# Non-Autonomous Complex Dynamical Systems

by

Han Peters

A dissertation submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
(Mathematics)
in The University of Michigan
2005

Doctoral Committee:

Professor John-Erik Fornæss, Chair
Professor Arthur Wasserman
Assistant Professor Lukas Geyer
Assistant Professor Mattias Jonsson
Assistant Professor Marc Newman

## ACKNOWLEDGEMENTS

Part of this thesis is joint work with Erlend Fornæss Wold, namely Chapters 5 and 6. I would like to thank John Erik Fornæss, Mattias Jonsson, Dror Varolin and Christian Wolf for inspiring me to work on the topics of this thesis. Also thanks to Al Taylor, Berit Stensønes and Ralph Spatzier for interesting discussions on these subjects, and to Hannah Robbins for proofreading.

# TABLE OF CONTENTS

# CHAPTER I

# Introduction

## 1.1 Introduction

The field of complex dynamical systems studies the iteration of complex differentiable mappings. The main objects of interest are the *orbits* of such mappings. For a given mapping $F : X \to X$, where $X$ is a complex manifold, the orbit of $p \in X$ is defined to be the sequence $p, F(p), F^2(p) := F(F(p)), \ldots$.

The simplest orbits are of course those of fixed points, $F(p) = p$. The point $p$ is called an *attracting fixed point* if the modulus of all eigenvalues of $F'(p)$ are strictly smaller than 1. It is easy to see that $p$ is an attracting fixed point if and only if there exists a neighborhood $\mathcal{N}$ of $p$ such that the orbit of any $z \in \mathcal{N}$ converges to $p$. The set of all points $z \in X$ whose orbits converge to $p$ is called the *basin of attraction* of $F$ at $p$:

$$\Omega = \{z \in X \mid F^n(z) \to p\}.$$

In complex analysis one fundamental question is to describe a given domain, or more generally a complex manifold, up to biholomorphic equivalence. It turns out that if the mapping $F$ is invertible then the basins of attraction are always biholomorphic to complex Euclidean space.

We may assume that $\mathcal{N}$ is diffeomorphic to the unit ball in $\mathbb{C}^k$ and that $F(\mathcal{N}) \subset\subset$

$\mathcal{N}$. Every orbit that converges to $p$ must at some point reach the neighborhood $\mathcal{N}$, and every orbit that reaches $\mathcal{N}$ at some point must converge to $p$. Hence we have that

$$\Omega = \bigcup_{n \geq 0} F^{-n}(\mathcal{N}).$$

Thus $\Omega$ is a monotone increasing union of balls, so $\Omega$ is diffeomorphic to $\mathbb{C}^k = \mathbb{R}^{2k}$.

It is harder to show that $\Omega$ is also biholomorphic to $\mathbb{C}^k$. If $F$ is an automorphism of the complex plane then this follows immediately, since all automorphisms of the complex plane are affine mappings. It is not feasible to give such an argument in higher dimensions, since the automorphism groups are much larger.

The two dimensional case was solved by Lattès [Lat11]. He showed that for any automorphism $F$ of a complex manifold of dimension 2 with an attracting fixed point at $p$ there exists a local conjugation $\phi^{-1}F\phi = G$ near $p$ where $G$ is a so-called *normal form*. To be more precise, $G$ is equal to either $L_{\lambda,\mu}$ or $E_{\lambda,k}$, where

$$L_{\lambda,\mu} : (x, y) \mapsto (\lambda x, \mu y), (\lambda, \mu \neq 0),$$

$$E_{\lambda,k} : (x, y) \mapsto (\lambda x, \lambda^k(y + x^k)), (\lambda \neq 0, k \in \mathbb{N}).$$

In either case the mapping $G$ is an automorphism of $\mathbb{C}^2$, and attracts all of $\mathbb{C}^2$ to the origin. It follows from the fact that $F$ and $G$ are locally conjugate that the basin of attraction of $F$ is biholomorphically equivalent to the basin of attraction of $G$, which is $\mathbb{C}^2$.

It can be much easier to choose specific automorphisms and show that the basin of attraction for these mappings is biholomorphic to $\mathbb{C}^2$. In the 1920's Fatou and Bieberbach constructed automorphisms of $\mathbb{C}^2$ that have more than one fixed point, including at least one attracting fixed point. Of course, the orbits of the other fixed points cannot converge to the attracting fixed point, therefore it follows that a basin of attraction of such an automorphism is a proper subdomain of $\mathbb{C}^2$ that is biholomorphically equivalent to $\mathbb{C}^2$. This discovery showed a fundamental difference between one dimensional complex conformal equivalences and higher dimensional biholomorphic equivalences, which at the time came as a complete surprise. Proper subdomains of $\mathbb{C}^2$ biholomorphic to $\mathbb{C}^2$ are now called Fatou-Bieberbach domains, and such domains have received significant interest.

The question of whether a basin of attraction is always biholomorphic to complex Euclidean space remained unsolved for quite a while. In the 1950's Sternberg [Ste57] showed the existence of a normal form for biholomorphic mappings in any dimension. In 1988 this result was proved again by Rosay and Rudin [RR88] and they used an argument similar to that of Lattès to prove the following:

**Theorem 1.1.** *Let $F$ be an automorphism of a complex manifold that has an attracting fixed point. Then the basin of attraction is biholomorphic to complex Euclidean space.*

We will consider the main ideas of the proof of this theorem more closely in the next chapter.

We will prove several generalizations of the above theorem to the *non-autonomous* setting. In non-autonomous dynamical systems the compositions of a sequence of endomorphisms are studied instead of the iteration of a single mapping. For a sequence of mappings $F_1, F_2, \ldots$, the orbit of a point $p$ is defined as $p, F_1(p), F_2(F_1(p)), \ldots$. Theorem 1.1 naturally suggests the following question:

**Question 1.2.** *Let $F_1, F_2, \ldots$ be a sequence of automorphisms of a complex manifold, all having a single attracting fixed point. Under what conditions is the basin of attraction biholomorphically equivalent to complex Euclidean space.*

By basin of attraction, we mean the set of points whose (non-autonomous) orbits converge to the fixed point.

In the second chapter we will give the background necessary for this thesis. We also show how the above question is motivated by a question about *stable manifolds*. A stable manifold is a generalization of a basin of attraction to the case where there is not a fixed point. Instead of studying the set of points whose orbits converge to a given point, one studies the set of points whose orbits converge to a given orbit. It turns out that there is a strong relationship between stable manifolds and non-autonomous basins of attraction: a stable manifold is biholomorphic to the basin of attraction of a sequence of automorphisms of $\mathbb{C}^k$, where $k$ is the dimension of the stable manifold. We will see in the next chapter that a stable manifold is always biholomorphic to complex Euclidean space if the following conjecture holds:

**Conjecture 1.3.** *Let $F_1, F_2, \ldots$ be a sequence of automorphism of $\mathbb{C}^k$. Assume that there exist $a, b \in \mathbb{R}$ with $0 < a < b < 1$ so that for any $z$ in the unit ball and any $n \in \mathbb{N}$ the following holds:*

$$a\|z\| \leq \|F_n(z)\| \leq b\|z\|.$$

*Then the basin of attraction of the origin is biholomorphic to $\mathbb{C}^k$.*

The above conjecture is the main focus of this thesis.

In the third chapter we give several examples of basins of attractions of sequences of biholomorphic mappings. The examples show that a basin of attraction of a sequence of biholomorphic mappings is not biholomorphic to $\mathbb{C}^k$ unless some assumptions are made on the rate at which different orbits converge to the attracting fixed point. We also give strong sufficient conditions on the rate of convergence of the orbits for the basin of attraction of a sequence of automorphisms to be biholomorphic to $\mathbb{C}^k$.

One natural question to ask is the following: Given a single automorphism with an attracting fixed point, is the basin of attraction of a sequence of automorphisms biholomorphic to $\mathbb{C}^k$ if all the automorphisms in the sequence are close enough to the original automorphism? In Chapter 4 we prove that this is indeed the case. Besides methods from the proof of Theorem 1.1, the proof of this result relies on two other ingredients. The first is the non-autonomous dynamics of matrices and the second is the actions of polynomial mappings on the space of polynomial mappings of some bounded degree.

In the fifth chapter we show that given any sequence of automorphisms with a common attracting fixed point, the basin of attraction is biholomorphic to $\mathbb{C}^k$ if the mappings are repeated often enough. For this result to hold, one needs a different definition of basin of attraction that we will call $\{r_j\}$*-calibrated basin of attraction*. This definition and the precise statement of the result are given in Chapter 5. The proof of this result relies heavily on the methods used in the proof of Theorem 1.1. For each of the given automorphisms, we will consider a local holomorphic conjugation to an automorphism that is in normal form. However, for the argument to hold we need that the conjugation mapping is a global automorphism. It follows from an interpolation result by Forstneric [For99] and Weickert [Wei97] that we may assume the conjugations to be global automorphisms.

In the sixth chapter we will study Fatou-Bieberbach domains and their boundaries. Stensønes [Ste97] showed that there exist Fatou-Bieberbach domains whose boundaries are smooth and hence of dimension 3. Wolf [Wol00] constructed for any $h$ in the interval $(3, 4)$ a Fatou-Bieberbach domain whose boundary has Hausdorff dimension $h$. Thus the only open questions are whether there exist Fatou-Bieberbach domains whose boundaries have Hausdorff dimension either exactly 4 or strictly smaller than 3. We will construct Fatou-Bieberbach domains whose boundaries have dimension 4 as basins of attraction of a sequence of automorphisms of $\mathbb{C}^2$. To show that these basins are biholomorphic to $\mathbb{C}^2$ we will use two theorems presented in the earlier chapters.

The last chapter is not directly related to the other chapters, although it also deals with non-autonomous complex dynamical systems. We will define several ergodic theoretical properties for the non-autonomous case and show that these properties hold for compact sequences of holomorphic self-maps of $\mathbb{P}^k$. The objects we will try to generalize are *ergodic and mixing measures*. Roughly speaking, these measures say something about how orbits starting at different parts of the space get mixed up.

In the non-autonomous case, it would be too restrictive to assume that there exists one measure that is invariant with respect to the whole sequence of mappings, so instead we will introduce *invariant sequences of measures*. It turns out that there exist quite natural generalizations of mixing and ergodicity for such sequences.

CHAPTER II

# Background information

## 2.1 Non-Autonomous Notation

Throughout this thesis we will use the product notation for a composition of mappings, for example $fg$ instead of $f \circ g$ and $f^2$ instead of $f \circ f$, and we will write $f^{-n}$ instead of $(f^{-1})^n$.

Let $f_1, f_2, \ldots$ be a sequence of endomorphisms of a complex manifold $X$. We will write $f(n)$ for the composition of the first $n$ mappings:

$$f(n) = f_n \cdots f_1.$$

We think of $f(n)$ as the mapping that takes us from stage 0 to stage $n$. When $z \in X$ is fixed we will often write $z_n$ for $f(n)(z)$. We will denote the mapping that takes us from stage $m$ to stage $n$ by $f(m, n)$:

$$f(m, n) = f_n f_{n-1} \cdots f_{m+1}.$$

We let $\mathrm{Aut}(\mathbb{C}^k)$ be the group of holomorphic automorphisms of $\mathbb{C}^k$, and write $\mathrm{Aut}_0(\mathbb{C}^k)$ for the subgroup of those automorphisms that fix the origin. We write $\mathcal{O}(\mathbb{C}^k)$ for the set of holomorphic functions $\phi : \mathbb{C}^k \to \mathbb{C}$.

We will denote the ball of radius $R$ centered at $p$ by $B(p, R)$, or $B(R)$ when $p = 0$, and for simplicity we will write $\mathbf{B}$ for the unit ball.

## 2.2 Complex Dynamical Systems in Several Variables

The complex differentiable maps that have received the most attention in complex dynamical systems are without question quadratic polynomials in the complex plane. Any quadratic polynomial is conjugate to a function $f_c = z^2 + c$ with $c \in \mathbb{C}$, so to study the dynamical behavior of quadratic polynomials it is sufficient to study the mappings $f_c$. The Julia set of $f_c$, denoted by $J_c$, is defined as the set of all $z \in \mathbb{C}$ for which the family $\{f^n \mid n \in \mathbb{N}\}$ is not a normal family in any neighborhood of $z$. For instance, for $f_0(z) = z^2$ it is easy to calculate that the Julia set is exactly the unit circle. The set of all constants $c \in \mathbb{C}$ for which the Julia set of $f_c$ is connected is called the *Mandelbrot set*. The connected component of the interior of the Mandelbrot set that contains 0 is called the *main cardioid* of the Mandelbrot set. It can be shown that $f_c$ has an attracting fixed point if and only if $c$ lies in the main cardioid of the Mandelbrot set. We will recall more results concerning the Mandelbrot set in the third section of this chapter.

In higher dimensions, the space of degree 2 polynomial endomorphisms is so large that it is very hard to get results like those in one dimension. However, in dimensions two and higher there exist polynomial mappings that are invertible and not linear. A complete classification of polynomial automorphisms in two complex variables was given by Friedland and Milnor [FM89]. It turns out that any polynomial automorphism of $\mathbb{C}^2$ is conjugate to an affine mapping, an elementary mapping $E$, or a composition of (generalized) *Hénon mappings* $H$, where

$$E(z,w) = (\alpha z - p(w), \beta w), \text{ and}$$
$$H(z,w) = (p(z) - \alpha w, z),$$

with $\alpha, \beta \in \mathbb{C} \setminus \{0\}$ and $p$ a polynomial. The dynamical behavior of affine mappings is quite trivial, and it is not hard to understand the dynamical behavior of elementary mappings either. Therefore the only polynomial automorphisms of $\mathbb{C}^2$ that are dynamically interesting are compositions of Hénon mappings, and the simplest ones are the Hénon mappings themselves.

Several of the results presented in this thesis rely upon dynamical properties of Hénon mappings, so we recall some of these properties here (see for instance [BS91] and [BS92] for more details).

