1 Introduction

The study of minimal surfaces is one of the classical areas of geometry, both for its beauty and its applications to other areas of Mathematics and Physics. Minimal surfaces are those surfaces in Euclidean 3–space that minimize the area compared to all other surfaces with the same boundary curve.

Already since the nineteenth century, it is known that there is an intimate relationship between the structure of these surfaces and complex analysis. Namely, let $f = (f_1\,,\, f_2\,,\, f_3) : U \to \mathbf{R}^3$ be a parametrization of a minimal surface by so–called *isothermal coordinates* $(x\,,\, y) \in U$, defined in some region U in the plane; this means that the partial derivative vectors of f with respect to these coordinates x and y have the same length and are orthogonal at every point of the surface. (For the existence of such coordinates and proofs of the following results, see e.g. [S], p. 387–397, or any classical treatment of minimal surfaces, as in [O]). Then the component functions f_j are *harmonic functions* with respect to these coordinates, i.e. they satisfy the Laplace equation

$$0 = \triangle f_j = \frac{\partial^2 f_j}{\partial x^2} + \frac{\partial^2 f_j}{\partial y^2}\ , \quad \text{for } j = 1\,,\, 2\,,\, 3\,.$$

But by complex analysis, this means that the f_j are locally the real parts of *holomorphic* functions $F_j : V \to \mathbf{C}$, where V is some subset of U, viewed as a region in the complex plane $\mathbf{C}$ with its complex coordinate $z = x + i\,y$. Thus, for every minimal surface in Euclidean 3–space, we can (locally) find a holomorphic map $F : V \to \mathbf{C}^3$, the so–called *holomorphic representative* of f in the given isothermal coordinates, such that

$$f = \sqrt{2}\,\mathrm{Re}(F)\ .$$

Here, the factor $\sqrt{2}$ is commonly introduced to make F isometric to f, if both are regarded as immersions from V (with the by f induced metric) into $\mathbf{C}^3 \cong \mathbf{R}^6$, for $\mathbf{R}^3 \cong \mathbf{R}^3 \times \{0\} \subset \mathbf{R}^6$ (compare also page 3).

But we have even more structure. If we take the *complex* derivative of F with respect to z, we can observe that $\varphi := \frac{\partial F}{\partial z} \in \mathbf{C}^3$ is a so–called *isotropic vector* of $\mathbf{C}^3$, which means the following. Taking the standard *symmetric* inner product in $\mathbf{C}^3$, i.e.

$$v \cdot w := \sum_{j=1}^{3} v_j\, w_j \quad \text{for } v\,,\, w \in \mathbf{C}^3$$

(**NO** complex conjugates!), and writing $v^2 := v \cdot v$, we find that

$$\varphi^2 = \frac{\partial F}{\partial z} \cdot \frac{\partial F}{\partial z} = 0 \,.$$

But it is fairly easy to describe (almost) all vectors $X \in \mathbf{C}^3$ that are isotropic in this sense. Namely, if λ is any non–zero complex number, and ξ any other complex number, it is easy to check that

$$\varphi = \lambda \left(\frac{1-\xi^2}{2} \,,\; i\,\frac{1+\xi^2}{2} \,,\; \xi \right) \tag{1}$$

is isotropic. On the other hand, if $\varphi = (\varphi_1 \,, \varphi_2 \,, \varphi_3)$ is isotropic, and if for the first two coordinates we have $\varphi_1 \neq i\,\varphi_2$, then we can always find a $\lambda \neq 0$ and a ξ as above such that φ has the form of (1); namely, simply take $\lambda := \varphi_1 - i\,\varphi_2$ and $\xi := \frac{\varphi_3}{\lambda}$. The map φ as in (1) is called the (local) **complex Gauss map** of our minimal surface with respect to the given isothermal coordinates.

We will show later (Lemma 1.3) that, by slightly rotating the minimal surface, we can always achieve $\varphi_1 \neq i\,\varphi_2$, at least locally. Summarizing, we thus have the following result:

Locally and up to isometry, every minimal surface can be parametrized with respect to isothermal coordinates $(x \,, y)$ *in the following way:*

$$\begin{aligned} f_1(x\,,y) &= \sqrt{2}\ \mathrm{Re} \int \lambda\,\frac{1-\xi^2}{2}\,dz \\ f_2(x\,,y) &= \sqrt{2}\ \mathrm{Re} \int i\,\lambda\,\frac{1+\xi^2}{2}\,dz \\ f_3(x\,,y) &= \sqrt{2}\ \mathrm{Re} \int \lambda\,\xi\,dz \end{aligned}$$

where $z = x + i\,y$*, and* λ *and* ξ *are certain holomorphic functions in* z*.*

Conversely, it is not hard to show that, for any two holomorphic functions ξ *and* λ *(the latter nowhere zero), the above formulas give a minimal surface in Euclidean* 3*–space.*

This parametrization is called a **Weierstrass representation** of the minimal surface. In fact, one does not have to rotate the surface to avoid points z

where we might have $\lambda(z) = 0$, if we allow ξ to be *meromorphic*, with poles precisely at the points where λ has zeroes, and their order being exactly half of the order of the zeroes of λ (see [S], page 395). However, we will later avoid this more general description.

