complex coordinate system $(z_1\,,\ldots,\,z_n)$ on M). Furthermore, we know from Theorem 1.1 that the Jacobian of F spans an isotropic subspace in $\mathbf{C}^{2n+1}$, i.e.

$$\frac{\partial F}{\partial z_j}\cdot\frac{\partial F}{\partial z_k}=0\,,\quad 1\le j\,,\,k\le n\,,$$

for each complex coordinate system on M. As in [D-G$_2$], we see that this means that the Jacobian of F must be a $(2n+1)\times n$ matrix of rank n that has the form

$$F_*=\left(\beta(z)+\sum_{j=1}^{n-1}w_j\,\gamma_j'(z)\,,\,\gamma_1(z)\,,\ldots,\,\gamma_{n-1}(z)\right)\,,\tag{2}$$

where $\gamma'(z)=\frac{\partial\gamma}{\partial z}$, and where $\beta\,,\,\gamma_j:U\to\mathbf{C}^{2n+1}$ are holomorphic in z and satisfy the additional constraint that

$$\text{span}\,\{\beta\,,\,\gamma_1\,,\,\gamma_1'\,,\ldots,\,\gamma_{n-1}\,,\,\gamma_{n-1}'\}\ \text{ is isotropic}\,.\tag{3}$$

Now, as noted on page 10, the largest possible dimension of an isotropic subspace in $\mathbf{C}^{2n+1}$ is n, and by (2) and (3) the holomorphic subbundle

$$E:=\text{span}\,\{\gamma_1\,,\ldots,\,\gamma_{n-1}\}\subset U\times\mathbf{C}^{2n+1}$$

is isotropic and already has rank $n-1$. This means that its osculating bundle $E':=\text{span}\{\gamma_j\,,\,\gamma_j'\,|\,1\le j\le n-1\}$, which by (3) is also isotropic, can only have rank $\le n$. But then Lemma 1 in [D-G$_2$] (page 239) tells us that either E contains a parallel subbundle (which means that f is reducible), or that there is a unique holomorphic line bundle $L\subset E$ such that $L^{(n-2)}$, the $(n-2)^{nd}$ osculating bundle of L, equals E. More explicitly, if in the latter case $\gamma:U\to\mathbf{C}^{2n+1}$ is a nowhere zero section in L, then

$$E=L^{(n-2)}=\text{span}\{\gamma(z)\,,\,\gamma'(z)\,,\ldots,\,\gamma^{(n-2)}\}\,.\tag{4}$$

From now on we will assume that f is *irreducible*, so in particular we have (4). Note that, in this case, we can conclude further that rank $E' = n$, and we have

$$E' = L^{(n-1)} = \operatorname{span}\{\gamma(z)\,,\, \gamma'(z)\,, \ldots,\, \gamma^{(n-1)}\}\ .$$

Thus, (3) gives us that we always have $\beta \in E'$; i.e. we can always find holomorphic functions $b_0\,, \ldots,\, b_{n-1} : U \to \mathbf{C}$ such that

$$\beta(z) = \sum_{j=0}^{n-1} b_j(z)\; \gamma^{(j)}(z)\ , \tag{5}$$

where b_{n-1} is never 0. (Note: By [D-G$_2$] (9), this shows that an irreducible f can never be *completely* ruled, so in particular that *an irreducible, nowhere flat minimal real Kähler hypersurface can never be complete*. In fact, this is also true for *non-minimal* Kähler hypersurfaces, and was first proved by Abe [A]; see also Corollary 2.7 in [D-G$_1$]). With this β, (2) now becomes

$$F_* = \left(\sum_{j=0}^{n-1} b_j(z)\; \gamma^{(j)}(z) + \sum_{j=1}^{n-1} w_j\; \gamma^{(j)}(z)\,,\, \gamma(z)\,,\, \gamma'(z)\,, \ldots,\, \gamma^{(n-2)}(z) \right)\ ,$$

and we have that

$$F(z\,,\, w_1\,, \ldots,\, w_{n-1}) = \sum_{j=0}^{n-1} \Big(\int b_j(z)\; \gamma^{(j)}(z)\, dz \Big)\ +\ \sum_{j=1}^{n-1} w_j\; \gamma^{(j-1)}(z)\ , \tag{6}$$

and $f(z\,,\, w_1\,, \ldots,\, w_{n-1}) = \sqrt{2}\ \operatorname{Re}(F)$. Writing $w_j = u_j + i\, v_j$, we thus obtain

$$\left.\begin{aligned} &\tfrac{1}{\sqrt{2}}\, f(z\,,\, u_1\,, \ldots,\, u_{n-1}\,,\, v_1\,, \ldots,\, v_{n-1}) \\ &= \operatorname{Re}\Big(\sum_{j=0}^{n-1} \int\, b_j(z)\, \gamma^{(j)}(z)\, dz \Big)\ +\ \sum_{j=1}^{n-1} \Big(u_j \operatorname{Re}\, \gamma^{(j-1)}(z) - v_j \operatorname{Im}\, \gamma^{(j-1)}(z) \Big)\ . \end{aligned}\right\} \tag{7}$$

This is a (local) **Weierstrass representation** of the minimal real Kähler hypersurface f.