For $R > 0$ one defines:

$$V^+ = \{(z,w) \in \mathbb{C}^2 \mid |z| \geq |w|, |z| \geq R\},$$
$$V^- = \{(z,w) \in \mathbb{C}^2 \mid |w| \geq |z|, |w| \geq R\},$$

and

$$V = \{(z,w) \in \mathbb{C}^2 \mid |z|, |w| \leq R\}.$$

One can choose $R$ large enough such that the following properties hold:

$$F(V^+) \subset V^+, \tag{2.1}$$
$$F^{-1}(V^-) \subset V^-. \tag{2.2}$$

The orbit of $z \in \mathbb{C}^2$ converges to infinity in positive time (or negative time) if and only if for some $n \in \mathbb{N}$ one has that $F^n(z) \in V^+$ (resp. $F^{-n}(z) \in V^-$). The partition of $\mathbb{C}^2$ given by the sets $V, V^+$ and $V^-$ is called a *filtration* for $F$, and it describes the global dynamics of the mapping $F$.

Next we recall the definitions of the Julia sets of a Hénon mapping $f$. We define the sets $K^+$ and $K^-$ as

$$K^{\pm} = \{z \in \mathbb{C}^2 \mid \{f^{\pm n}(z)\} \text{ is bounded}\}.$$

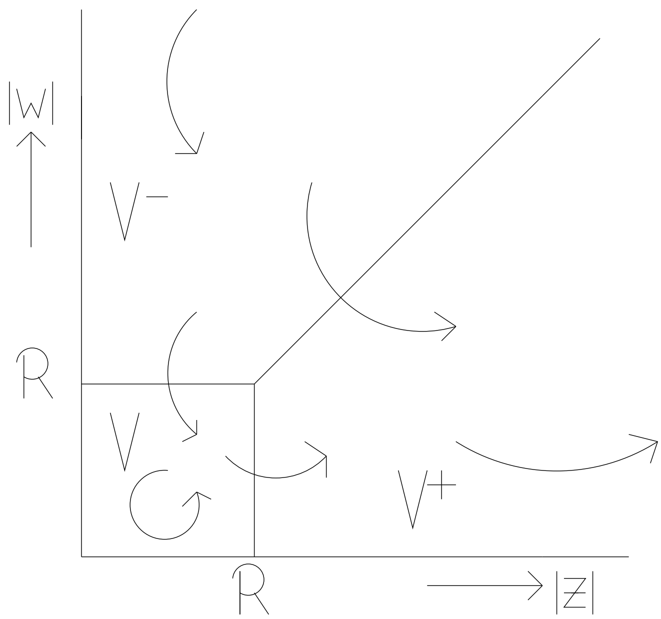


Figure 2.1: Filtration for the Hénon mapping

It follows from the filtration properties discussed in Chapter 2 that $z \in K^+$ if and only if $f^n(z) \in V$ for all $n$ large enough.

We define the forward and backward Julia sets $J^+$ and $J^-$ as $J^\pm = \partial K^\pm$, and the Julia set as $J = J^+ \cap J^-$.

## 2.3 Basins of Attraction: The Method of Rosay and Rudin

We will use the methods used by Rosay and Rudin [RR88] in the proof of Theorem 1.1 for several of the results in this thesis, so we present the proof of this theorem in some more detail in this section.

A mapping $G = (g_1, \cdots, g_k) : \mathbb{C}^k \to \mathbb{C}^k$ is called a *lower triangular polynomial mapping* if there exist constants $s_j \in \mathbb{C}$ and polynomial mappings $h_2, \ldots, h_k$ with $h_j(0) = 0$ such that $g_1(z) = s_1 z_1$ and

$$g_j(z) = s_j z_j + h_j(z_1, \cdots, z_{j-1}),$$

for every $j = 2, \ldots, k$. If $0 < |s_j| < 1$ for every $j$ then $G$ is a polynomial automorphism of $\mathbb{C}^k$ with an attracting fixed point at the origin.

Let $F$ be an automorphism of $\mathbb{C}^k$ with an attracting fixed point at $p \in \mathbb{C}^k$. Let $\Omega$ denote the basin of attraction as in the introduction. Without loss of generality we may assume that $F'(p)$ is a lower triangular matrix, with diagonal terms $\lambda_1, \ldots, \lambda_k$ that satisfy $|\lambda_1| \geq \cdots \geq |\lambda_k|$. It was shown in [RR88] that there exists a lower triangular polynomial mapping $G$, and for every $d \in \mathbb{N}$ a polynomial mapping $\phi_d = \mathrm{Id} + h.o.t.$ so that

$$\phi_d F \phi_d^{-1} = G + O(\|z\|^d). \tag{2.3}$$

The only monomials $z^\alpha = z_1^{\alpha_1} \cdots z_k^{\alpha_k}$ that can appear in the $j$-th coordinate function of the polynomial mapping $G$ are those for which $\lambda_1^{\alpha_1} \cdots \lambda_{j-1}^{\alpha_{j-1}} = \lambda_j$. In particular we have that if $|\lambda_1|^2 < |\lambda_k|$ then $G = F'(p)$.

Since $G'(0) = F'(p)$, we have that $|s_j| = |\lambda_j| < 1$ for every $j$. The following lemma was proved in [RR88]:

**Lemma 2.1.** *For every compact $K \subset \mathbb{C}^k$, there exists an $N \in \mathbb{N}$ such that for every $n \geq N$ one has $G^n(K) \subset \mathbf{B}$. Also, there exists a $\gamma > 1$ such that for any $w, w' \in \mathbf{B}$ and every $n \in \mathbb{N}$, one has that*

$$\|G^{-n}(w) - G^{-n}(w')\| \leq \gamma^n \|w - w'\|.$$

This means that in some sense the mapping $G$ behaves very much like a linear mapping with an attracting fixed point, and in particular the basin of attraction for $G$ at the origin is all of $\mathbb{C}^k$.

The last step in the proof is to show that for $d$ large enough the maps $\Psi_n = G^{-n}\phi_d F^n$ converge locally uniformly to a biholomorphic mapping from $\Omega$ to $\mathbb{C}^k$. Let $K \subset\subset \Omega$, and $n$ large enough such that $F^n(K)$ lies in a small neighborhood of $p$.

Notice that for any $z \in K$ we have

$$\Psi_{n+1}(z) - \Psi_n(z) = G^{-n}(G^{-1}\phi_d F^{n+1}(z)) - G^{-n}(\phi_d F^n(z)),$$

so it follows from Lemma 2.1 that

$$\|\Psi_{n+1}(z) - \Psi_n(z)\| \leq \gamma^n \|G^{-1}\phi_d F^{n+1}(z) - \phi_d F^n(z))\|. \tag{2.4}$$

It now follows from Equation (2.3) that if $d$ is chosen large enough then the right hand side of Equation (2.4) is smaller than $C\alpha^n$, with $\alpha < 1$ independent of $K$ and $C > 0$. It follows that the sequence $\Psi_n$ is a Cauchy sequence, and converges uniformly on $K$ to a holomorphic mapping $\Psi$ that extends to all of $\Omega$.

It is a well known fact that the uniform limit of a sequence of biholomorphic mappings is either injective or degenerate everywhere. In this case we have that $\Psi'_n(0) = I$ for all $n \in \mathbb{N}$, so $\Psi$ is biholomorphic. Also, the limit mapping satisfies the functional equation $G\Psi = \Psi F$. As $F(\Omega) = \Omega$, the functional equation gives us

$$G\Psi(\Omega) = \Psi(\Omega).$$

As $G$ attracts all of $\mathbb{C}^k$ to the origin, it follows that $\Psi(\Omega) = \mathbb{C}^k$ which completes the proof.

### 2.4 Stable Manifolds

We will now show how one could solve an open conjecture about stable manifolds by studying basins of attraction of sequences of automorphisms.

The definition of the basin of attraction still makes sense in the case that $p$ is a fixed point that is not attracting. If the modulus of one or more of the eigenvalues of $F'(p)$ equals 1 the situation is very difficult and we will not discuss it here (but see for instance the work of Ueda [Ued86], Abate [Aba02], Weickert [Wei98] and Hakim

[Hak98]). A periodic point $p$ of order $d$ is called *hyperbolic* if none of the moduli of the eigenvalues of $(F^d)'(p)$ are equal to 1. In the case of a hyperbolic periodic point $p$ we may as well replace $F$ with $F^d$ so that $p$ is a fixed point.

Let $p$ be a hyperbolic periodic point and assume that there are $l$ eigenvalues of $F'(p)$ whose moduli are strictly smaller than 1 (corresponding to the "attracting" directions), and $k-l$ eigenvalues whose moduli are strictly larger than 1 (the "repelling" directions).

For $\epsilon > 0$ define

$$\Omega(\epsilon) = \{z \in X \mid d(F^n(z), p) < \epsilon \ \forall n \in \mathbb{N}\}.$$

If we choose $\epsilon$ small enough then the dynamics of $F$ in the $\epsilon$-neighborhood of $p$ is very similar to the dynamics of the map $F'(p)$. It is not hard to see that an orbit that stays in $B(p, \epsilon)$ must actually converge to $p$. Therefore we have that

$$\Omega = \bigcup_{n \geq 1} F^{-n}(\Omega(\epsilon)). \tag{2.5}$$

One can show that for $\epsilon$ small enough $\Omega(\epsilon)$ is a holomorphic graph over the tangent space $T_p(\Omega)$ near $p$ ( see Theorem 9.3.2 in [MNTU00]). It follows from (2.5) that $\Omega$ is a complex manifold of dimension $l$. Hence $F|_\Omega$ is an automorphism of $\Omega$ with an attracting fixed point at $p$ and the basin of attraction (which is equal to $\Omega$) is biholomorphic to $\mathbb{C}^l$.

The definition of the basin of attraction does not make sense if the point $p \in X$ is not fixed by $F$, but instead we can study the set of all points whose orbits converge to the orbit of the point $p$. This set is called the stable set, and denoted by

$$W_p^s = \{z \in X \mid d(F^n(z), F^n(p)) \to 0\}. \tag{2.6}$$

As we have seen above, the situation of a periodic point is not really different from the case of a fixed point, but when $p$ is not periodic the situation is far more difficult.

An automorphism $F$ of $X$ is said to act *hyperbolically* on a compact invariant set $K \subset X$ if there exists a continuous splitting of the tangent bundle $T(X)|_K = E^s \oplus E^u$, ($E^s$ and $E^u$ consist of the *stable* and *unstable* directions) with the following three properties:

(i) $E^s_p$ and $E^u_p$ have constant rank for all $p \in K$, say $l$ and $k - l$.

(ii) $dF(E^s_p) = E^s_{f(p)}$ and $df(E^u_p) = E^u_{f(p)}$ for all $p \in K$.

(iii) there exist positive constants $C$ and $\rho$, with $\rho < 1$, so that, for all $p \in K$ and all $n \geq 0$

$$\|dF^n|_{E^s_p}\| \leq C\rho^n \text{ and } \|dF^n|_{E^u_p}\| \geq C^{-1}\rho^{-n}. \tag{2.7}$$

If $F$ acts hyperbolically on $K$ and $p \in L$ then we can define the local stable set of $p$ as follows:

$$W^s_p(\epsilon) = \{z \in X \mid d(F^n(z), F^n(p)) < \epsilon \; \forall n \in \mathbb{N}\}.$$

We have that

$$W^s_p = \bigcup_{n \geq 1} F^{-n}(W^s_p(\epsilon)),$$

similarly to the case of a hyperbolic fixed point.

Again one can show that for small $\epsilon$ the local stable set $W^s_p(\epsilon)$ is a holomorphic graph over $E^s_p$ near $p$, making $W^s_p$ a complex manifold. The set $W^s_p$ is called the *stable manifold* of $F$ through $p$ (similarly one can define the unstable manifold, usually denoted by $W^u_p$).

The following natural conjecture was posed by Bedford [Bed]:

**Conjecture 2.2.** *Let $F$ be an automorphism of a complex manifold $X$, and suppose that $F$ acts hyperbolically on a compact invariant subset $K$ of $X$. Then for every $p \in K$ we have that $W^s_p$ is biholomorphic to complex Euclidean space.*

It was proved by Jonsson and Varolin [JV02] that Conjecture 2.2 holds almost everywhere with respect to any invariant probability measure supported on $K$.

One may be able to show that Conjecture 2.2 holds by solving a related problem in non-autonomous dynamics. Fix a complete Riemannian metric $d(\cdot,\cdot)$ on $X$. Following [JV02] we define

$$\mathcal{W}^s := \bigsqcup_{p\in K} W_p^s$$

as well as

$$E_p^s(\epsilon) := \{v \in E_p^s \mid \|v\| < \epsilon\} \text{ and } E^s(\epsilon) := \bigcup_{p\in K} E_p^s(\epsilon).$$

For $p \in K$ let $\mathrm{dist}_p$ be the distance on $W_p^s$ associated to the metric on $X$. For $x, y$ in $\mathcal{W}^s$ we define $\mathrm{dist}(x,y) = \mathrm{dist}_p(x,y)$ if $x, y \in W_p^s$ for some $p \in K$. If $x \in W_p^s$ and $y \in W_q^s$ for some $p \neq q \in K$, then we define $\mathrm{dist}(x,y) = \mathrm{dist}_p(x,p) + d(p,q) + \mathrm{dist}_q(q,y)$. Similarly we define a distance on $E^s$: $\mathrm{dist}(v,w) = \|v - w\|$ when $v, w \in E_p^s$ and $\mathrm{dist}(v,w) = \|v\| + d(p,q) + \|w\|$ when $v \in E_p^s$ and $w \in E_q^s$. When we talk about the topologies on $\mathcal{W}^s$ and $E^s$ then we mean those induced by the distances $\mathrm{dist}(\cdot,\cdot)$.