Another phenomenon of minimal surfaces in Euclidean 3–space is that they always allow so–called *associated families*, which are also called "isometric deformations". These are one–parameter families of minimal isometric immersions from some two–dimensional parameter manifold into $\mathbf{R}^3$ that are not congruent to each other, but all have the same (real) Gauss map. Using Weierstrass representations, they are extremely easy to describe. We have that if F is a holomorphic representative of a minimal isometric immersion f with respect to an isothermal coordinate system (as desribed above), then the associated family $\{f_\theta \,|\, \theta \in \mathbf{R}\}$ to $f = f_0$ is given by

$$f_\theta = \sqrt{2}\,\mathrm{Re}(e^{i\,\theta}\,F)\ .$$

The classical example of such an associated family is the isometric deformation of the helicoid into the catenoid. (For a nice picture of this deformation see the June/July 1999 issue of the Notices of the American Mathematical Society, Volume 46, Number 6, page 649.)

The associated family $\{f_\theta \,|\, \theta \in \mathbf{R}\}$ of a minimal isometric immersion $f : U \to \mathbf{R}^3$ also gives a very simple way to express a holomorphic representative F of f without reference to an isothermal coordinate system; namely, we can write

$$F = \frac{1}{\sqrt{2}}\,(f \oplus f_{-\pi/2}) \cong \frac{1}{\sqrt{2}}\,(f + i\,f_{-\pi/2})\ ,$$

where we identify $\mathbf{R}^3 \oplus \mathbf{R}^3 \cong \mathbf{C}^3$, with respect to the standard complex structure $J(u\,,\,v) = (-v\,,\,u)$ on $\mathbf{R}^3 \oplus \mathbf{R}^3$; see e.g. [L], page 143[1].

As mentioned above, associated families of minimal surfaces are the classical counterexamples to the fact that the Gauss map of an isometric immersion $f : M \to \mathbf{R}^N$ (i.e. the map that assigns to each point $p \in M$ the

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[1] There, F is defined to be $\frac{1}{\sqrt{2}}\,(f_{\pi/2} \oplus f) \cong \frac{1}{\sqrt{2}}\,(f_{\pi/2} + i\,f)$, which means that we would have $f = \sqrt{2}\,\mathrm{Im}(F)$. This differs from our holomorphic representative simply by a multiplicative factor of i.

image of its tangent space $f_* T_p M$ in Euclidean space) does, in general, *not* determine its image $f(M)$ up to congruence. In 1985, M. Dajczer and D. Gromoll asked the question if there are other examples of this kind. In [D-G$_1$], they prove that there is, in fact, a wider class of isometric immersions which display this behavior, namely **circular Kähler manifolds**, i.e. isometric immersions $f : M \to Q^N$ from a Kähler manifold M into a space of constant curvature whose second fundamental form α satisfies

$$\alpha(JX\, , \, Y) = \alpha(X\, , \, JY)$$

for all vector fields X and Y on M, J being the complex structure on M. It is easy to see that "f circular" always implies "f minimal", which in general means that the second fundamental form α of f has vanishing trace, i.e.

$$\mathrm{tr}(\alpha) := \sum_{j=1}^{n} \alpha(X_j\, , \, X_j) = 0$$

for every orthonormal basis frame $X_1\, , \ldots , \, X_n$ on M. Dajczer and Gromoll show that circular immersions always allow associated families, defined at least on any simply connected open subset of M. More explicitly, we have that for any fixed point $p_0 \in M$, f_θ is given by the line integral

$$f_\theta(p) = \int_{p_0}^{p} f_* \circ J_\theta \ ,$$

where $J_\theta := \cos\theta\, I + \sin\theta\, J$ (I being the identity tensor on TM; see [D-G$_1$], formula (1.15) on page 17). Note that, in particular, we have that

$$(f_{-\pi/2})_* = -f_* \circ J \ . \tag{2}$$

Then Dajczer and Gromoll are able to prove that essentially all local examples of non–congruent isometric immersions with the same Gauss map are of this kind.

In this dissertation, we will focus on the Euclidean case, i.e. on so–called **minimal real Kähler submanifolds**. These are minimal isometric immersions $f : M \to \mathbf{R}^N$ from a Kähler manifold into an Euclidean space. As M. Dajczer and L. Rodríguez show in [D-R$_1$], for these immersions "circular" and "minimal" mean exactly the same thing, whereas for immersions into spaces of constant, non–zero curvature, "circular" is far more restrictive than "minimal"; namely, M has to be a *surface* for f to be circular in this case (see Proposition 1.8 on page 16 in [D-G$_1$]).