Comparing this parametrization of f with the Gauss parametrization in (1), we see that, in these coordinates, the first term in (7) corresponds to $(h \cdot g + \nabla h)(z)$, and the second term to a normal vector to g in S^{2n}. Thus, if we have a convenient way to find such Weierstrass representations – i.e. maps γ as in (4) – we can also easily find (local) examples for pseudoholomorphic maps $g : U \longrightarrow S^{2n}$: they are simply the normal vectors to $\operatorname{span}\{\,\mathrm{Re}\,\gamma^{(j)}(z)\,,\,\mathrm{Im}\,\gamma^{(j)}(z)\,|\,0 \leq j \leq n-1\}$ in $\mathbf{R}^{2n+1}$, viewed as a function of $z = x + i\,y \cong (x\,,\,y)$, which represent *isothermal* coordinates on U (compare page 1).

Now it is in fact rather easy to find such a γ. We only have to mimic the construction on page 237 in [D-G$_2$]. Thus, let U be any simply connected domain in $\mathbf{C}$. Start with any *non–zero* holomorphic function[1] $\alpha_0 : U \to \mathbf{C}$, and let $\phi_0 := \int \alpha_0(z)\,dz$ (or just start with any non-constant holomorphic ϕ_0). Assuming that the maps $\alpha_r\,,\,\phi_r : U \to \mathbf{C}^{2r+1}$ have been defined for some $0 \leq r \leq n-1$, choose any *nowhere zero* function $\mu_{r+1} : U \to \mathbf{C}$, and set

$$\alpha_{r+1} := \mu_{r+1} \begin{pmatrix} \dfrac{1-\phi_r^2}{2} \\ i\,\dfrac{1+\phi_r^2}{2} \\ \phi_r \end{pmatrix}, \text{ and } \quad \phi_{r+1} := \int \alpha_{r+1}(z)\,dz\ , \tag{8}$$

where $\phi_r^2 = \phi_r \cdot \phi_r$ with respect to the standard symmetric inner product in $\mathbf{C}^{2r+1}$. Then,

$$\gamma := \alpha_n$$

is the section of L for which we are looking; i.e. if we use it in (7) above, then the so defined f will be a minimal isometric immersion from the Kähler manifold $M := (U \times W,\, f^*<.\,,.>)$ into $\mathbf{R}^{2n+1}$, where W is some open subset of the origin in $\mathbf{C}^{n-1}$ and $<.\,,.>$ is the standard Euclidean metric in $\mathbf{R}^{2n+1}$.

The proof that this method works is *exactly* the same as for the case of codimension 2 described in [D-G$_2$]. In fact, it is not hard to show that *every* minimal real Kähler hypersurface *must locally* be of this form (up to renumbering the coordinates in $\mathbf{R}^{2n+1}$).

Note that the first step in (8) is *exactly* the classical way to find minimal surfaces in $\mathbf{R}^3$ via their Weierstrass representation (compage page 2). Thus, if we simply "continue this construction to higher dimensions" (as given in (8)), what we obtain are exactly minimal real Kähler hypersurfaces.

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[1] The only difference to the procedure in [D-G$_2$] is that there, α_0 is a function with values in $\mathbf{C}^2$. Also, note that if $\alpha_0 \equiv 0$, we could choose all integration constants in (8) to be zero, and the hypersurface would be part of $\mathbf{R}^{2n} \times \{0\}$, and thus flat (and reducible).

Let us now consider two examples, both in the simplest possible case of a minimal real Kähler hypersurface in $\mathbf{R}^5$, so that $n = 2$.