Recall Proposition 1.1 of [JV02]:

**Proposition 2.3.** *There exist $\epsilon > 0$ and a continuous mapping $\chi : E^s(\epsilon) \to \mathcal{W}^s$ which maps each $E_p^s(\epsilon)$ biholomorphically into $W_p^s$, maps the zero vector $0_p \in E_p^s$ to $p \in W_p^s$, and satisfies $d(\chi|_{E_p^s})0_p = id_{E_p^s}$.*

By replacing $F$ with a high iterate of $F$ if necessary, we may assume that $C = 1$ in Equation (2.7), and identify $E_p^s(\epsilon)$ with the unit ball in $\mathbb{C}^k$ for every $p \in K$. Let $p_0 = p, p_1, p_2, \ldots$ be the orbit of $p \in K$. Then the map

$$f_n(z) = \chi^{-1} \circ F \circ \chi,$$

maps $E_{p_{n-1}}^s(\epsilon)$ biholomorphically into $E_{p_n}^s(\epsilon)$. Therefore we may take the $f_n's$ to be biholomorphic mappings from $\mathbf{B}$ into $\mathbf{B}$ that fix the origin and are uniformly attracting, i.e. there exist real numbers $a, b$ with $0 < a < b < 1$ such that

$$a\|z\| \leq \|f_n(z)\| \leq b\|z\|$$

for every $n \in \mathbb{N}$ and every $z \in \mathbf{B}$.

Recall the definition of the *tail-space* of the sequence $\{f_n\}$ from [FS04]: let $\mathcal{S}_{\{f_n\}}$ be the set of all sequences $(z_j, z_{j+1}, \cdots)$ with $z_n = f_n(z_{n-1})$ for $n = j+1, j+2, \ldots$. Define an equivalence relation $\sim$ on $\mathcal{S}_{\{f_n\}}$ by $(z_j, z_{j+1}, \cdots) \sim (w_l, w_{l+1}, \cdots)$ if and only if $z_n = w_n$ for some $n \in \mathbb{N}$. The tail space of the sequence $\{f_n\}$, is the set $\Omega_{\{f_n\}} := \mathcal{S}_{\{f_n\}}$ equipped with the equivalence relation $\sim$.

We define a map $\Phi : W_p^s \to \Omega_{\{f_n\}}$ as follows:

For $z \in W_p^s$, let $n \in \mathbb{N}$ be large enough such that $F^n(z) \in W_{p_n}^s(\epsilon)$, and let $\Phi(z)$ be the equivalence class of $(\chi^{-1}F^n(z), \chi^{-1}F^{n+1}(z), \cdots)$. It is not hard to check that this definition of $\Phi$ is independent of the choice of $n$ and gives a well-defined, biholomorphic mapping from $W_p^s$ onto $\Omega_{\{f_n\}}$.

It follows from [FS04] that $\Omega_{\{f_n\}}$ is biholomorphic to the basin of attraction of a sequence of automorphisms $\{F_n\}$ of $\mathbb{C}^l$ that satisfy

$$a\|z\| \leq \|F_n(z)\| \leq b\|z\|,$$

for every $n \in \mathbb{N}$ and every $z \in \mathbf{B}$. This means that Conjecture 2.2 can be answered positively by proving the following:

**Conjecture 2.4.** *Let $F_1, F_2, \ldots$ be a sequence of automorphisms of $\mathbb{C}^k$ that fix the origin. Suppose that there exist $a, b \in \mathbb{R}$ that satisfy $0 < a < b < 1$, such that for every $n \in \mathbb{N}$ and every $z \in \mathbf{B}$ we have:*

$$a\|z\| \leq \|F_n(z)\| \leq b\|z\|.$$

*Then the basin of attraction of the sequence $F_1, F_2, \ldots$ is biholomorphic to $\mathbb{C}^k$.*

## 2.5 Fatou-Bieberbach Domains and Dimension Theory

Recall that a Fatou-Bieberbach domain is a proper subset of $\mathbb{C}^2$ that is biholomorphically equivalent to $\mathbb{C}^2$. Stensønes [Ste97] recently showed that there exist Fatou-Bieberbach domains whose boundaries are $\mathcal{C}^\infty$ smooth. She does not construct these Fatou-Bieberbach domains as basins of attraction, but rather as the set of points where a sequence of biholomorphic mappings converges.

Before we state the next result we recall the definitions of Hausdorff dimension and upper box dimension. Let $K$ be a subset of $\mathbb{R}^n$, for simplicity we will assume that $K$ is compact. For $\epsilon > 0$ we write $\beta_\epsilon$ for the set of all coverings $\{B_i\}$ of $K$ with balls of radius at most $\epsilon$. For $h \geq 0$ we define

$$\vartheta_h^\epsilon(K) = \inf_{\beta_\epsilon} \sum (\text{diam}(B_i))^h \,, \text{ and}$$

$$\mu_h(K) = \lim_{\epsilon \to 0} \vartheta_h^\epsilon(K).$$

$\mu_h(K)$ is called the Hausdorff measure of $K$ (of dimension $h$).

The Hausdorff dimension of $K$, written as $\dim_H(K)$, is defined as the unique value $h \geq 0$ such that $\mu_{h'}(K) = 0$ for all $h' > h$ and $\mu_{h'}(K) = \infty$ for all $0 \leq h' < h$.

A different concept often used to measure the dimension of sets is called *upper box dimension*. Instead of covering the set $K$ with balls of radius at most $\epsilon$, let $\mathcal{B}_\epsilon$ be the set of all coverings of $K$ with balls of radius exactly $\epsilon$. The definition of $\gamma_h^\epsilon(K)$ is exactly the same, so

$$\gamma_h^\epsilon(K) = \epsilon^h \inf_{\mathcal{B}_\epsilon} \#\{B_i\}.$$

The limit of $\gamma_h^\epsilon(K)$ as $\epsilon \to 0$ does not necessarily exist, but the upper box content of $K$ (of dimension $h$) is defined as

$$\overline{\mu}(K) = \limsup_{\epsilon \to 0} \gamma_h^\epsilon(K).$$

Again, the upper box dimension of $K$, written as $\overline{\dim}_B(K)$, is defined as the unique value $h \geq 0$ such that $\overline{\mu}_{h'}(K) = 0$ for all $h' > h$ and $\overline{\mu}_{h'}(K) = \infty$ for all $0 \leq h' < h$.

The Hausdorff (or upper box) dimension of a non-compact set is the supremum of the Hausdorff (resp. upper box) dimensions of all its compact subsets.

While the upper box dimension of $K$ may in general be larger than the Hausdorff dimension of $K$, for many sets the two concepts of dimension agree. Upper box dimension is often used instead of Hausdorff dimension because it is better suited for computer approximations.

Wolf [Wol00] proved the following result:

**Theorem 2.5.** *For any $h \in (3, 4)$ there exists a Hénon mapping $f(z, w) = (aw, az + w^2 + c)$ such that the boundary of the basin of attraction of the unique attracting fixed point of $f$ has Hausdorff dimension $h$.*

In fact, it follows from the proof of Theorem 4.1 in [Wol00] that the dimension of $J^+$ is equal to $h$ near any point $p \in J^+$.

The proof of Theorem 2.5 relies on the work of Fornæss and Sibony ([FS92]), who studied the dynamics of Hénon maps of the form $f_{c,a}(z, w) = (z^2 + c + aw, az)$ where $c$ is in the main cardioid of the Mandelbrot set and the constant $a > 0$ is very small. The complement of the forward Julia set of $f_{c,a}$ consists of exactly two connected components, called Fatou components, namely the basins of attraction of the unique attracting fixed point and a point at infinity. The idea is that as $a$ approaches 0, the dynamics of $f_{c,a}$ becomes similar to the dynamics of the map $f_c : \mathbb{C} \to \mathbb{C}$ defined by $f_c(z) = z^2 + c$. It turns out that the intersection of the Julia set of $f_{c,a}$ with the $w = 0$ plane is very close to the Julia set of the map $f_c$. So an estimate on the dimension of the Julia set of $f_{c,a}$ can be obtained by the use of the following theorem by Shishikura ([Shi98]:

**Theorem 2.6.** *There exists a residual (hence dense) subset $\mathcal{S}$ of the boundary of the main cardioid of the Mandelbrot set such that the Hausdorff dimension of the Julia set of $f_c$ is equal to 2 for any $c \in \mathcal{S}$. As a consequence, the Hausdorff dimension of the Julia set of a polynomial $f_c$ with c in the main cardioid of the Mandelbrot set can be arbitrarily close to 2.*

Wolf uses the last part of the Theorem 2.6 to obtain Hausdorff dimension arbitrarily close to 4 in the proof of Theorem 2.5. The theorem is then obtained by showing that the dimension can also be arbitrarily close to 3, and a continuity argument about the dependence of the Hausdorff dimension of $J^+_{f_{c,a}}$ as a function of the parameters $a$ and $c$.

## CHAPTER III

# Examples and Counterexamples

### 3.1 Basins not biholomorphic to $\mathbb{C}^k$

More generally than Conjecture 2.4 we can ask if the basin of attraction of a given sequence of automorphisms of $\mathbb{C}^k$ with a common attracting fixed point at the origin is biholomorphic to $\mathbb{C}^k$. It is easy to see that in general the answer to this question is no:

**Example 3.1.** Let $\{f_n\}$ be a sequence of automorphisms of $\mathbb{C}^k$ given by $f_n(z) = c_n z$ with $c_n \in (0,1)$, and assume that $\Pi_{n\in\mathbb{N}} c_n > 0$. We have that $f(n)(z) = \Pi_{j\leq n} c_j \cdot z$, so the basin of attraction of the sequence $f_1, f_2, \ldots$ is $\{0\}$.

**Example 3.2.** Let $\{f_n\}$ be a sequence of automorphisms of $\mathbb{C}^2$ given by $f_n(z,w) = (\frac{1}{2}z + 2w, \frac{1}{2}w)$ for $n$ even, and $f_n(z,w) = (\frac{1}{2}z, 2z + \frac{1}{2}w)$ for $n$ odd. It is easy to see that 0 is an attracting fixed point for every $f_n$. However, for $n$ even the mapping $f_n f_{n-1} = (4\frac{1}{4}z + w, z + \frac{1}{4}w)$ has a hyperbolic fixed point at the origin, with one repelling and one attracting direction. Therefore, the basin of attraction of the sequence $f_1, f_2, \ldots$ is one dimensional, and in particular not biholomorphic to $\mathbb{C}^2$.

In the Examples 3.1 and 3.2 there exist orbits that approach the attracting fixed point arbitrarily well but do not converge to the origin. This situation cannot happen in the setting of Conjecture 2.4 because the constant $b < 1$ gives an upper bound for the rate at which orbits that have reached the unit ball converge to the origin. The following result (from [For04]) shows that this condition is not sufficient:

**Theorem 3.3.** *Let $a_1, a_2, \ldots$ be complex numbers such that $0 < |a_n| < 1$ and such that $|a_{n+1}| < |a_n|^t$ for some fixed $t > 2$. For $j \in \mathbb{N}$ let $f_j$ be a polynomial automorphism of $\mathbb{C}^2$ defined by $f_j(z, w) = (z^2 + a_j w, a_j z)$. Then the basin of attraction of the sequence $f_1, f_2, \ldots$ is not biholomorphic to $\mathbb{C}^2$.*

It is shown in [For04] that the basin of attraction in Theorem 3.3 is an example of a so-called *short* $\mathbb{C}^2$, i.e. an increasing union of holomorphic balls whose Kobayashi metric vanishes identically but that is not biholomorphic to $\mathbb{C}^2$.

Theorem 3.3 suggests that to ensure that a basin of attraction is biholomorphic to $\mathbb{C}^k$, there should also be a lower bound for the rate at which orbits converge to the origin. In Conjecture 2.4 we have such a lower bound because of the constant $a > 0$.

For the following theorem, recall that $\Omega$ is defined as $\{z \in \mathbb{C}^2 \mid f(n)(z) \to 0\}$.

**Theorem 3.4.** *Let $f_j \in \mathrm{Aut}_0(\mathbb{C}^2)$ be defined by $f_j(x, y) = (x^2 + a_j y, a_j x)$ where $a_j \in (0, 1)$ for every $j \in \mathbb{N}$ and $a_j \to 1$ as $j \to \infty$. Then $\Omega$ is not open, and in particular not biholomorphic to $\mathbb{C}^2$.*

*Proof.* Let $\Omega_\infty$ be the set of points in $\mathbb{C}^2$ whose orbits converge to infinity. Since the constants $a_n$ converge to 1 as $n \to \infty$, there exists one filtration, as defined in Section 2.2, which works for all the mappings $f_j$. For instance we may assume that $a_n > \frac{1}{2}$ for all $n \in \mathbb{N}$, and then $R = 4$ will suffice for the definition of $D, V^+$ and $V^-$. This means that if $z \in \Omega_\infty$ then there is some $z_n = (x_n, y_n)$ such that $|x_n| > 4$ and $|x_n| > |y_n|$. Also if $z = (x, y)$ with $|x| > 4$ and $|x| > |y|$, then we have that

$\|f_n(z)\| > \|z\| + 1$. Therefore

$$\Omega_\infty = \bigcup_{n\geq 0} f(n)^{-1}\left(\{z = (x,y) \in \mathbb{C}^2 \mid |x| > 4, |x| > |y|\}\right),$$

and in particular, $\Omega_\infty$ is open.

Let $z = (x,y) \in \mathbb{R}^2_+ := \{((x,y) \in \mathbb{R}^2 \mid x, y \geq 0\}$. We write $\|z\|_1$ for $x+y$. Assume that $z_n$ does not converge to 0. Then there exists an $\epsilon > 0$ such that $\|z_n\|_1 > \epsilon$ for arbitrarily large $n \in \mathbb{N}$. Take $j$ so large that for all $n \geq j$ we have

$$(a_n)^2(1 + \frac{1}{4}\epsilon^2) > 1 + \frac{1}{5}\epsilon^2,$$

and let $n \geq j$ with $\|z_n\|_1 > \epsilon$. Notice that

$$f_{n+2}f_{n+1}(z_n) = ((x_n^2 + a_{n+1}y_n)^2 + a_{n+2}a_{n+1}x_n, a_{n+2}x_n^2 + a_{n+2}a_{n+1}y_n),$$

and therefore

$$\begin{aligned}
\|f_{n+2}f_{n+1}(z_n)\|_1 &\geq a_{n+1}^2 y_n^2 + a_{n+2}x_n^2 + a_{n+2}a_{n+1}(x_n + y_n) \\
&> \min(a_{n+1}, a_{n+2})^2(x_n + y_n + x_n^2 + y_n^2) \\
&\geq \min(a_{n+1}, a_{n+2})^2(1 + \frac{1}{4}\epsilon^2)\|z\|_1 > (1 + \frac{1}{5}\epsilon^2)\|z\|_1
\end{aligned}$$

Thus $\|z_n\|_1$ converges to infinity and $z \in \Omega_\infty$, so

$$\mathbb{R}^2_+ = (\Omega \cap \mathbb{R}^2_+) \cup (\Omega_\infty \cap \mathbb{R}^2_+).$$

We have that $\Omega_\infty \cap \mathbb{R}^2_+$ is a nonempty relatively open subset of $\mathbb{R}^2_+$, which means $(\Omega \cap \mathbb{R}^2_+)$ is a proper relatively closed subset of $\mathbb{R}^2_+$. Since $0 \in \Omega$, it follows that $\Omega$ is not open. □

### 3.2 Basins biholomorphic to $\mathbb{C}^k$

In the previous section we have seen several examples of sequences of automorphisms that have basins of attraction not biholomorphic to $\mathbb{C}^k$. Fortunately, none of these examples satisfy the conditions in Conjecture 2.4. In this section we will present some examples of basins of attraction that are biholomorphic to $\mathbb{C}^k$.