In even codimension, it is particularly easy to find examples for minimal real Kähler submanifolds. As shown in [D_2], page 139, every *holomorphic* isometric immersion $f : M^{2n} \to \mathbf{C}^N$ from a Kähler manifold M (of *complex* dimension n) into a complex vector space $\mathbf{C}^N$ will become a minimal real Kähler submanifold, if we view $\mathbf{C}^N \cong \mathbf{R}^{2N}$ as Euclidean $2N$–space. This means, of course, that f will have even *real* codimension $2(N-n)$. But one can also find minimal real Kähler manifolds in odd codimensions; in fact, in the same article [D-G_1] mentioned above, Dajczer and Gromoll classify all real Kähler *hypersurfaces*, i.e. in *real* codimension one, minimal and non–minimal. For more on real Kähler hypersurfaces, see Chapter 2.

Allowing associated families is not the only phenomenon that minimal real Kähler submanifolds and minimal surfaces in Euclidean 3–space have in common. As with minimal surfaces, minimal real Kähler immersions $f : M \to \mathbf{R}^N$ always have (local) holomorphic representatives $F : U \to \mathbf{C}^N$, where U is a suitable, usually simply connected open subset of M. They can be defined using the associated families $\{f_\theta : U \to \mathbf{R}^N \,|\, \theta \in \mathbf{R}\}$ of f discussed above, namely:

$$F = \frac{1}{\sqrt{2}}\,(f \oplus f_{-\pi/2}) \cong \frac{1}{\sqrt{2}}\,(f + i\, f_{-\pi/2}) \tag{3}$$

(see e.g. [D-G_1], formula (1.17)[2]). Again we identify $\mathbf{R}^N \oplus \mathbf{R}^N \cong \mathbf{C}^N$, with respect to the standard complex structure $J(u\,,\, v) = (-v\,,\, u)$ on $\mathbf{R}^N \oplus \mathbf{R}^N$. And in fact, the analogy goes further. As with minimal surfaces, the complex Gauss map for a minimal real Kähler submanifold will be a *holomorphic, isotropic map*; we will clarify in Theorem 1.1 below exactly what we mean by that. This will, at least in principle, allow us to find "Weierstrass representations" for minimal real Kähler submanifolds in general. In [D-G_2] for instance, Dajczer and Gromoll used such representations to describe the structure of *complete* minimal real Kähler submanifolds in codimension two. (For more on this case see the remarks below, and also Chapter 2.)

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[2] Note: In most articles on these topics, we find that F is taken to be $\frac{1}{\sqrt{2}}\,(f + i\, f_{\pi/2})$. However, the image of the (complex) Jacobian F_* of F would then consist of *anti*–holomorphic vectors, whereas we want to work with holomorphic vectors here. Compare the remarks and the footnote after Theorem 1.1.

In the years after 1985, several articles were published concerning the structure of real Kähler submanifolds, in particular in low codimensions. Many of these results rely on the fact that, unless the submanifold is the image of a holomorphic map as described above (with respect to some complex structure on the Euclidean space), it usually has "plenty of (relative) nullity". Recall that the (relative) nullity space $\triangle_p$ of an isometric immersion $f : M \to \mathbf{R}^N$ at a point $p \in M$ is the degeneracy space of the second fundamental form α of M in the tangent space T_pM at p:

$$\triangle_p = \{ v \in T_pM \,|\, \alpha(v\,, ...) = 0 \} \,.$$

Its dimension is called the *index of (relative) nullity*: $\nu_f(p) = \dim \triangle_p$. This index is locally constant, and on every open subset U of M where it is constant, $\{\triangle_p \,|\, p \in U\}$ is a subbundle of TM. Moreover, it is well–known that on such an open set U, $\triangle$ forms an involutive distribution whose leaves are totally geodesic submanifolds of M, and furthermore that f maps these leaves into affine subspaces of $\mathbf{R}^N$; see e.g. [D$_2$], pages 67 to 70. For real Kähler submanifolds, $\triangle$ is often rather high–dimensional, if the map is not already holomorphic in the above mentioned sense. For example, Takahashi showed in [T] (see also [A]) that for a (not necessarily minimal) hypersurface immersion $f : M^{2n} \to \mathbf{R}^{2n+1}$ of a Kähler manifold M, we must have that $\nu_p(f) \geq 2n-2$ for all $p \in M$, which is as large as possible if f is not flat at p.