Example 2.1: Start with $\alpha_0 := 1$, and thus $\phi_0 = z$, if we set the integration constant equal to zero. Now choose $\mu_1 := 6$, and obtain

$$\alpha_1 = \frac{\mu_1}{2}\begin{pmatrix} 1-\phi_0^2 \\ i\,(1+\phi_0^2) \\ 2\,\phi_0 \end{pmatrix} = \begin{pmatrix} 3-3\,z^2 \\ 3\,i\,(1+z^2) \\ 6\,z \end{pmatrix} ,$$

and by integrating (and setting all integration constants equal to zero),

$$\phi_1 = \int \alpha_1(z)\,dz = \begin{pmatrix} 3\,z - z^3 \\ i\,(3\,z+z^3) \\ 3\,z^2 \end{pmatrix} .$$

For the next step, we need to calculate

$$\phi_1^2 = (3\,z-z^3)^2 + i^2(3\,z+z^3)^2 + (3\,z^2)^2 = -3\,z^4 .$$

Setting $\mu_2 := 2$, we obtain (since here $n = 2$)

$$\gamma = \alpha_2 = \frac{\mu_2}{2}\begin{pmatrix} 1-\phi_1^2 \\ i\,(1+\phi_1^2) \\ 2\,\phi_1 \end{pmatrix} = \begin{pmatrix} 1+3\,z^4 \\ i\,(1-3\,z^4) \\ 6\,z-2\,z^3 \\ i\,(6\,z+2\,z^3) \\ 6\,z^2 \end{pmatrix} .$$

If in (6) and (7) we now choose $b_0 := 0$ and $b_1 := 1$, we see that

$$f(z,\,w_1) = \sqrt{2}\;\operatorname{Re}\Big(\int 1\cdot\gamma'(z)\,dz + w_1\cdot\gamma(z)\Big) = \sqrt{2}\;\operatorname{Re}((1+w_1)\cdot\gamma(z)) \quad (9)$$

(setting yet another integration constant equal to zero), and if we write $z = x + i\,y$ and $w_1 = w = u + i\,v$, we finally obtain that

$$f(x\,,\,y\,,\,u\,,\,v) = \sqrt{2}\,(1+u)\,\operatorname{Re}\,\gamma(x+i\,y) - \sqrt{2}\,v\;\operatorname{Im}\,\gamma(x+i\,y)$$

$$= \sqrt{2}(1+u)\cdot\begin{pmatrix} 1+3\,x^4-18\,x^2\,y^2+3\,y^4 \\ 12\,x^3\,y-12\,x\,y^3 \\ 6\,x-2\,x^3+6\,x\,y^2 \\ -6\,y-6\,x^2\,y+2\,y^3 \\ 6\,x^2-6\,y^2 \end{pmatrix} - \sqrt{2}\,v\cdot\begin{pmatrix} 12\,x^3\,y-12\,x\,y^3 \\ 1-3\,x^4+18\,x^2\,y^2-3\,y^4 \\ 6\,y-6\,x^2\,y+2\,y^3 \\ 6\,x+2\,x^3-6\,x\,y^2 \\ 12\,x\,y \end{pmatrix}$$

is a minimal real Kähler hypersurface in $\mathbf{R}^5$, defined for all $(x\,,\,y\,,\,u\,,\,v)$ in some neighborhood $U\times W$ of the origin in $\mathbf{R}^4$ $(U\,,\,W$ open in $\mathbf{R}^2)$.

We can also use this example for f to obtain an example for a pseudoholomorhic surface $g: U \to S^4$ in the 4–sphere.[2] Note that since here

$$f_*T_{(z\,,0)}U = \ \mathrm{span}\{\mathrm{Re}\gamma(z)\,,\,\mathrm{Im}\gamma(z)\,,\,\mathrm{Re}\gamma'(z)\,,\,\mathrm{Im}\gamma'(z)\,\}\ ,$$

we only need to calculate $\gamma'(z)$ and then determine a normal vector to f at $(z\,,\,0)$. We have

$$\gamma'(z) = 6\begin{pmatrix} 2\,z^3 \\ -2\,i\,z^3 \\ 1-z^2 \\ i\,(1+z^2) \\ 2\,z \end{pmatrix}\ , \quad \text{and thus}$$

$$\tfrac{1}{6}\,\mathrm{Re}\ \gamma'(x,\,y) = \begin{pmatrix} 2\,x^3-6\,x\,y^2 \\ 6\,x^2\,y-2\,y^3 \\ 1-x^2+y^2 \\ -2\,x\,y \\ 2\,x \end{pmatrix}\ , \quad \tfrac{1}{6}\,\mathrm{Im}\ \gamma'(x,\,y) = \begin{pmatrix} 6\,x^2\,y-2\,y^3 \\ -2\,x^3+6\,x\,y^2 \\ -2\,x\,y \\ 1+x^2-y^2 \\ 2\,y \end{pmatrix}\ .$$

To find the required normal vector, one probably wants to use a computer algebra system (or otherwise plenty of time and patience). In any case, one finally arrives at the result

$$g(x\,,\,y) = \frac{1}{a^3+9\,a^2+a+1}\begin{pmatrix} 2\,(x^2-y^2)\,(a+3) \\ 4\,x\,y\,(a+3) \\ 2\,x\,(3\,a^2-1) \\ 2\,y\,(3\,a^2-1) \\ a^3-9\,a^2-a+1 \end{pmatrix}$$

where $a := x^2+y^2 = ||(x\,,\,y)||^2$.