**Example 3.5.** Let $f_n \in \mathrm{Aut}_0(\mathbb{C}^k)$ be defined by $f_n(z, w) = (a_n w + z^2, a_n z)$, and assume that there exist $a, b \in \mathbb{R}$ such that $0 < a \leq a_n \leq b < 1$ for all $n \in \mathbb{N}$. Then the conditions in Conjecture 2.4 are satisfied. It is a result due to Fornæss [For04] that in this case the basin of attraction of the sequence $f_1, f_2, \ldots$ is biholomorphic to $\mathbb{C}^k$.

It is relatively simple to prove Theorem 1.1 in the case where the eigenvalues $\lambda_1, \ldots, \lambda_k$ of $F'(0)$ satisfy the equation

$$|\lambda_i|^2 < |\lambda_j|, \tag{3.1}$$

for any $i, j \in 1, \ldots k$. In this case, the normal form of $F$ is just the linear mapping $F'(0)$, and the sequence of mappings $F'(0)^{-n} \circ F^n$ converges uniformly on compact subsets of the basin of attraction to a biholomorphic mapping from the basin of attraction onto $\mathbb{C}^k$. A similar argument can be used in the case of a sequence of automorphism that satisfies a condition like Equation (3.1):

**Theorem 3.6.** *Let $f_1, f_2, \ldots \in \mathrm{Aut}_0(\mathbb{C}^k)$ and let $a, b \in \mathbb{R}$ satisfy $0 < b^2 < a < b < 1$. If for every $n \in \mathbb{N}$ and every $z \in \mathbf{B}$ we have*

$$a\|z\| \leq \|F_n(z)\| \leq b\|z\|,$$

*then the basin of attraction of the sequence $\{f_n\}$ is biholomorphic to $\mathbb{C}^k$.*

The proof of this theorem can be found in [Wol04]. Notice that the hypotheses in Conjecture 2.4 are identical to those in the above theorem, except for the condition $b^2 < a$. This is a very strong restriction and it makes the above theorem less interesting from a dynamical point of view. However, Theorem 3.6 is extremely useful in the construction of Fatou-Bieberbach domains, as we shall see in Chapter 6 (but see also [Wol04]).

# CHAPTER IV

# Basins of Attraction of Slightly Perturbed Mappings

## 4.1 Small perturbations

Let $F$ be an automorphism of $\mathbb{C}^k$ with an attracting fixed point at the origin, and let $\mathcal{N}$ be a small neighborhood of the origin contained in the basin of attraction with $F(\mathcal{N}) \subset\subset \mathcal{N}$. As we have already noted in earlier chapters, we have the following equality:

$$\Omega = \bigcup_{n \in \mathbb{N}} F^{-n}(\mathcal{N}),$$

which we might as well use as the definition for the basin of attraction, $\Omega$.

If the mapping $F$ is slightly perturbed (to, say, $\tilde{F}$) then 0 might no longer be a fixed point. However, for small enough perturbations we still get $\tilde{F}(\mathcal{N}) \subset\subset \mathcal{N}$, so the basin of attraction of $\tilde{F}$ is equal to $\bigcup_{n \in \mathbb{N}} \tilde{F}^{-n}(\mathcal{N})$, which makes this definition in some sense more stable.

Now let $f_1, f_2, \ldots$ be a sequence of automorphisms of $\mathbb{C}^2$ that all satisfy $f_n(\mathcal{N}) \subset\subset \mathcal{N}$. For this chapter only we define the basin of attraction of $f_1, f_2, \ldots$ as the set

$$\bigcup_{n \in \mathbb{N}} f(n)^{-1}(\mathcal{N}). \tag{4.1}$$

The main result of this chapter is the following generalization of Theorem 1.1:

**Theorem 4.1.** *Let $F$ be an automorphism of $\mathbb{C}^k$ which has an attracting fixed point at the origin. Then there exists an $\epsilon > 0$ such that for any sequence $f_1, f_2, \ldots$ of* *automorphisms that satisfy* $\|F(z) - f_n(z)\| < \epsilon$ *for all* $n \in \mathbb{N}$ *and* $z \in \mathbf{B}$*, one has that the basin of attraction of the sequence* $f_1, f_2, \ldots$ *is biholomorphic to* $\mathbb{C}^k$*.*

If we choose $\epsilon$ small enough, there exists a small neighborhood $\mathcal{N}$ of 0 with $f_n(\mathcal{N}) \subset\subset \mathcal{N}$ for all $n \in \mathbb{N}$. Notice that the definition of the basin of attraction is independent of the choice of $\mathcal{N}$, as long as $\mathcal{N}$ is sufficiently small. Also notice that if the maps $f_n$ do fix the origin, the basin of attraction defined in (4.1) is exactly the same as the usual basin of attraction.

In the next section we will introduce the main ingredient in the proof of Theorem 4.1, namely Theorem 4.5, which is interesting in its own right. In the third section we will prove that Theorem 4.5 implies Theorem 4.1 and in the last two sections we will prove Theorem 4.5

### 4.2 Lower Triangular Form

Recall the following result (called QR factorization) from linear algebra:

**Theorem 4.2.** *Let* $A$ *be a* $n \times n$ *matrix with complex coefficients. Then there exists a unitary matrix* $U$ *such that* $UA$ *is lower triangular.*

It follows that for any automorphism $F \in \mathrm{Aut}_0(\mathbb{C}^k)$ we can find a unitary matrix $U \in U_k(\mathbb{C})$ such that $(U \circ F)'(0)$ is lower triangular. If $F'(0)$ is already lower triangular then we say that $F$ is in *lower triangular form*.

*Remark* 4.3. Given some sequence $F_1, F_2, \ldots \in \mathrm{Aut}_0(\mathbb{C}^k)$ we can find unitary matrices $U_1, U_2 \ldots$ such that $U_1F_1, U_2F_2U_1^{-1}, U_3F_3U_2^{-1}, \ldots$ are all in lower triangular form. If all the maps $F_n$ are attracting at the origin, we define $\tilde{F}_n = U_nF_nU_{n-1}^{-1}$. Since $\tilde{F}(n) = U_nF(n)$, the basin of attraction of the sequence $\{\tilde{F}_n\}$ is exactly equal to the basin of the original maps, so we may assume our original maps were already in lower triangular form.

Let $F \in \text{Aut}_0(\mathbb{C}^k)$ be in lower triangular form. We will say that F is *correctly ordered* if $F'(0)$ has diagonal entries (from upper left to lower right) $\lambda_1, \ldots, \lambda_k$ that satisfy the following condition:

$$|\lambda_j||\lambda_i| < |\lambda_l|,$$

for $l \leq j$ and any $i$. Note in particular that $|\lambda_j| < 1$ for every $j$, and that $|\lambda_j|^2 < |\lambda_l|$ for $j \geq l$. Note also that if the eigenvalues satisfy the ordering $1 > |\lambda_1| \geq \cdots \geq |\lambda_k|$ then $F$ is correctly ordered, but for $F$ to be correctly ordered the sequence $|\lambda_1|, \ldots, |\lambda_k|$ may fail to be decreasing by a relatively small amount.

**Definition 4.4.** Let $\mathcal{F} \subset \text{Aut}_0(\mathbb{C}^k)$ be a family of correctly ordered automorphisms. We say that $\mathcal{F}$ is *uniformly attracting* if there exist $a, b \in \mathbb{R}$ with $0 < a < b < 1$ so that for every $F \in \mathcal{F}$ and every $z \in \mathbf{B}$,

$$a\|z\| \leq \|F(z)\| \leq b\|z\|.$$

Additionally we require that there exists a uniform $\xi < 1$ such that

$$|\lambda_i||\lambda_j| \leq \xi|\lambda_l|, \tag{4.2}$$

for $l \leq j$ and any $i$.

At the end of this chapter we will prove the following theorem:

**Theorem 4.5.** *Let $F_1, F_2, \ldots \in \text{Aut}_0(\mathbb{C}^k)$ be a uniformly attracting sequence of correctly ordered automorphisms. Then the basin of attraction of $F_1, F_2, \ldots$ is biholomorphic to $\mathbb{C}^k$.*

## 4.3 Perturbed Basins Biholomorphic to $\mathbb{C}^k$

We will now prove that Theorem 4.1 follows from Theorem 4.5. The proof consists of several steps, and in each step we may decrease the value of $\epsilon$.

We may assume that $F$ is such that $F'(0)$ is a lower triangular matrix with diagonal entries (from upper left to lower right) $\lambda_1, \ldots, \lambda_k$ that satisfy $1 > |\lambda_1| \geq \ldots \geq |\lambda_k|$. By conjugating with an affine mapping we can guarantee that the off-diagonal terms of $F'(0)$ are arbitrarily small.

Since $F'(0)$ is arbitrarily close to a diagonal matrix, we choose the neighborhood of the origin $\mathcal{N}$ so small that $F$ is contracting on $\mathcal{N}$, i.e. that there exists some $\theta < 1$ such that for all $x, y \in \mathcal{N}$ we have that $\|F(x) - F(y)\| \leq \theta\|x - y\|$. We can make $\epsilon$ small enough such that every $f_n$ is also contracting on $\mathcal{N}$ (where the uniform constant $\theta < 1$ is slightly increased if necessary). Let $x_0 = 0, x_1, x_2, \ldots$ be the orbit of 0, i.e. $x_n = f_n(x_{n-1})$. We have that $f_n(\mathcal{N}) \subset\subset \mathcal{N}$ for every $n \in \mathbb{N}$, so that $x_n \in \mathcal{N}$ for every $n \in \mathbb{N}$. Let $T_n$ be the translation of $\mathbb{C}^k$ that maps $x_n$ to 0. Then define $\tilde{f}_n = T_n \circ f_n \circ T_{n-1}^{-1}$. We have that $\tilde{f}_n(0) = 0$ for all $n$, 0 is an attracting fixed point for every map $f_n$, and the maps $\tilde{f}_n$ are still arbitrarily close to the original map $F$. Since $\tilde{f}(n) = T_n f(n)$, we have that the basin of attraction of the sequence $f_1, f_2, \ldots$ as defined in (4.1) is exactly equal to the basin of attraction of the sequence $\tilde{f}_1, \tilde{f}_2, \ldots$. Therefore, we may as well assume that all the maps $f_n$ have 0 has an attracting fixed point, and we will use the standard definition for the basin of attraction of the sequence $f_1, f_2, \ldots$.

We first prove Theorem 4.1 in the simpler case $k = 2$. We may assume that $|\lambda_2|$ is strictly smaller than $|\lambda_1|$, otherwise the result follows easily from Theorem 3.6. The fact that $F'(0)$ is lower diagonal means exactly that $(0, 1)$ is an eigenvector of $F'(0)$. Let $\Phi$ be the action on $\mathbb{P}^1$ induced by the mapping $F'(0)$. It follows that $\Phi([0 : 1]) = [0 : 1]$, and the multiplier of $\Phi$ at $[0 : 1]$ is exactly $\frac{\lambda_1}{\lambda_2}$. Hence $[0 : 1]$ is a repelling fixed point, and there exists an arbitrarily small neighborhood of $[0 : 1]$, say $\mathcal{M}$ such that $\mathcal{M} \subset\subset \Phi(\mathcal{M})$.

Let $\phi_n$ be the action on $\mathbb{P}^1$ induced by $f_n'(0)$. We can make sure that $\phi_n$ is close enough to $\Phi$ so that $\mathcal{M} \subset\subset \phi_n(\mathcal{M})$ for all $n \in \mathbb{N}$. We then have $\mathcal{M} \supset\supset \phi_1^{-1}(\mathcal{M}) \supset\supset \phi(2)^{-1}(\mathcal{M}) \cdots$, so it follows that

$$\bigcap_{n\in\mathbb{N}} \phi(n)^{-1}(\mathcal{M}) \neq \emptyset.$$

Let $v \in \bigcap \phi(n)^{-1}(\mathcal{M})$ (in fact $v$ is unique), and let $v_0 = v, v_1, v_2, \ldots$ be the orbit of $v$, i.e. $\phi_n(v_{n-1}) = v_n$, thus $v_n \in \mathcal{M}$ for every $n \in \mathbb{N}$.

Let $U_n$ be a unitary $2 \times 2$ matrix that maps some representative of $v_n$ in $\mathbb{C}^2$ of unit length onto $(0, 1)$, and define $\tilde{f}_n := U_n f_n U_{n-1}^{-1}$. This makes $(0, 1)$ an eigenvector of every map $\tilde{f}_n'(0)$, which means $\tilde{f}_n'(0)$ is lower triangular. The basin of the sequence $\tilde{f}_1, \tilde{f}_2, \ldots$ is exactly equal to $U_0(\Omega)$, so it is in particular biholomorphic to $\Omega$. Because the unitary matrices $U_n$ are arbitrarily close to the identity matrix, we have that $\tilde{f}_n$ is arbitrarily close to $F$, and it follows that the sequence $\tilde{f}_1, \tilde{f}_2, \ldots$ satisfy the properties in Theorem 4.5 and we are done.