Perhaps the strongest result of this kind for codimension two was published by Dajczer in [D$_1$], where he uses the theory of flat bilinear forms developed by Moore (see [M]) to prove that if the nullity of a (not necessarily minimal) isometric immersion $f : M^{2n} \to \mathbf{R}^{2n+2}$ from a Kähler manifold M into Euclidean $(2n+2)$–space is everywhere less than $2n-4$, then f must be holomorphic with respect to some complex structure on $\mathbf{R}^{2n+2}$. This means that if such an immersion is non–holomorphic, then it must locally be an affine vectorbundle of rank at least $2n-4$ over an at most four–dimensional Kähler submanifold of M.

Shortly afterwards, Dajczer and Rodríguez where able to analyze the structure of such isometric immersions with $\nu_p(f) \geq 2n-4$ for all $p \in M$ in codimension two, given that they are *minimal* and that the underlying Kähler manifold M is *complete*; see [D-R$_2$]. The key to their result is that for *complete* M, one can always find one more "complex direction" in which M is ruled. This means that a complete isometric immersion

$f : M^{2n} \rightarrow \mathbf{R}^{2n+2}$ will either stem from a holomorphic map; or that f is a *cylinder*, i.e. $f = f_1 \times \mathrm{id}_{\mathbf{R}^{2n-4}}$, where $f_1 : \tilde{M}^4 \rightarrow \mathbf{R}^6$ is a minimal isometric immersion from a Kähler submanifold $\tilde{M}$ of M into Euclidean 6–space; or that f is *completely complex ruled*, i.e. M is an affine vectorbundle of rank $2n-2$ over a two–dimensional Kähler submanifold $\tilde{M}$ of M, and f maps the rulings into affine subspaces of $\mathbf{R}^{2n+2}$. But then the image of the submanifold $\tilde{M}$ of M under f is nothing but a minimal surface in $\mathbf{R}^{2n+2}$, and as Harvey and Osserman demonstrated in [H-O], most results concerning the structure of minimal surfaces in 3–spaces can be generalized verbatim to those in an arbitrary Euclidean space of dimension larger than 3; see also [L]. Finally, in [D-G$_2$] Dajczer and Gromoll use this idea to give Weierstrass representations for all minimal isometric immersions $f : M^{2n} \rightarrow \mathbf{R}^{2n+2}$ where M is *complete* and f is irreducible (i.e. not a cylinder) and non–holomorphic. Their method allows the explicit construction of examples for such immersions.

However, some questions remain open. For instance, it is not clear how non–holomorphic isometric immersions $f : M^{2n} \rightarrow \mathbf{R}^{2n+2}$ may look *locally* if M is *not complete*. And our nullity condition $\nu_p(f) \geq 2n-4$ tells us nothing if M is *four*–dimensional ($n = 2$), i.e. in the case of a minimal isometric immersion $f : M^4 \rightarrow \mathbf{R}^6$. In fact, to the knowledge of the author, only a few examples in this case were known up to this point; see [F] and [D-G$_3$]. Very recently, Arezzo, Pirola, and Solci were able to give entire series of examples (see [A-P-S]). But a classification of those submanifolds had not been established until this time.

The main goal of this dissertation is exactly to give a *complete local classification of four–dimensional, minimal real Kähler submanifolds in codimension two*, at least away from certain isolated "singularities" in the manifold. This will be established in Chapter 4, where we utilize a parametrization for two–dimensional isotropic subspaces in $\mathbf{C}^N$ that is based on "Weierstrass formulas" very similar to the one in (1). The latter will be developed in Chapter 3. Chapter 2 contains an "addendum" to Dajczer and Gromoll's work [D-G$_2$], namely that the methods developed in this article can be used almost verbatim to explicitly construct minimal Kähler hypersurfaces. One interesting consequence of this is that we will be able to explicitly write down formulas for so–called "superminimal surfaces" in Euclidean spheres, a topic that was first studied by E. Calabi in 1968 (see [C]) and is still an active area of research in algebraic geometry.

The following theorem contains the clarification promised on page 5, and is the backbone of the Weierstrass representation for minimal real Kähler submanifolds. It is well–known in the literature and can be proven in a straightforward (if tedious) fashion, e.g. by expressing all given conditions in a complex chart of the Kähler manifold. In their recent article mentioned above, Arezzo, Pirola, and Solci have given a very elegant proof of this theorem, using differential forms (see [A-P-S]).