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[2] Actually, for a minimal surface in the 4-sphere, being pseudoholomorphic is equivalent to being *superminimal*. See [Loo], page 8, or [D-G$_1$], page 18.

Remark: Once that we have a candidate for a pseudoholomorphic map $g : U \longrightarrow S^{2n}$, it is not too difficult to check if it is indeed pseudoholomorphic. If $\partial = \frac{1}{2}\,(\frac{\partial}{\partial x} - i\,\frac{\partial}{\partial y})$, where $(x\,,\,y)$ are isothermal coordinates on U, then according to Calabi [C] – or better: [Loo], page 7 – we only need to check that $\partial^j g \cdot \partial^j g = 0$ for $1 \leq j \leq n$. The author did, in fact, use a computer algebra system to successfully double–check our g above for pseudoholomorphicity.

Example 2.2: Here, we start with $\alpha_0 := \frac{1}{2}\,e^{\frac{z}{2}}$, and thus $\phi_0 = e^{\frac{z}{2}}$ and $\phi_0^2 = e^z$ (setting this and all following integration constants equal to zero again). Choosing $\mu_1 := 2$, we find

$$\alpha_1 = \frac{\mu_1}{2}\begin{pmatrix} 1 - \phi_0^2 \\ i\,(1 + \phi_0^2) \\ 2\,\phi_0 \end{pmatrix} = \begin{pmatrix} 1 - e^z \\ i\,(1 + e^z) \\ 2\,e^{\frac{z}{2}} \end{pmatrix} .$$

Integrate to obtain

$$\phi_1 = \int \alpha_1(z)\,dz = \begin{pmatrix} z - e^z \\ i\,(z + e^z) \\ 4\,e^{\frac{z}{2}} \end{pmatrix} ,$$

so that $\phi_1^2 = (z - e^z)^2 + i^2(z + e^z)^2 + (4\,e^{\frac{z}{2}})^2 = 16\,e^z - 4\,z\,e^z$. Setting $\mu_2 := 2$ gives

$$\gamma = \alpha_2 = \frac{\mu_2}{2}\begin{pmatrix} 1 - \phi_1^2 \\ i\,(1 + \phi_1^2) \\ 2\,\phi_1 \end{pmatrix} = \begin{pmatrix} 1 - 16\,e^z + 4\,z\,e^z \\ i\,(1 + 16\,e^z - 4\,z\,e^z) \\ 2\,z - 2\,e^z \\ i\,(2\,z + 2\,e^z) \\ 8\,e^{\frac{z}{2}} \end{pmatrix} .$$

Again, we choose $b_0 := 0$ and $b_1 := 1$, and have the same general form of f as in (9). Finally, we obtain another example for a minimal real Kähler manifold in $\mathbf{R}^5$, namely

$$\begin{aligned} f(x\,,\,y\,,\,u\,,\,v) \;&=\; \sqrt{2}\,(1+u)\;\;\mathrm{Re}\;\gamma(x+i\,y) - \sqrt{2}\;\;v\;\;\mathrm{Im}\;\gamma(x+i\,y) \\ &=\; \sqrt{2}\,(1+u)\cdot\begin{pmatrix} 1+e^{x}\;\cos y\,(4\,x-16)-4\,y\,e^{x}\;\sin y \\ 4\,y\,e^{x}\;\cos y + e^{x}\;\sin y\,(4\,x-16) \\ 2\,x-2\,e^{x}\;\cos y \\ -2\,y-2\,e^{x}\;\sin y \\ 8\,e^{\frac{x}{2}}\cos\left(\frac{y}{2}\right) \end{pmatrix} \\ &\qquad\qquad - \sqrt{2}\;\;v\cdot\begin{pmatrix} 4\,y\,e^{x}\;\cos y + e^{x}\;\sin y\,(4\,x-16) \\ 1-e^{x}\;\cos y\,(4\,x-16)+4\,y\,e^{x}\;\sin y \\ 2\,y-2\,e^{x}\;\sin y \\ 2\,x+2\,e^{x}\;\cos y \\ 8\,e^{\frac{x}{2}}\sin\left(\frac{y}{2}\right) \end{pmatrix}. \end{aligned}$$