We will now use a similar argument to prove Theorem 4.1 in the general case. We will denote by $G(m, k)$ the Grassmanian of all $m$-dimensional linear subspaces of $\mathbb{P}^k$.

**Lemma 4.6.** *Let $A$ be a lower triangular $k \times k$ matrix whose diagonal entries $\lambda_1, \lambda_2, \ldots \lambda_k$ satisfy $|\lambda_1| \geq \cdots \geq |\lambda_k|$. Suppose that $l \in \{1, \ldots k-1\}$ is such that $|\lambda_l| > |\lambda_{l+1}|$, and let $L$ be the linear subspace of $\mathbb{P}^{k-1}$ defined by*

$$L = \{[z_1 : \cdots : z_k] \in \mathbb{P}^{k-1} \mid z_1 = \cdots = z_l = 0\}.$$

*Then there exists an arbitrarily small neighborhood $\mathcal{M} \subset G(k-l-1, k-1)$ of $L$ such that $A(\mathcal{M}) \supset\supset \mathcal{M}$.*

*Proof.* First assume that $A$ is diagonal. For $X \in G(k-l-1, k-1)$ close enough to

$L$ we can write

$$X = \{[z_1 : \cdots : z_k] \in \mathbb{P}^{k-1} \mid z_1 = \sum_{j \geq l} \epsilon_{1,j} z_j, \ldots, z_l = \sum \epsilon_{l,j} z_j\}.$$

We define $\mathrm{d}(X, L) = \max |\epsilon_{i,j}|$.

For small $\delta > 0$ let $\mathcal{M}_\delta = \{X \in G(k-l-1, k-1) \mid \mathrm{d}(X, L) < \delta\}$. Since $A$ is diagonal and $|\lambda_l| > |\lambda_{l+1}|$ for all $i \leq l$ and $j \geq l+1$, we have that

$$A(\mathcal{M}_\delta) \supset \mathcal{M}_{\frac{|\lambda_l|}{|\lambda_{l+1}|}\delta} \supset\supset \mathcal{M}_\delta$$

Fix some $\delta$. Then $\mathcal{M}_\delta$ will also work for arbitrarily small perturbations of the diagonal matrix $A$.

We can conjugate any $A$ with an invertible linear mapping $T$ such that $T^{-1}AT$ is lower triangular and arbitrarily close to a diagonal mapping (whose diagonal entries are exactly those of $A$). Hence the set $\mathcal{M} = T(\mathcal{M}_\delta)$ will suffice for $A$. $\square$

We continue the proof of Theorem 4.1. Let $l \in \{1, \ldots, k-1\}$ be such that $|\lambda_l| > |\lambda_{l+1}| = |\lambda_k|$, and let $\Phi$ be the action of $F'(0)$ on $\mathbb{P}^{k-1}$ (as in the 2-dimensional case, we may assume that such an $l$ exists, otherwise we are done by Theorem 3.6). It follows from Lemma 4.6 that there exists a small neighborhood $\mathcal{M} \in G(k-l-1, k-1)$ of $L$ (where $L$ is as in Lemma 4.6), such that $\Phi(\mathcal{M}) \supset\supset \mathcal{M}$. As in the 2-dimensional case, we denote by $\phi_n$ the action of $f_n$ on $\mathbb{P}^{k-1}$. We can decrease the value of $\epsilon$ if necessary to get that $\phi_n(\mathcal{M}) \supset\supset \mathcal{M}$ for every $n \in \mathbb{N}$, so we can find an orbit of linear subspaces $L_0, L_1, \ldots$ in $\mathcal{M}$ with $\phi_n(L_{n-1}) = L_n$ for every $n \in \mathbb{N}$.

We denote by $U_n$ a unitary matrix arbitrarily close to the identity that maps a representative of $L_n$ in the unit sphere in $\mathbb{C}^k$ onto the set

$$T = \{z \in \mathbb{C}^k \mid \|z\| = 1, z_1 = \cdots = z_l = 0\},$$

and we define the maps $\tilde{f}_n$ by

$$\tilde{f}_n = U_n f_n U_{n-1}^{-1}.$$

We then have that for every $v \in T$ there exists a $c \in (0,1)$ such that $\tilde{f}_n'(0)(v) \in cT$, and therefore we have that for any $n \in \mathbb{N}$ the entries in the $p$-th rows and $q$-th columns of the matrix $f_n'(0)$ are equal to 0 for $p \leq l$ and $q \geq l+1$. Since the matrices $U_n$ are arbitrarily close to the identity, we have that $\tilde{f}_n'(0)$ is arbitrarily close to $F'(0)$.

We may also assume that the $(k-l) \times (k-l)$ blocks in the lower right corner of the matrices $\tilde{f}_n'(0)$ are lower triangular, since we can apply QR-factorizations as in Remark 4.3 to these $(k-l) \times (k-l)$ blocks. The diagonal entries in these blocks must be arbitrarily close to $|\lambda_k|$ in absolute value, since we made sure that $F'(0)$ is arbitrarily close to a diagonal matrix and we have since only composed $F$ with unitary matrices arbitrarily close to the identity. Also, the off-diagonal terms in the $(k-l) \times (k-l)$ block in the lower right corner of the matrices $\tilde{f}_n'(0)$ are arbitrarily small.

Since the last $k-l$ columns of the matrices $\tilde{f}_n'(0)$ are already lower triangular, and the corresponding eigenvalues are the smallest in absolute value, we can restrict ourselves to the first $l$ dimensions and apply the same arguments to the next eigenvalues of $F'$ which are equal in absolute value.

In finitely many steps, we get that all the maps $\tilde{f}_n'(0)$ are lower triangular matrices with diagonal entries arbitrarily close to the original diagonal entries in absolute value, and with off-diagonal terms arbitrarily small. Therefore, all the maps in the sequence $f_1, f_2, \ldots$ are correctly ordered.

We constructed the unitary matrices $U_0, U_1, \ldots$ such that for every $n \in \mathbb{N}$ we have $\tilde{f}(n) = U_n f(n) U_0^{-1}$. It follows that the basin of attraction of the sequence $\tilde{f}_1, \tilde{f}_2, \ldots$ is equal to the image of the basin of attraction of the original sequence $f_1, f_2 \ldots$ under the map $U_0^{-1}$, which is a biholomorphism. Thus the sequence $\tilde{f}_1, \tilde{f}_2, \ldots$ satisfies the conditions in Theorem 4.5 and the proof of Theorem 4.1 is completed. □

## 4.4 Main Ideas for Theorem 4.5

In the proof of Theorem 4.5, we will construct a sequence of maps $\Psi_n := G(n)^{-1} \circ X_n \circ F(n)$ which converges to a biholomorphism $\Psi : \Omega \to \mathbb{C}^k$. Here the $G_n$'s are *lower triangular polynomial mappings*, and the $X_n$'s are polynomial mappings whose linear parts are the identity. To be more specific, we will start with some choice for $X_1$ and then define $X_n = [G_n \circ X_{n-1} \circ F_n^{-1}]_d$. Here $d$ is some large integer and we mean by $[\cdot]_d$ that we discard all terms of degree $d+1$ and higher. It follows immediately that

$$\|G_n^{-1} X_n F_n(z) - X_{n-1}(z)\| = O(\|z\|^{d+1}). \tag{4.3}$$

The challenge is to choose a bounded sequence of lower triangular polynomial mappings $\{G_n\}$ so that there exists a bounded orbit of polynomial mappings $\{X_n\}$ for $d$ arbitrarily large. We will first show that we can do this in some simpler cases before we complete the proof of theorem 4.5.

Recall that a polynomial mapping $G = (g_1, \ldots g_k)$, of $\mathbb{C}^k$ with $G(0) = 0$ is called lower triangular if

$$g_j(z) = c_j z_j + h_j(z_1, \ldots z_{j-1}),$$

for all $j \in \{1, \ldots, k\}$.

**Lemma 4.7.** *Let $G_1, G_2, \ldots$ be a sequence of lower triangular polynomial mappings of some fixed degree, whose coefficients are uniformly bounded. Then we have the following:*

*(a) The degrees of the maps $G(n)$ are bounded, and there is a constant $\beta < \infty$ so that*

$$G(n)(\mathbf{B}) \subset B(\beta^n)$$

*(b) If there exists a $\theta < 1$ such that $|c_i| < \theta$ for all $G_j$, then $G(n)(z) \to 0$ uniformly on compact subsets of $\mathbb{C}^k$, and for every $R > 0$ and $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that for all $n \geq N$ we have that*

$$G(n)(B(R)) \subset B(\epsilon).$$

This lemma is similar to Lemma 1 in the appendix of [RR88], as is its proof.

We say that $F_1, F_2, \ldots$ with $F_n(z) = a_n z + b_n$ is a uniformly bounded sequence of expending affine maps if the constants $a_n$ and $b_n$ are bounded from above, and that the $a_n$'s are bounded from below by some $c > 1$.

**Lemma 4.8.** *Every uniformly bounded sequence of expanding affine maps $F_1, F_2, \ldots$ has a unique bounded orbit $z_0, z_1, z_2, \ldots$.*

*Proof.* Since the sequence is uniformly bounded and expanding, there exists a constant $R$ such that

$$F_i(B(R)) \supset\supset B(R)$$

for all $i \in \mathbb{N}$. Now we have that

$$B(R) \supset\supset F(1)^{-1}B(R) \supset\supset F(2)^{-1}B(R) \supset\supset \cdots,$$

so the intersection, $\bigcap F(n)^{-1}B(R)$, contains a unique point $z_0$. This means that the sequence $z_0, z_1, z_2, \ldots$, where $z_n = F_n(z_{n-1})$, is a bounded orbit. Indeed, $|z_n| < R$ for all $n \in \mathbb{N}$.

It is clear that the point $z_0$ is independent of $R$ (as long ass $R$ is large enough, so $z_0, z_1, \ldots$ is indeed the unique bounded orbit. $\square$

The argument showing that there exist bounded sequences $G_n$ and $X_n$ is somewhat complicated. To make it clearer we will first prove the existence in the one-dimensional case where the argument is much easier.

Let $f_1, f_2, \ldots$ be a sequence of polynomials with uniformly bounded coefficients of the form $f_n(z) = \lambda_n z + h.o.t.$, where $|\lambda_n| > \theta > 1$, and let $g_n(z) = \lambda_n^{-1} z$. Let $\mathcal{P}_d$ be the space of polynomials of the form $z + h.o.t.$ of degree at most $d$. We define the map $\phi_n : \mathcal{P}_d \to \mathcal{P}_d$ by $\phi_n(X) = [g_n \circ X \circ f_n]_d$.

**Proposition 4.9.** *There exists a unique bounded sequence $X_0, X_1, \ldots \in \mathcal{P}_d$ with $X_n = \phi_n(X_{n-1})$.*

*Proof.* We will use induction on the degree $d$. For degree 1 the statement is clear since we have $X_n(z) = z$ for all $n$.

Suppose we have constructed a bounded orbit $Y_n$ that satisfies condition (4.3) for $d - 1 \geq 1$. We will add terms of degree $d$ to the $Y_n$'s so that the sequence satisfies condition (4.3) for $d$. Let us say $X_0 = Y_0 + c_0 z^d$ for some $c_0 \in \mathbb{C}$. Then it is easy to see that $X_1 = \phi_1(X_0) = Y_1 + c_1 z^d$, where $c_1$ is equal to $\lambda_1^{d-1} c_0$ plus some constant depending linearly on the coefficients of $Y_0$ and the higher order terms of $f_1$ (which are uniformly bounded by assumption). Continuing inductively we get a uniformly bounded sequence of expanding affine maps $\tilde{\psi}_n : \mathbb{C} \to \mathbb{C}$ that take $c_{n-1}$ to $c_n$. It now follows from Lemma 4.8 that there is a unique $c_0$ such that the sequence $c_0, c_1, \ldots$ is bounded and so it follows by induction that we can find a unique bounded sequence $X_n$ for any $d \in \mathbb{N}$. $\square$

Now that we have shown the idea in the one-dimensional case, we will show that we can get bounded sequences $\{G_n\}$ and $\{X_n\}$ in the two dimensional case for degree 2, where we can make explicit calculations. We let $F_n(x, y) = (\lambda_n x, \mu_n y + a_n x) + h.o.t.$,

$(F_n^{-1})(x,y) = (\lambda_n^{-1}x, \mu_n^{-1}y + b_n x) + h.o.t.$ and let $G_n(x,y) = (\lambda_n x, \mu_n y + a_n x + d_n x^2)$ for some constants $d_n \in \mathbb{C}$ to be chosen later. We will also set $X_n(x,y) = (x + \alpha_n y^2 + \beta_n xy + \gamma_n x^2, y + \delta_n y^2 + \epsilon_n xy + \zeta_n x^2)$, so we can identify the map $X_n$ with $(\alpha_n, \beta_n, \gamma_n, \delta_n, \epsilon_n, \zeta_n) \in \mathbb{C}^6$. Consider the map $X_{n-1} \mapsto X_n = [G_n \circ X_{n-1} \circ F_n^{-1}]_2$. We have that

$$\begin{aligned}
\alpha_n &= \lambda_n \mu_n^{-2} \alpha_{n-1} + l_{1,n}, \\
\beta_n &= \mu_n^{-1} \beta_{n-1} + l_{2,n}(\alpha_{n-1}), \\
\gamma_n &= \lambda_n^{-1} \gamma_{n-1} + l_{3,n}(\alpha_{n-1}, \beta_{n-1}), \\
\delta_n &= \mu_n^{-1} \delta_{n-1} + l_{4,n}(\alpha_{n-1}), \\
\epsilon_n &= \lambda_n^{-1} \epsilon_{n-1} + l_{5,n}(\alpha_{n-1}, \beta_{n-1}, \delta_{n-1}), \text{ and} \\
\zeta_n &= \mu_n \lambda_n^{-2} \zeta_{n-1} + l_{6,n}(\alpha_{n-1}, \dots, \epsilon_{n-1}) + d_n \lambda_n^{-2}.
\end{aligned}$$

Here the $l_{i,n}$ are linear maps depending only on the coefficients of $F_n^{-1}$ (which are uniformly bounded) and on the given variables.