Theorem 1.1: *Let* $f : M^{2n} \to \mathbf{R}^N$ *be a minimal isometric immersion from a Kähler manifold* M *into Euclidean* N*–space. Furthermore, let* $(z_1, \ldots, z_n)$ *be a complex chart of* M *on some open (and without loss of generality simply connected) subset* U *of* M*, and define the maps* $\varphi_j : U \to \mathbf{C}^N$ *for* $j = 1, \ldots, n$ *by*

$$\varphi_j := \sqrt{2}\ \frac{\partial f}{\partial z_j} = \frac{1}{\sqrt{2}} \left(\frac{\partial f}{\partial x_j} - i\, \frac{\partial f}{\partial y_j} \right) ,$$

where $z_j = x_j + i\, y_j$*. Then these* φ_j *satisfy the following conditions:*

(a) *For each point* $p \in U$*, the vectors* $\varphi_1(p), \ldots, \varphi_n(p)$ *are linearly independent in* $\mathbf{C}^N$ *;*

(b) φ_j *is holomorphic for* $j = 1, \ldots, n$ *;*

(c) $\operatorname{span}\{\varphi_1, \ldots, \varphi_n\}$ *is an isotropic subspace of* $\mathbf{C}^N$*, i.e.*

$$\varphi_j \cdot \varphi_k = 0 \quad \textit{for all} \ \ 1 \le j\,,\, k \le n\,,$$

where " $\cdot$ *" is the standard* symmetric *inner product in* $\mathbf{C}^N$*:*

$$(v_1, \ldots, v_N) \cdot (w_1, \ldots, w_N) := \sum_{j=1}^{N} v_j\, w_j \ ;$$

(d) $\dfrac{\partial \varphi_j}{\partial z_k} = \dfrac{\partial \varphi_k}{\partial z_j}$ *for all* $1 \le j\,,\, k \le n$ *("Integrability Conditions").*

Furthermore, if $F : U \to \mathbf{C}^N$ *is a holomorphic representative of* f *on* U *as described in* (3)*, then we have*

$$\frac{\partial F}{\partial z_j} = \varphi_j \quad \textit{for all} \ \ j = 1, \ldots, n\,. \tag{4}$$

Conversely, let U be a simply connected open subset of $\mathbf{C}^n$, and $\varphi_1, \ldots, \varphi_n : U \to \mathbf{C}^N$ be maps that satisfy conditions (a) *through* (d) *as above. Then there is a holomorphic map $F : U \to \mathbf{C}^N$ such that* (4) *is satisfied, and if $f : M \to \mathbf{R}^N$ is defined by*

$$f := \sqrt{2}\, \mathrm{Re}(F) \ ,$$

then $M := (U\,,\, f^* <\,,>)$ is a Kähler manifold and f is a minimal isometric immersion from M into Euclidean N–space whose holomorphic representative is F.

The map $\varphi := \mathrm{span}\{\varphi_1, \ldots, \varphi_n\}$ from U into the complex Grassmannian $\mathrm{Gr}_n(\mathbf{C}^N)$ of all complex n–spaces in $\mathbf{C}^N$ is more correctly what one calls the **complex Gauss map**[3] of the Kähler manifold M over U. It is independent of the choice of the complex chart of M, and thus a holomorphic map on all of M. Note that we always have

$$\varphi(p) = F_*\, T_pM = f_*\, T_pM^{(1,0)} \ , \tag{5}$$

since $\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_n}$ is a basis for the subspace of all *holomorphic* vectors in $(T_pM)^{\mathbf{C}} = T_pM \otimes_{\mathbf{R}} \mathbf{C}$. Therefore, (a) means exactly that the holomorphic representative F of our immersion f is itself immersive, regarded as a *holomorphic* map between *complex* manifolds.

To see how the factor $\sqrt{2}$ behaves in these formulas, let us briefly check (4). Using (2) and (3), we find

$$\begin{aligned}
\frac{\partial F}{\partial z_j} &= \frac{1}{\sqrt{2}} \left(\frac{\partial f}{\partial z_j} + i\, \frac{\partial f_{-\pi/2}}{\partial z_j} \right) \\
&= \frac{1}{2\sqrt{2}} \left(\frac{\partial f}{\partial x_j} - i\, \frac{\partial f}{\partial y_j} - i\, f_* \left(J \frac{\partial}{\partial x_j} - i\, J \frac{\partial}{\partial y_j} \right) \right) \\
&= \frac{1}{2\sqrt{2}} \left(\frac{\partial f}{\partial x_j} - i\, \frac{\partial f}{\partial y_j} - i\, \frac{\partial f}{\partial y_j} - i \left(i\, \frac{\partial f}{\partial x_j} \right) \right) \\
&= \frac{1}{\sqrt{2}} \left(\frac{\partial f}{\partial x_j} - i\, \frac{\partial f}{\partial y_j} \right) = \varphi_j \ .
\end{aligned}$$

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[3] Again, usually one looks at the *antiholomorphic* Gauss map $\varphi = \mathrm{span}\{\frac{\partial F}{\partial \bar z_1}, \ldots, \frac{\partial F}{\partial \bar z_n}\} = f_*\, T_pM^{(0,1)}$ here. But since we will later work with this map quite extensively, we will prefer having a *holomorphic* Gauss map.

We will use Theorem 1.1 shortly to give simple examples of how to construct minimal real Kähler submanifolds. However, before we do this, we want to investigate isotropic subspaces in $\mathbf{C}^N$ a little closer.