As in Example 2.1, this f gives rise to another example of a pseudoholomorhic surface $g : U \longrightarrow S^4$ in the 4–sphere. We have

$$\gamma'(z) = \begin{pmatrix} -12\,e^{z}+4\,z\,e^{z} \\ i\,(12\,e^{z}-4\,z\,e^{z}) \\ 2-2\,e^{z} \\ i\,(2+2\,e^{z}) \\ 4\,e^{\frac{z}{2}} \end{pmatrix} \quad ,$$

and thus

$$\mathrm{Re}\;\gamma'(x,\,y) = \begin{pmatrix} e^{x}\;\cos y\,(4\,x-12)-4\,y\,e^{x}\;\sin y \\ 4\,y\,e^{x}\;\cos y + e^{x}\;\sin y\,(4\,x-12) \\ 2-2\,e^{x}\;\cos y \\ -2\,e^{x}\;\sin y \\ 4\,e^{\frac{x}{2}}\cos\left(\frac{y}{2}\right) \end{pmatrix}$$

and

$$\operatorname{Im}\,\gamma'(x,\,y) = \begin{pmatrix} 4\,y\,e^x\,\cos y + e^x\,\sin y\,(4\,x - 12) \\ -e^x\,\cos y\,(4\,x - 12) + 4\,y\,e^x\,\sin y \\ -2\,e^x\,\sin y \\ 2 + 2\,e^x\,\cos y \\ 4\,e^{\frac{x}{2}}\,\sin\left(\frac{y}{2}\right) \end{pmatrix}.$$

It appears somewhat daunting to try to find the normal vector to the space $\operatorname{span}\{\operatorname{Re}\gamma(z)\,,\,\operatorname{Im}\gamma(z)\,,\,\operatorname{Re}\gamma'(z)\,,\,\operatorname{Im}\gamma'(z)\,\}$, but one can make this task slightly easier (even for a computer algebra system) by doing some linear algebra on this basis, thereby finding the following new basis vectors:

$$\begin{pmatrix} 1 - 4\,e^x\,\cos y \\ -4\,e^x\,\sin y \\ 2\,x - 2 \\ -2\,y \\ 4\,e^{\frac{x}{2}}\,\cos\left(\frac{y}{2}\right) \end{pmatrix}, \begin{pmatrix} -4\,e^x\,\sin y \\ 1 + 4\,e^x\,\cos y \\ 2\,y \\ 2\,x - 2 \\ 4\,e^{\frac{x}{2}}\,\sin\left(\frac{y}{2}\right) \end{pmatrix}, \begin{pmatrix} 2\,x - 6 \\ 2\,y \\ -1 + e^{-x}\,\cos y \\ e^{-x}\,\sin y \\ 2\,e^{-\frac{x}{2}}\,\cos\left(\frac{y}{2}\right) \end{pmatrix}, \begin{pmatrix} 2\,y \\ -2\,x + 6 \\ -\,e^{-x}\,\sin y \\ 1 + e^{-x}\,\cos y \\ -2\,e^{-\frac{x}{2}}\,\sin\left(\frac{y}{2}\right) \end{pmatrix}.$$

Again, we employ a computer algebra system to calculate the normal vector, and find

$$g(x\,,\,y) = \frac{2\,e^{\frac{x}{2}}}{4\,e^{2x} + e^x\,b(x,y) + 1} \begin{pmatrix} 2\,\cos(\frac{y}{2})\,(e^x + x - 2) - 2\,y\,\sin(\frac{y}{2}) \\ 2\,y\,\cos(\frac{y}{2}) + 2\,\sin(\frac{y}{2})\,(e^x + x - 2) \\ \cos(\frac{y}{2})\,(4\,x\,e^x - 8\,e^x - 1) + 4\,y\,e^x\,\sin(\frac{y}{2}) \\ -4\,y\,e^x\,\cos(\frac{y}{2}) + \sin(\frac{y}{2})\,(4\,x\,e^x - 8\,e^x - 1) \\ \frac{1}{2}\,e^{-\frac{x}{2}}\,(4\,e^{2x} - e^x\,b(x,y) + 1) \end{pmatrix},$$

where $b(x,y) := 4\,x^2 + 4\,y^2 - 16\,x + 17$. As in Example 2.1, the author double–checked that $\partial^j g \cdot \partial^j g = 0$ for $j = 1, 2$. Actually, it took a fairly fast computer system several minutes to complete this task.