It follows from equation (4.2) that $|\lambda_n \mu_n^{-2}| > 1$ for any $n$, and therefore we get a uniformly bounded sequence of expanding affine maps $\alpha_{n-1} \mapsto \alpha_n$. Hence by Lemma 4.8, we can find $\alpha_0$ such that the sequence $\alpha_0, \alpha_1, \dots$ is bounded. Having fixed the $\alpha_n$'s, we can use the same argument for the $\beta_n$'s, since we also have that $|\mu_n^{-1}| > 1$ for all $n$. After we fix the $\beta_n$'s, we can similarly fix the $\gamma_n$'s, the $\delta_n$'s, and finally the $\epsilon_n$'s.

We cannot use the same argument for the $\zeta_n$'s, since we may not have that $|\mu_n \lambda_n^{-2}| > 1$. However, we can choose $\zeta_0 = 0$, and then choose the constants $d_n$ such that $\zeta_n = 0$ for every $n$. This gives us the bounded sequences $\{G_n\}$ and $\{X_n\}$.

## 4.5 Basins of Uniformly Attracting Automorphisms

In this section we will give the complete proof of Theorem 4.5. The argument that we use to construct bounded sequences $\{X_n\}$ and $\{G_n\}$ for higher dimensions and higher degrees is essentially the same as the argument for degree 2 polynomial mappings in $\mathbb{C}^2$ that we gave in the previous section.

We will write $\lambda_{n,1}, \ldots, \lambda_{n,k}$ for the diagonal entries of $F_n'(0)$.

**Proposition 4.10.** *For any $d \geq 2$ we can find a bounded sequence of polynomial mappings $X_0, X_1, \ldots$, where $X_n = I_k + h.o.t.$, and a bounded sequence of lower triangular polynomial mappings $G_1, G_2, \ldots$ with $G_n'(0) = F_n'(0)$, such that (4.3) holds for any $n \geq 1$.*

*Proof.* We will construct bounded sequences $X_1, X_2, \ldots$ and $G_1, G_2, \ldots$ such that the following equation holds:

$$X_n = [G_n X_{n-1} F_n^{-1}]_d, \tag{4.4}$$

for every $n \in \mathbb{N}$. Notice that this will imply that Equation (4.3) holds for every $n \in \mathbb{N}$.

We write

$$X_n(z) = (x_{n,1}(z), x_{n,2}(z), \ldots, x_{n,k}(z)),$$

and

$$x_{n,j} = \sum_{\alpha} c_{n,j,\alpha} z^{\alpha},$$

where $\alpha$ is a $k$-tuple and $z^{\alpha} = z_1^{\alpha_1} \cdots z_k^{\alpha_k}$. We refer to $c_{n,j,\alpha} z^{\alpha}$ as a term of degree $|\alpha| = \alpha_1 + \cdots + \alpha_k$, index $j$, and power $\alpha$. We similarly write $g_{n,j,\alpha} z^{\alpha}$ for the term of the mapping $G_n$ of degree $|\alpha|$, index $j$ and power $\alpha$.

For two $k$-tuples $\alpha$ and $\beta$ with $|\alpha| = |\beta|$ we will write $\alpha > \beta$ if $\alpha$ has a higher lexicographical ordering than $\beta$. That is, $\alpha > \beta$ if and only if there is some $j \in \{1, \ldots, k\}$ such that $\alpha_j < \beta_j$ and $\alpha_i = \beta_i$ for $i \in \{1, \ldots j-1\}$ (so that $z_1^d$ comes first in the alphabet and $z_k^d$ comes last).

We will use induction on the degree and index and reverse induction on the power to fix all the terms of the sequences $\{X_n\}$ and $\{G_n\}$.

Let $\alpha$ be some $k$-tuple, and let $j \in \{1, \ldots, k\}$. Suppose that we have fixed all the terms of degree up to $|\alpha| - 1$ in the sequences $\{X_n\}$ and $\{G_n\}$, such that (4.4) holds for $d = |\alpha| - 1$, and assume that the corresponding coefficients are uniformly bounded. Also assume that we have fixed all terms of degree $|\alpha|$ and index up to $j-1$ in the sequences $\{X_n\}$ and $\{G_n\}$ such that (4.4) holds for index $1, \ldots, j-1$ and $d = |\alpha|$. Finally, assume that all terms of degree $|\alpha|$, index $j$ and powers $\beta > \alpha$ in the sequences $\{X_n\}$ and $\{G_n\}$ are fixed such that (4.4) holds for the terms of degree $|\alpha|$, index $j$ and power $\beta$.

We will continue to choose the terms of index $j$ and power $\alpha$ in the sequences $\{X_n\}$ and $\{G_n\}$. It is clear that after fixing the $c_{n,j,\alpha}$'s and $g_{n,j,\alpha}$'s, Equation (4.4) will still hold for $d = |\alpha| - 1$. Since the linear parts of the $G_n$'s are lower triangular, it also follows that (4.4) will still hold for degree $d = |\alpha|$ and index $1, \ldots, j-1$. Furthermore, it follows from the fact that $F_n'(0)$ is lower triangular that (4.4) will still hold for the terms of degree $d = |\alpha|$, index $j$ and all powers $\beta$ with $\beta > \alpha$.

Once some $c_{n-1,j,\alpha}$ is chosen, it follows from (4.4) that we must have

$$c_{n,j,\alpha} = \lambda_{n,j} \lambda_n^{-\alpha} c_{n-1,j,\alpha} + g_{n,j,\alpha} \lambda_n^{-\alpha} + C_{n,j,\alpha} \tag{4.5}$$

where the constant $C_{n,j,\alpha}$ depends on the terms of $f_n^{-1}$ (which are uniformly bounded for any fixed degree) and the terms of $G_n$ and $X_{n-1}$ that we have already fixed (which are also uniformly bounded per hypothesis).

If the term of index $j$ and power $\alpha$ is lower triangular (i.e. if $\alpha_i = 0$ for $i \in \{j, \dots k\}$), then we can choose $c_{n,j,\alpha} = 0$ for all $n \in \mathbb{N}$ and $g_{n,j,\alpha} = -C_{n,j,\alpha}\lambda_n^\alpha$ so that (4.5) holds for every $n \in \mathbb{N}$. It follows that the constants $g_{n,j,\alpha}$'s are bounded.

If the term of index $j$ and power $\alpha$ is not lower triangular, it follows from the hypothesis that the sequence $\{F_n\}$ is uniformly attracting that $|\lambda_{n,j}\lambda_n^{-\alpha}| > \xi > 1$ (where $\xi$ is as in Definition 4.4). We also have that $g_{n,j,\alpha} = 0$ for all $n \in \mathbb{N}$, and it follows that the sequence of maps given by $c_{n-1,j,\alpha} \mapsto c_{n,j,\alpha}$ is a uniformly bounded sequence of affine maps on $\mathbb{C}$ that are all expanding. It then follows from Lemma 4.8 that there exists a bounded sequence $c_{n,j,\alpha}$.

So whether the term is lower triangular or not, we can always choose bounded sequences $\{x_{n,j,\alpha}\}$ and $\{g_{n,j,\alpha}\}$ such that (4.4) is satisfied.

The proposition follows by induction. □

Now let $p$ be so large that for every $n \in \mathbb{N}$ the eigenvalues $\lambda_{n,1}, \dots \lambda_{n,k}$ of $F_n'(0)$ satisfy $|\lambda_{n,i}|^p < \xi|\lambda_{n,j}|$ for $i, j \in \{1, \dots, k\}$ where $\xi < 1$ as in Definition 4.4. We can do this because the sequence $\{F_n\}$ is uniformly attracting. We construct sequences $\{G_n\}$ and $\{X_n\}$ as in Proposition 4.10 for $d = p$. It follows from part (a) of Lemma 4.7 that there exists a $\gamma > 1$ such that

$$\|G(n)^{-1}(w) - G(n)^{-1}(w')\| \leq \gamma^n \|w - w'\|,$$

for any $w, w' \in \mathbf{B}$. Recall that we assumed that there is a constant $b < 1$ such that $\|F_n(z)\| < b\|z\|$ for any $z \in \mathbf{B}$, and fix an integer $q$ such that $\gamma b^q < \alpha < 1$. We can change the mappings $X_n$ by adding higher order terms such that (4.3) holds for $d = q + 1$. Since we chose $p \in \mathbb{N}$ such that $|\lambda_i|^p < \xi|\lambda_j|$ holds for all $i, j$, we have that $|\lambda_{n,j}\lambda_n^{-\alpha}| > \xi$ in (4.5) even for the terms that are lower triangular. This means we can guarantee that the altered sequence $\{X_n\}$ is bounded without changing the sequence $\{G_n\}$, and therefore we have that

$$\|G_n^{-1} X_n F_n(z) - X_{n-1}(z)\| \leq C\|z\|^{q+1}, \tag{4.6}$$

for some $C > 0$ independent of $n \in \mathbb{N}$ and every $z \in \mathbf{B}$.

The proof of Theorem 4.5 now follows quickly from an argument similar to that of the proof of Theorem 1.1 in the appendix of [RR88]:

Define the maps $\Psi_n : \Omega \to \mathbb{C}^k$ by

$$\Psi_n := G(n)^{-1} \circ X_n \circ F(n).$$

We will show that the maps $\Psi_n$ converge uniformly on compact subsets of $\Omega$ to a biholomorphic map from $\Omega$ onto $\mathbb{C}^k$.

Since the higher degree terms of the mappings $\{X_n\}$ are uniformly bounded there is a radius $r < 1$ such that all the maps $X_n$ are invertible on $B(r)$.

Let $K$ be a compact set in $\Omega$, let $m \in \mathbb{N}$ such that $F(m)(K) \subset B(r)$, and let $n \geq m$. Then for all $z \in B(r)$ we have that $\|F(m,n)(z)\| \leq b^{n-m}\|z\|$, where $b$ is as in Definition 4.4.

We notice that

$$\|\Psi_{n+1}(z) - \Psi_n(z)\| = \| \left(G(n)^{-1} G_{n+1}^{-1} X_{n+1} F_{n+1} - G(n)^{-1} X_n\right) F(n)(z)\| \tag{4.7}$$

$$\leq C\gamma^n (b^{n-m}\|F(m)(z)\|)^{q+1} < \tilde{C}\alpha^n \|F(m)(z)\|^{q+1}. \tag{4.8}$$

Since the sequence $\{\alpha^n\}$ is summable, the maps $\Psi_n$ converge uniformly on compact subsets of $\Omega$ to a holomorphic map $\Psi$. Also, for any compact subset $K$ of $\Omega$, there is a large $N \in \mathbb{N}$ such that for $n \geq N$, $\Psi_n$ is biholomorphic on K. It is a well known fact that the limit of a convergent sequence of biholomorphic mappings is either injective or degenerate everywhere. But we have $\Psi_n'(0) = I$ for all $n$, and therefore $\Psi'(0) = I$ so the limit map $\Psi$ is injective.

To prove surjectivity of the map $\Psi$, we may assume that we have chosen $r$ so small that

$$r^q \sum_{n=1}^{\infty} (C\alpha^n) < \frac{1}{2}.$$

It follows from estimates (4.7) and (4.8) that for $z \in B(r)$ and for any $n > m \geq 1$ we have that

$$\|G(m,n)^{-1} X_n F(m,n)(z) - z\| \leq \frac{\|z\|}{2}$$

Now let $K_m$ be the compact subset of the basin of attraction such that $F(m)(K_m) = B(r)$. Then we have that $G(m,n)^{-1} X_n F(n)(K_m) \supset B(r/2)$ for any $n > m$. It follows from part (b) of Lemma 4.7 that for every $R > 0$ there exists an $N \in \mathbb{N}$ such that for $m \geq N$ we have that

$$B(R) \subset G(m)^{-1} \left( B(r/2) \right).$$

Therefore $B(R) \subset \Psi(K_m)$ for large enough $m$, and thus we have that $\Psi(\Omega)$ contains balls around the origin with arbitrarily large radii. This completes the proof. $\square$

CHAPTER V

# Basins of Repeated Automorphisms

## 5.1 Repeated Automorphisms

In this chapter we will study sequences of maps in $\mathrm{Aut}_0(\mathbb{C}^k)$ that are repeated many times, and their basins of attraction. When a sequence of integers $n_1, n_2, \ldots$ is given and the maps $f_j$ are iterated $n_j$ times we will write $F_j = f_j^{n_j}$. We also use the notation $N_j = n_j + \cdots + n_1$, and as before we will write

$$F(j) = F_j \cdots F_1 \text{ and } F(n,m) = F(m)F(n)^{-1}.$$

For $n = N_j + m$, where $m < n_{j+1}$, we also let

$$f(n) = f_{j+1}^m F(j) \text{ and } f(p,q) = f(q)f(p)^{-1}.$$

Notice that $f(n)$ is the composition of $n$ maps $f_j$, while $F(n)$ is the composition of $n$ maps $F_j$.

Let $f_1, f_2, \ldots \in \mathrm{Aut}_0(\mathbb{C}^k)$ and assume that every $f_n$ is attracting at 0. Suppose that there exist non-negative real numbers $r_1, r_2, \ldots$ that are uniformly bounded from above such that $f_j(B(r_j)) \subset B(r_{j+1})$ for all $j \in \mathbb{N}$. Then we define the $\{r_j\}$-*calibrated basin of attraction* of the sequence $f_1, f_2, \ldots$ as

$$\Omega(\{r_j\}) = \bigcup f(j)^{-1} B(r_{j+1}).$$

*Remark* 5.1. If $f(j)(z) \in B(r_{j+1})$ guarantees that the orbit of $z$ converges to 0, then the $\{r_j\}$-calibrated basin of attraction is a subset of the basin of attraction. If we also have that the radii $\{r_j\}$ are uniformly bounded from below then $\Omega(\{r_j\}) = \Omega$. However, if the radii $r_j$ converge to 0, then there may be orbits that converge to 0 but that do not reach the balls $B(r_j)$ at the appropriate stage. This situation can indeed occur; for instance let $f_1, f_2, \ldots$ be a sequence of automorphisms that satisfy the hypotheses of Theorem 3.4, and let $r_1, r_2, \ldots > 0$ be such that $B(r_j)$ is contained in the basin of attraction of the mapping $f_j$. It follows from Theorem 5.2 below that there exist $n_1, n_2, \ldots \in \mathbb{N}$ such that the $\{r_j\}$-calibrated basin of attraction of the sequence $f_1^{n_1}, f_2^{n_2}, \ldots$ is biholomorphic to $\mathbb{C}^k$. However, the sequence $f_1, \ldots, f_1, f_2, \ldots, f_2, f_3, \ldots$, where $f_j$ is repeated $n_j$-times, still satisfies the hypotheses of Theorem 3.4, so the basin of attraction of this sequence is not biholomorphic to $\mathbb{C}^k$ and in particular not equal to the $\{r_j\}$-calibrated basin of attraction of the same sequence.