First, in general we say that a subvectorspace V of $\mathbf{C}^N$ is **isotropic** if we have $V^2 := V \cdot V = \{0\}$, which, of course, means that

$$v \cdot w = \sum_{j=1}^{N} v_j\, w_j = 0$$

for all $v = (v_1 \,, \ldots, \, v_N)$, $w = (w_1 \,, \ldots, \, w_N) \in V$. Geometrically, the most important property of isotropic subspaces in $\mathbf{C}^N$ is that they are exactly the ones that "stem from orthonormal systems in $\mathbf{R}^N$". More exactly, we have:

Lemma 1.2: *Let V be an n–dimensional complex subvectorspace of $\mathbf{C}^N$. Then we have that*

$$Z_1 = \frac{1}{\sqrt{2}}\,(X_1 + i\,Y_1)\,, \ldots, \, Z_n = \frac{1}{\sqrt{2}}\,(X_n + i\,Y_n)$$

(with $X_j \,, Y_j \in \mathbf{R}^N$) is an isotropic, *Hermitean orthonormal basis of V if and only if $X_1 \,, \ldots, \, X_n \,, Y_1 \,, \ldots, \, Y_n$ is a Euclidean orthonormal system in $\mathbf{R}^N$.*

Note that this lemma immediately implies that the maximal dimension of an isotropic subspace of $\mathbf{C}^N$ is $[\frac{N}{2}]$, where $[x]$ is the largest integer less than or equal to x.

Proof: If $Z_1 \,, \ldots, \, Z_n$ is an Hermitean orthonormal system, and if we denote the regular Hermitean product on $\mathbf{C}^N$ by $< .\,, . >$, we have that

$$\begin{aligned} \delta_{jk} &= \,<Z_j \,, Z_k> = Z_j \cdot \overline{Z_k} \\ &= \tfrac{1}{2}\,(<X_j \,, X_k> \,+\, <Y_j \,, Y_k>) + \tfrac{i}{2}\,(-<X_j \,, Y_k> \,+\, <Y_j \,, X_k>)\,. \end{aligned}$$

On the other hand, if V is isotropic, we have

$$0 = Z_j \cdot Z_k = \tfrac{1}{2}\,(<X_j \,, X_k> \,-\, <Y_j \,, Y_k>) + \tfrac{i}{2}\,(<X_j \,, Y_k> \,+\, <Y_j \,, X_k>)\,.$$

Adding these equations and then separating real from imaginary parts gives $<X_j \,, X_k> = \delta_{jk}$ and $<Y_j \,, X_k> = 0$, whereas subtracting leads to $<Y_j \,, Y_k> = \delta_{jk}$ and $<X_j \,, Y_k> = 0$, i.e $X_1 \,, \ldots, \, X_n \,, Y_1 \,, \ldots, \, Y_n$ is an orthonormal system in $\mathbf{R}^N$. The converse is clear from the equations above.

Another fact that we will need later is that congruent minimal real Kähler immersions have congruent holomorphic representatives:

Lemma 1.3: *Let* f *and* $\tilde{f} : M^n \to \mathbf{R}^N$ *be two minimal real Kähler immersions from* M *into Euclidean* N*–space, and let* F *and* $\tilde{F} : U \to \mathbf{C}^N$ *be (local) holomorphic representatives of* f *and* $\tilde{f}$*, respectively. Assume that* f *and* $\tilde{f}$ *are congruent, i.e.* $\tilde{f} = A \circ f + b$ *for an* $A \in O(N)$ *and* $b \in \mathbf{R}^N$*. Then we have* $\tilde{F} = A \circ F + c$*, where we view* A *as a complex* $N \times N$*–matrix, and where* $c \in \mathbf{C}^N$ *.*

Proof: Obviously, it suffices to show that $\tilde{F}_* = A \circ F_*$. By (3), we have that $F = \frac{1}{\sqrt{2}} \, (f + i \, f_{-\pi/2})$, and thus by (2) that $F_* = \frac{1}{\sqrt{2}} \, (f_* - i \, f_* \circ J)$, and analogously for $\tilde{F}$. But since $\tilde{f}_* = A \circ f_*$, we have

$$\tilde{F}_* = \frac{1}{\sqrt{2}} \, \left(\tilde{f}_* - i \, \tilde{f}_* \circ J \right) = A \circ F_* \, .$$

We will end this chapter with the promised simple examples of how we can use Theorem 1.1 to construct minimal real Kähler submanifolds.

Example 1.4: For some fixed integers $m > n \geq 1$ and $N \geq 2m$, choose a basis $\mathbf{X}_1 , \ldots , \mathbf{X}_m$ of some *fixed* m–dimensional isotropic subspace of $\mathbf{C}^N$. Let $\mathbf{C}_0 \in \mathbf{C}^N$ be another constant vector. For $j = n+1 , \ldots , m$, let $g_j(z_1 , \ldots , z_n) : U \to \mathbf{C}$ be some holomorphic function in the n complex variables $z_1 , \ldots , z_n$, defined on some open subset U of $\mathbf{C}^n$, and define the map $F : U \to \mathbf{C}^N$ by

$$F(z_1, \ldots , z_n) := \sum_{k=1}^{n} z_k \, \mathbf{X}_k \quad + \quad \sum_{k=n+1}^{m} g_k(z_1 , \ldots , z_n) \, \mathbf{X}_k \quad + \quad \mathbf{C}_0 \ . \tag{6}$$

Then we have that for $j = 1 , \ldots , n$,

$$\frac{\partial F}{\partial z_j} = \mathbf{X}_j + \sum_{k=n+1}^{m} \frac{\partial g_k}{\partial z_j} (z_1 , \ldots , z_n) \, \mathbf{X}_k \ ,$$

and it is easy to see that the $\varphi_j := \frac{\partial F}{\partial z_j}$ indeed satisfy conditions (a) through (d) in Theorem 1.1. Therefore, $f := \sqrt{2} \, \mathrm{Re}(F) : M \to \mathbf{R}^N$, where the manifold $M := (U \, , \, f^*{<}\, , {>})$ has *real* dimension $2n$, is a minimal real Kähler immersion. Since we can view F as the "graph" of the holomorphic map $(g_{n+1}, \dots, g_m)$ "with respect to the isotropic basis $\mathbf{X}_1, \dots, \mathbf{X}_m$", we will say that such an immersion f is **generated by an isotropic graph**.

It is interesting to note that in *even codimensions*, we have basically only rediscovered the examples for minimal real Kähler submanifolds that we encountered on page 5, since we have the following

Proposition 1.5: *Let M^{2n} be a Kähler manifold, and $f : M \to \mathbf{R}^{2N}$ be an isometric immersion that is also a holomorphic map with respect to some complex structure on $\mathbf{R}^{2N}$ (so by the remarks on page 5: f is, in particular, a minimal immersion). Then for each point in M, there is a neighborhood U of this point and a complex chart $(z_1, \dots, z_n)$ on U such that with respect to this chart, $f|_U$ is generated by an isotropic graph.*

Conversely, every minimal real Kähler immersion $f : M^{2n} \to \mathbf{R}^{2N}$ that is generated by an isotropic graph in even codimension *is holomorphic with respect to a suitably chosen complex structure J on $\mathbf{R}^{2N}$.*

Proof: The main tool of this proof is a fact that Calabi discovered first for minimal surfaces in even–dimensional Euclidean spaces (see [C], or [L], page 165), and that Rigoli and Tribuzy generalized for minimal real Kähler immersions (see [R-T], Theorem 4 on page 517)[4]:

A minimal isometric immersion $f : M^{2n} \to \mathbf{R}^{2N}$ is holomorphic with respect to some complex structure J on $\mathbf{R}^{2N}$ if and only if there is a fixed *isotropic subspace V of $\mathbf{C}^{2N}$ such that the image of the complex Gauss map of f is contained in V, i.e.*

$$f_* T_p M^{(1,0)} \subset V \quad \text{for all } p \in M \ .$$

Now assume that f is holomorphic with respect to some complex structure J on $\mathbf{R}^{2N}$. Let V be the isotropic subspace of $\mathbf{C}^N$ mentioned above, and let m denote the (complex) dimension of V (so: $n \le m \le N$). Taking an Hermitean orthonormal basis $\mathbf{X}_1, \dots, \mathbf{X}_m$ of V, we obviously can write any holomorphic representative $F : W \to \mathbf{C}^{2N}$ of f in any given complex coordinate system $(\tilde{z}_1, \dots, \tilde{z}_n)$ on $W \subset M$ as

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[4] In their article, Rigoli and Tribuzy work with the *antiholomorphic* Gauss map $f_* T_p M^{(0,1)}$, so their theorem had to be slightly adapted here

$$F(\tilde{z}_1\,,\dots,\,\tilde{z}_n) = \sum_{j=1}^{m} \tilde{g}_j(\tilde{z}_1\,,\dots,\,\tilde{z}_n)\,\mathbf{X}_j + F(p_0)\;, \tag{7}$$

where the $\tilde{g}_j$ are certain holomorphic functions defined on W, and where p_0 is some fixed point in W. But since F is immersive (see (5)), the rank of the map $\tilde{G} := (\tilde{g}_1\,,\dots,\,\tilde{g}_m) : W \to \mathbf{C}^m$ is also equal to n everywhere. Given a fixed point p in W, we can thus find indices $j_1\,,\dots,\,j_n \in \{1\,,\dots,\,m\}$ such that the $n \times n$–matrix

$$\left(\frac{\partial\,\tilde{g}_{j_k}}{\partial\,\tilde{z}_j}(p)\right)_{j,\,k=1,\dots,\,n}$$