The main result of this chapter is the following:

**Theorem 5.2.** *Let $f_1, f_2, \ldots$ be a sequence in $\mathrm{Aut}_0(\mathbb{C}^k)$, and let $r_1, r_2, \ldots > 0$ be such that $B(r_j) \subset\subset \Omega_j$, where $\Omega_j$ is the basin of attraction of the mapping $f_j$. Then there exist large enough integers $n_1, n_2, \ldots$ such that the $\{r_j\}$-calibrated basin of attraction of the sequence $f_1^{n_1}, f_2^{n_2}, \ldots$ is biholomorphic to $\mathbb{C}^k$.*

To prove this result we will make will use of the ideas in the proof of Theorem 1.1 outlined in Chapter 2, as well as the following Theorem, due to Forstnerič [For99] and Weickert [Wei97] independently:

**Theorem 5.3.** *For $k \geq 2$ let $P = (P_1, \ldots, P_k)$ be a polynomial endomorphism of $\mathbb{C}^k$, where $P'(0)$ is invertible. Let $d \geq \max_i(\deg(P_i))$. Then there exists $\phi \in \mathrm{Aut}(\mathbb{C}^k)$*

*such that the d-jet of $\phi$ at* 0 *equals* $P$.

This result will be used to extend the local conjugations used in the proof of Theorem 1.1 to global automorphisms of $\mathbb{C}^k$.

## 5.2 Basins of Repeated Automorphisms Biholomorphic to $\mathbb{C}^k$

In this section, we will be working with invertible maps $T_j, g_j$ and $G_j$, where $G_j = g_j^{n_j}$. We will use the notation

$$G(j) = T_j^{-1} G_j T_j \cdots T_1^{-1} G_1 T_1, \text{ and } G(n,m) = G(m)G(n)^{-1},$$

and for $n = N_j + m$ we let

$$g(n) = T_{j+1}^{-1} g_j^m T_{j+1} G(j) \text{ and } g(p,q) = g(q)g(p)^{-1}.$$

Usually we will be working with inverse mappings for the $G$'s, so then we can think of $G(n,m)^{-1}$ as the map that takes us back from stage $m$ to stage $n$. The definition for $G(n)$ may not be entirely consistent with the definition for $F(n)$, but this will not cause any problems.

*Remark* 5.4. Before we give the proof of Theorem 5.2, we must first show that $\Omega(\{r_j\})$ is well defined. Recall that we have chosen the radii $\{r_j\}$ such that $B(r_j) \subset\subset \Omega_j$ for every $j$. Therefore we can choose $n_j$ large enough such that $F_j(B(r_j)) \subset B(r_{j+1})$, so that the $\{r_j\}$-calibrated basin of attraction is well-defined.

By choosing $n_j$ larger if necessary, we can also guarantee that $\|F_j(z)\| \leq \frac{1}{2}\|z\|$ for any $j \in \mathbb{N}$ and $z \in B(r_j)$, and hence we can get $\Omega(\{r_j\}) \subset \Omega$. Note that if there exists a radius $r > 0$ such that $B(r) \subset \Omega_j$ for every $j \in \mathbb{N}$ then Theorem 5.2 implies that the usual basin of attraction $\Omega$ is biholomorphic to $\mathbb{C}^k$.

We may decrease the values of the $r_j$'s in the proof, but we can always increase the integers $\{n_j\}$ later such that the definition of the basin of attraction with the smaller $r_j$'s is equivalent to the definition of the basin of attraction with the original $r_j$'s.

*Proof of Theorem 5.2:* As discussed in Chapter 2, it follows from Lemma 3 in the appendix of [RR88] that for every $j \in \mathbb{N}$ we can find:

(i) A polynomial automorphism $g_j$ of $\mathbb{C}^k$ that is linearly conjugate to a lower triangular automorphism, with $g_j(0) = 0$ and $g_j'(0) = f_j'(0)$.

(ii) For $m_j$ as large as we desire, a polynomial map $\phi_j : \mathbb{C}^k \to \mathbb{C}^k$ with $\phi_j(0) = 0, \phi_j'(0) = I$, for which the following equation holds:

$$g_j^{-1}\phi_j f_j - \phi_j = O(\|z\|^{m_j}).$$

It follows from Theorem 5.3 that for every $j \in \mathbb{N}$ we can find an automorphism $T_j$ with $T_j - \phi_j = O(\|z\|^{m_j})$, so we get the equation:

$$T_j^{-1}g_j^{-1}T_j f_j - I = O(\|z\|^{m_j}).$$

Furthermore there exist constants $\gamma_j$ such that

$$\|g_j^{-n}(w) - g_j^{-n}(w')\| \leq \gamma_j^n \|w - w'\|$$

for all $w, w'$ in $\mathbf{B}$ and all natural numbers $n$. Also recall that the $g_j$'s are attracting at 0, and that the basins of the maps $g_j$ are all of $\mathbb{C}^k$.

By replacing $f_j$ by a high iterate of $f_j$ if necessary, we may assume that $\|f_j(z)\| \leq \frac{1}{2}\|z\|$ for all $j \in \mathbb{N}$ and all $z \in B(r_j)$. We now choose the $m_j$'s so large that

$$\frac{\gamma_j}{2^{m_j}} < 1.$$

and $m_{j+1} > m_j$ for every $j \in \mathbb{N}$. It then follows from equation (6) of the proof of the theorem in the appendix of [RR88] that there exist constants $C_j$ so that for every $j$ and every $z \in B(r_j)$ the following holds:

$$\|T_j^{-1}g_j^{-n-1}T_j f_j^{n+1}(z) - T_j^{-1}g_j^{-n}T_j f_j^n(z)\| \leq C_j(\frac{\gamma_j}{2^{m_j}})^n \|z\|^{m_j}.$$

Since $(\frac{\gamma_j}{2^{m_j}})^n$ is summable in $n$ and the other terms do not depend on $n$, we can decrease the radii $r_j$ if necessary such that for $z \in B(r_j)$ we have

$$\sum_{n\geq 0} \|T_j^{-1} g_j^{-n-1} T_j f_j^{n+1}(z) - T_j^{-1} g_j^{-n} T_j f_j^n(z)\| \leq \frac{1}{2}\|z\|^{m_j-1}. \tag{5.1}$$

It follows that for $z \in B(r_j)$ and any $m$ larger than $n$ we get

$$\|T_j^{-1} g_j^{-m} T_j f_j^m(z) - T_j^{-1} g_j^{-n} T_j f_j^n(z)\| \leq \frac{1}{2}\|z\|^{m_j-1}. \tag{5.2}$$

We now inductively construct the integers $n_j$. Suppose that we have already constructed $n_1$ through $n_{j-1}$. At this stage we have fixed the automorphism $G(j-1)$, and thus the set $T_j G(j-1)(B(2^j))$ is a fixed bounded set. The basin of attraction of the lower triangular mapping $g_j$ is equal to $\mathbb{C}^k$, therefore we can choose $n_j$ large enough so that

$$T_j(G(j-1)(B(2^j))) \subset g_j^{-n_j} T_j(B(\frac{r_{j+1}}{2})), \tag{5.3}$$

and we can enlarge $n_j$ if necessary so that

$$f_j^{n_j}(B(r_j)) \subset B(r_{j+1}). \tag{5.4}$$

It follows from (5.1), (5.2), (5.4) and the fact that we chose $m_{j+1}$ strictly larger than $m_j$ that we can increase $n_j$ if necessary and choose all subsequent $n_{j+1}, n_{j+2}, \ldots$ large enough so that for every $m$ and every $z \in B(r_j)$ the following inequality will hold throughout the construction:

$$\|g(N_{j-1}, m)^{-1} f(N_{j-1}, m)(z) - z\| \leq \|z\|^{m_j-1}. \tag{5.5}$$

We will now show that with these choices the basin $\Omega(\{r_j\})$ is biholomorphic to $\mathbb{C}^k$.

Let $K_j = F(j)^{-1}(B(r_{j+1}))$. Clearly $F(j)(K_j) = B(r_{j+1})$, and thus we have that

$$\|g(N_j, m)^{-1} f(m)(z) - F(j)(z)\| \leq \|F(j)(z)\|^{m_{j+1}-1}$$

for every $z \in K_j$. It follows that $H_m = g(m)^{-1} f(m)$ is bounded on every $K_j$, so we can find a subsequence of $\{H_m\}$ that converges uniformly on $K_1$, say to a map $H$. We have that $K_1 \subset K_2 \subset \ldots$ and thus we can apply the same argument to every $K_j$, and by a diagonal argument, we see that $H$ extends to $\bigcup K_j = \Omega(\{r_j\})$. We claim that $H$ maps $\Omega(\{r_j\})$ biholomorphically onto $\mathbb{C}^k$.

As we stated in the previous chapter, the limit map of a sequence of biholomorphic maps is either degenerate everywhere or it is one to one. $H$ is one to one since $H'_m(0) = I$ for all $m$.

We still need to show that $H$ is surjective. Notice that it follows from (5.2) and (5.4) that

$$g(N_j, m)^{-1} f(m)(K_j) \supset B(\frac{r_{j+1}}{2})$$

for all $m$. Therefore by (5.3), $H_n(K_j)$ contains $B(2^j)$ for all $n \geq N_j$, and thus $B(2^j) \subset H(K_j)$, which means $H$ maps $\Omega(\{r_j\})$ biholomorphically onto $\mathbb{C}^k$. This completes the proof. $\square$

# CHAPTER VI

# Boundaries of Fatou-Bieberbach Domains

## 6.1 Fatou-Bieberbach Domains

The results of Stensønes and Wolf discussed in Chapter 2 suggest two open questions about the Hausdorff dimension of the boundary of a Fatou-Bieberbach domain:

i) Is it possible for the dimension of the boundary to be less than 3?

ii) Is it possible that the dimension of the boundary is exactly 4?

We will show that the answer to question (i) is no if we assume that the Fatou-Bieberbach domain is Runge. It is unknown whether there exist Fatou-Bieberbach domains that are not Runge, but a Fatou-Bieberbach domain that is the basin of attraction of a sequence of automorphisms is always Runge.

We will answer question (ii) by proving the following result:

**Theorem 6.1.** *There exists a Fatou-Bieberbach domain $\Omega \subset \mathbb{C}^2$ with $\partial\Omega = \partial\overline{\Omega}$ whose boundary $\partial\Omega$ has non-zero 4-dimensional Hausdorff measure near any of its points.*

It will be clear from the proof that we can get the equivalent result in higher dimensions. In fact, all the results in this section will hold in $\mathbb{C}^k$ for $k \geq 2$, we only work in $\mathbb{C}^2$ because of convenience and because Fatou-Bieberbach domains are traditionally defined in $\mathbb{C}^2$.

Before we prove Theorem 6.1, we will construct a Fatou-Bieberbach domain whose boundary has upper box dimension 4. This basin will be the limit of a sequence of Fatou-Bieberbach domains for whom the dimensions of the boundaries approach 4. We will use Wolf's result, Theorem 2.5, to obtain the sequence of Fatou-Bieberbach domains in this construction, and then use Theorem 5.2 to show that the limit domain is also a Fatou-Bieberbach domain. We will not be able to show that the Hausdorff dimension of the limit set is 4; somehow upper box dimension works better in conjunction with the limiting construction that we will use.

The Fatou-Bieberbach domain that we will construct in the proof of Theorem 6.1 will also be the basin of attraction of a sequence in $\mathrm{Aut}_0(\mathbb{C}^k)$. However, we will not construct the boundary as the limit of lower dimensional sets. Instead we will start with a set of Hausdorff dimension 4, and construct a basin of attraction whose boundary includes this set.

## 6.2 Upper Box Dimension 4

For two compact subsets $A, B \subset \mathbb{C}^k$, we use the usual definition for the Hausdorff distance between $A$ and $B$, namely

$$d_H(A, B) = \max\{d(A, B), d(B, A)\},$$

where

$$d(A, B) = \sup_{x \in A} \inf_{y \in B} d(x, y).$$

Let $h_1, h_2, \ldots$ be an increasing sequence of real numbers that converge to 4, let $f_1, f_2, \ldots$ be a sequence of Hénon maps as in Theorem 2.5 so that the Hausdorff dimension near any point in the Julia set of $f_j$ is equal to $\frac{h_j+4}{2}$.

By conjugating the maps $f_j$ with translations we may assume that each $f_j$ has an attracting fixed point at 0. Of course, the Hausdorff-dimension properties of the maps are unchanged by this conjugation. For every map there is a small ball centered at 0, say $B(r_j)$, such that the action of $f_j$ is contracting on $B(r_j)$.

In this section we will denote by $J_j^+$ the Julia set of $f_j$, and we let $R_1, R_2, \ldots \in \mathbb{R}$ be an increasing sequence with $\lim_{j\to\infty} R_j = \infty$ so that $R_j$ defines a filtration $D_j, V_j^+, V_j^-$ for the map $f_j$.

In the theorem below we will construct integers $n_1, n_2, \ldots$ and study the forward Julia set $J^+$ of the sequence $f_1^{n_1}, f_1^{n_2}, \ldots$. As in Chapter 5 we will write $F_j$ for $f_j^{n_j}$.

**Theorem 6.2.** *We can choose integers $n_1, n_2, \ldots$ large enough so that the upper box dimension of $J^+$ is equal to 4 near any point in $J^+$.*

*Proof.* First we take the $n_j$'s so large that $F_j(B(r_j)) \subset B(r_{j+1})$, $F_j(V_j^+) \subset V_{j+1}^+$, and any orbit $z_1, z_2, \ldots$ with $z_j \in B(r_{j+1})$ (or in $V_{j+1}^+$) converges to the origin (or to the attracting point at infinity respectively).