has rank n. Assume without loss of generality that $j_1 = 1\,,\dots,\,j_n = n$. Now set

$$\Phi := (\tilde{g}_1\,,\dots,\,\tilde{g}_n)^{-1}\;,$$

and let the domain of Φ be some open set $\tilde{U} \subset \mathbf{C}^n$ that Φ maps biholomorphically onto some open neighborhood U of p in W (the existence of such an open set is guaranteed by the "holomorphic" Inverse Mapping Theorem; see e.g. [G-R], page 17). Thus, $\Phi^{-1} : U \to \tilde{U}$ is a complex chart for M, and changing coordinates to $(z_1\,,\dots,\,z_n) \in \tilde{U}$, we see that

$$\begin{aligned}\tilde{G} \circ \Phi(z_1\,,\dots,\,z_n) \;&=\; (\tilde{g}_1(\Phi(z_1\,,\dots,\,z_n))\,,\dots,\,(\tilde{g}_m(\Phi(z_1\,,\dots,\,z_n))\\ &=\; (z_1\,,\dots,\,z_n\,,\,g_{n+1}(z_1\,,\dots,\,z_n)\,,\dots,\,g_m(z_1\,,\dots,\,z_n))\;,\end{aligned}$$

for the holomorphic functions $g_j := \tilde{g}_j \circ \Phi$ $(j = n+1\,,\dots,\,m)$. Inserting them into (7), we see that in the "complex coordinate system" $(z_1\,,\dots,\,z_n)$, the holomorphic representative of f has the same form as in (6), which proves the first part of our claim.

Conversely, if f is given as in Example 1.4, then the complex subspace $V := \operatorname{span}\{\mathbf{X}_1\,,\dots,\,\mathbf{X}_m\}$ of $\mathbf{C}^{2N}$ is isotropic, and by (5) and (6) it is clear that the image of the Gauss map of f always lies in V. Thus f is holomorphic with respect to some complex structure on $\mathbf{R}^{2N}$ by the criterion given above.

Another way to construct simple examples is to look at "isotropic cylinders" in the following sense:

Example 1.6: Take a *fixed* isotropic subspace V of $\mathbf{C}^N$ of dimension $n < [\frac{N}{2}]$. Further, take a basis $\mathbf{X}_1\,,\dots,\,\mathbf{X}_n$ of V. Denote the "isotropic orthogonal complement" of V by $V^{\perp^\bullet}$, where $A \perp^\bullet B$ for A, $B \in \mathbf{C}^N$ means by definition that $A \cdot B = 0$ with respect to the standard symmetric inner product in $\mathbf{C}^N$. Note that it is clear from the general theory of bilinear forms that since "$\cdot$" is *non–degenerate*, the dimension of $V^{\perp^\bullet}$ is $N - n$. However, since V is supposed to be isotropic, in our case we always have that $V \subset V^{\perp^\bullet}$.

Now choose any complement W of V in $V^{\perp^\bullet}$, so that $V^{\perp^\bullet} = V \oplus W$, and take any nowhere zero *isotropic* map $\mathbf{Z} : U \to W$ which is holomorphic in the one complex variable $w \in U \subset \mathbf{C}$. (It is not hard to show that any subspace of $\mathbf{C}^N$ of dimension at least two contains non-zero isotropic vectors, and that by our assumption on the dimension of V, $\dim W$ has to be at least two.) Then it is easy to check that the partial derivatives $\varphi_j := \frac{\partial F}{\partial z_j}$ and $\varphi_{n+1} := \frac{\partial F}{\partial w}$ of the map

$$F(z_1, \ldots, z_n, w) := \sum_{j=1}^{n} z_j \mathbf{X}_j \quad + \quad \int \mathbf{Z}(w)\, dw$$

satisfy conditions (a) through (d) in Theorem 1.1, and thus that the map $f := \sqrt{2}\,\mathrm{Re}(F) : M := (\mathbf{C}^n \times U\,,\, f^* \!< , >) \to \mathbf{R}^N$ is a minimal real Kähler immersion. Because F has a similar structure as a *cylinder* in a Euclidean space (where all coordinates and functions would be real), we will say that such an immersion f is **generated by an isotropic cylinder**.

Note that since we cannot assume that the $\mathbf{X}_j$ and $\mathbf{Z}$ are always contained in a *fixed* isotropic subspace of $\mathbf{C}^N$, we may very well obtain *non–holomorphic* immersions here (see the criterion in the proof of Proposition 1.5). For the same reason, Lemma 1.2 implies that we also cannot expect that f itself will split as a cylinder in $\mathbf{R}^N$.

The real reason why we introduce Examples 1.4 and 1.6 is that they will later reemerge naturally as special cases in our classification of four–dimensional minimal real Kähler submanifolds of $\mathbf{R}^6$.