We will choose the sequence $n_1, n_2, \ldots$ inductively. Suppose that we have fixed $n_1, \ldots, n_{j-1}$.

Let $I_j$ be the forward Julia set of the sequence $F_1, F_2, \ldots F_{j-1}, f_j, f_j, \ldots$. We have $z \in I_j$ if and only if $F(j-1)(x) \in J_j^+$, so $I_j = F(j-1)^{-1} J_j^+$, and therefore we have that the Hausdorff dimension of $I_j$ near any point of $I_j$ is equal to $\dim_H(J_j^+) = \frac{h_j+4}{2}$.

Let $\mathcal{B}_j$ be a finite collection of open balls of radius $(\frac{1}{2})^j$ covering $I_j \cap D_j$ so that every element of $\mathcal{B}_j$ intersects $I_j$. Since the Hausdorff dimension of $I_j$ is strictly larger than $h_j$ near any point in $I_j$, we can choose some small $\hat{\epsilon}_j > 0$ such that $\gamma_{h_j}^{\hat{\epsilon}_j}(B \cap I_j) > 2^{j+1}$ for any $B \in \mathcal{B}_j$.

Now take $\epsilon_j < \hat{\epsilon}_j$ so that

$$\left(\frac{\hat{\epsilon}_j}{\epsilon_j}\right)^{h_j} < 2,$$

and let $\delta_j < \hat{\epsilon}_j - \epsilon_j$.

We define the set $K_j$ by

$$K_j = \{z \in D_j \mid d(z, I_j) \geq \delta_j\}.$$

For every $z \in K_j$, we have that $f_j^n F(j-1)(z)$ converges either to 0 or to the attracting point at infinity. $K_j$ is compact, so we can choose $n_j \in \mathbb{N}$ such that $F(j)(z) = f_j^{n_j} F(j-1)(z)$ lies in $B(r_{j+1})$ or $V_{j+1}^+$ for all $z \in K_j$. It follows that we can further increase $n_j$ if necessary so that for any $z$ in the compact set $I_j \cap D_j$ there exist $x, y \in B(z, \delta_j)$ such that $F(j)(x) \in B(r_{j+1})$ and $F(j)(y) \in V_{j+1}^+$. Therefore we will have that $d_H(I_j \cap D_j, J^+ \cap D_j) < \delta_j$.

Now let $z \in J^+$, $h \in (0, 4)$, and $\epsilon > 0$. It will suffice to show that the upper-box dimension of $J^+ \cap B(z, \epsilon)$ is larger than or equal to $h$.

Let $j \in \mathbb{N}$ be large enough so that we have $h_j > h$, $z \in D_j$ and $\epsilon > 3(\frac{1}{2})^j$. We can choose $B \in \mathcal{B}_j$ with $z \in B$, and since the radius of $B$ is exactly $(\frac{1}{2})^j$ we have that $B \subset B(z, \epsilon)$. Let $\{B_i\}$ be an $\epsilon_j$ covering of $J^+ \cap B(z, \epsilon)$. We will write $\tilde{B}_i$ for the ball with the same center as $B_i$ but with radius $\hat{\epsilon}_j$. Since $\delta_j < \hat{\epsilon}_j - \epsilon_j$ and $d_H(I_j \cap D_j, J^+ \cap D_j) < \delta_j$ we have that $\{\tilde{B}_i\}$ is an $\hat{\epsilon}_j$-covering of $B \cap I_j$, and thus $\hat{\epsilon}_j^{h_j} \#\{\tilde{B}_i\} > 2^{j+1}$. Since $\hat{\epsilon}_j^{h_j} / \epsilon_j^{h_j} < 2$ we have that

$$\gamma_h^{\epsilon_j}(J^+ \cap B(z, \epsilon)) > \gamma_{h_j}^{\epsilon_j}(J^+ \cap B(z, \epsilon)) > 2^j. \tag{6.1}$$

Because Equation (6.1) holds for all $j \gg 0$ , we have that $\mu_h(J^+ \cap B(z, \epsilon)) = \infty$, which completes the proof. □

We can now show:

**Corollary 6.3.** *There exists a Fatou-Bieberbach domain whose boundary has upper box dimension* 4 *near any boundary point.*

*Proof.* It follows from Theorem 5.2 that we can enlarge the integers $n_1, n_2, \ldots$ obtained in Theorem 6.2 if necessary to get that the $\{r_j\}$-calibrated basin of attraction of the sequence $f_1^{n_1}, f_2^{n_2}, \ldots$ is biholomorphic to $\mathbb{C}^k$. Since there are orbits converging to infinity, we have that $\Omega(\{r_j\})$ is a proper subset of $\mathbb{C}^2$ and thus a Fatou-Bieberbach domain.

In the proof of Theorem 6.2 we made sure that every open ball in $\mathbb{C}^k$ contained a point whose orbit either converged to the origin or to the attracting point at infinity. This means that there are only two Fatou-components one of which is $\Omega(\{r_j\})$. It follows from the proof of Theorem 6.2 that every ball centered at a point in $J^+$ contains an element of $\partial\Omega(\{r_j\})$, so $\partial\Omega(\{r_j\}) = J^+$ and we are done. □

It would be interesting to know whether the Hausdorff dimension of the boundary of $\Omega(\{r_k\})$ is also equal to 4.

### 6.3 Hausdorff Dimension 4

In the proof of Theorem 6.1 we will need to approximate local biholomorphic mappings by global automorphisms of $\mathbb{C}^k$. We will use a slight generalization of Theorem 2.3 in [FR93], which is itself a generalization of fundamental results by Andersén [And90] and Andersén and Lempert [AL92]. The generalization that we will use is the following:

**Theorem 6.4.** *Let $K_1, K_2, ..., K_m$ be pairwise disjoint polynomially convex compact sets in $\mathbb{C}^k$ whose union is polynomially convex, and assume that $K_1, K_2, .., K_l$ are star-shaped ($l \leq m$). Let $\phi_i \in \mathrm{Aut}(\mathbb{C}^k)$ be automorphisms for $i = 1, ..., l$ so that the sets $K_i' = \phi_i(K_i)$ and the sets $K_{l+1}, ..., K_m$ are pairwise disjoint, and their union is polynomially convex. Let $\epsilon > 0$. Then there exists an automorphism $\phi \in \mathrm{Aut}(\mathbb{C}^k)$, so that $\|\phi(z) - \phi_i(z)\| < \epsilon$ for all $z \in K_i$ and $i = 1, .., l$, and $\|\phi(z) - z\| < \epsilon$ for all*

*$z \in K_i$ and $i = l+1, .., m$.*

The proof of this theorem is a small modification of the proof of Theorem 2.3 in [FR93], and we give a brief outline of how one makes this modification:

Define $K_0 = \cup_{i=l+1}^{m} K_i$. Choose $R > 0$ so that $K_0 \subset B(R)$, and let $p_i$ denote the contracting center of the star-shaped set $K_i$ for $i = 1, ..., l$ ($p_i$ is such that $tz+(1-t)p_j$ is in $K_i$ for every $t \in [0,1]$ and every $z \in K_i$). Now choose appropriate $\mathcal{C}^2$ paths from the $p_i$'s to separate points outside of $\overline{B(R)}$, and use these to define an isotopy of biholomorphisms. It is clear that the arguments from the proof of Theorem 2.3 in [FR93] give an automorphism that is arbitrarily close to the identity on $K_0$ and maps the other $K_i$'s outside of $B(R)$. Now we can apply Theorem 2.3 in [FR93] to find an automorphism that moves $K_0$ far away from the other $K_i$'s and is close to the identity on the images of the other $K_i$'s. If the image of $K_0$ is far enough, we can use the inverses of our paths to map the images of the other $K_i$'s approximately back to their original positions. Now we can apply Theorem 2.3 again to get a single automorphism that approximates $\phi_i$ well on each $K_i$ for $i = 1, ...m$ and stays close to the identity on the image of $K_0$ (notice that the $K_i$'s may no longer be star-shaped, so we might have to pass to slightly bigger compact sets). Now repeat the same procedure as above to map the image of $K_0$ approximately back to $K_0$.

*Remark* 6.5. We recall and prove a few basic facts about polynomially convex compact sets that we will need later in this section:

(i) The union of a polynomially convex compact set and a finite set of points is polynomially convex.

(ii) If $K_1 \cup K_2$ is polynomially convex and compact, $K_1 \cap K_2 = \emptyset$, and $K_1' \subset K_1$ is polynomially convex and compact, then $K_1' \cup K_2$ is polynomially convex.

(iii) A polynomially convex compact set has a neighborhood basis consisting of polynomially convex compact sets.

To show (i), let $q \in \mathbb{C}^k \setminus (K \cup \{p\})$ and $f \in \mathcal{O}(\mathbb{C}^k)$ so that $f(p) = 0$ and $f(q) \neq 0$. Since $K$ is polynomially convex, there exists a $g \in \mathcal{O}(\mathbb{C}^k)$ so that $\|g\|_K < 1$ and so that $g(q) = 1$. If $m$ is large enough then the product $g^m \cdot f$ separates $q$ from $\{p\} \cup K$ (that is, $|g^m(z) \cdot f(z)| > \|g^m \cdot f\|_{\{p\} \cup K}$), and the result follows by induction.

To prove (ii), let $p \in \mathbb{C}^k \setminus (K_1' \cup K_2)$. We may assume that $p \in K_1 \setminus K_1'$. There exists an $f \in \mathcal{O}(\mathbb{C}^k)$ separating $p$ from $K_1'$. Let $g \in \mathcal{O}K_1 \cup K_2$ be defined by $g = f$ on $K_1$ and $g = 0$ on $K_2$. The Oka-Weil Theorem (see for instance [Ran86], page 220) tells us that we can approximate $g$ by polynomial mappings, and good enough approximations will separate $p$ from $K_1' \cup K_2$.

To prove (iii), observe that a polynomially convex compact set has a neighborhood basis $U_1 \supset\supset U_2 \supset\supset ...$ consisting of analytic polyhedra defined by entire functions, i.e. a Runge and Stein neighborhood basis (see for instance page 71 of [Ran86]), which means that the sets $\{\widehat{\overline{U}_i}\}$ satisfy (iii).

Theorem 8.5 in [RR88] states that for any sequence of points $\{p_j\} \subset \mathbb{C}^k$ and any strictly convex compact set $K$ such that $\{p_j\} \cap K = \emptyset$ one can find a Fatou-Bieberbach Domain $\Omega \subset \mathbb{C}^k \setminus K$ so that $\{p_j\} \subset \Omega$. We can use Theorem 6.4 and Theorem 3.6 to prove a slight generalization of this theorem:

**Theorem 6.6.** *Let $K$ be a polynomially convex compact subset of $\mathbb{C}^k$, and let $\{p_j\}_{j \in \mathbb{N}} \subset \mathbb{C}^k \setminus K$ be countable. Then there exists a Fatou-Bieberbach domain $\Omega$ so that $\{p_j\} \subset \Omega \subset \mathbb{C}^k \setminus K$.*

*Proof.* We may assume that $K$ does not intersect $\mathbf{B}$. Let A be the linear map defined by

$$A : (z_1, \cdots, z_k) \to (\frac{z_1}{2}, \cdots, \frac{z_k}{2})$$

Theorem 3.6 implies that there exists a $\delta > 0$ so that if $\{f_j\}_{j\in\mathbb{N}} \subset \mathrm{Aut}_0(\mathbb{C}^k)$ is a sequence of automorphisms with $\|f_j - A\|_{\overline{B}} < \delta$ for all $j \in \mathbb{N}$, then $\Omega$, the basin of attraction of $f_1, f_2, \ldots$, is biholomorphic to $\mathbb{C}^k$.

We will construct a sequence of automorphisms by induction, with induction hypothesis $I_j$ as follows: We have constructed automorphisms $\{f_1, ..., f_j\}$ so that:

$$\|f_i - A\|_{\overline{B}} < \delta \text{ for } i = 1, \ldots, j, \tag{6.2}$$

$$f(j)(p_i) \subset B \text{ for } i = 1, \ldots, j, \tag{6.3}$$

$$f(j)(K) \subset \mathbb{C}^k \setminus \overline{\mathbf{B}}. \tag{6.4}$$

$I_1$ is satisfied by assuming $K$ to be far enough away from the unit ball, letting $p_1$ be the origin, and defining $f_1 = A$. Now suppose $I_j$ is satisfied. Theorem 6.4 says that for any $\epsilon > 0$ there exists a $\psi \in \mathrm{Aut}_0(\mathbb{C}^k)$ with $\|\psi - A\|_{\overline{B}} < \epsilon$, and $\|\psi - \mathrm{Id}\|_{f(j)(K)} < \epsilon$. We also get a $\phi \in \mathrm{Aut}_0(\mathbb{C}^k)$ with $\|\phi - \mathrm{Id}\|_{f(j)(K)\cup\overline{B}} < \epsilon$ and $\phi(f(j)(p_{j+1})) \in \psi^{-1}(\mathbf{B})$. But then $I_{j+1}$ is satisfied by taking $\epsilon$ sufficiently small and setting $f_{j+1} = \psi\phi$.

Now by (6.2) and our earlier choice of $\delta$, $\Omega$ is biholomorphic to $\mathbb{C}^k$, and (6.3) and (6.4) imply that $\Omega$ satisfies the claims of the theorem. □

**Corollary 6.7.** *There exists a Fatou-Bieberbach domain $\Omega \subset \mathbb{C}^2$ for which $\partial\Omega$ has non-zero 4-dimensional Hausdorff measure.*

*Proof.* Let $D = (\overline{D})^\circ \subset\subset \mathbb{C}$ be a simply connected open set where $\partial D$ has non-zero 2-dimensional Hausdorff measure. Then $K = \overline{D \times D} \subset \mathbb{C}^2$ is a polynomially convex compact set whose boundary has non-zero 4-dimensional Hausdorff measure (see for instance Theorem 8.10 in [Mat95]). Pick a countable set $\{p_j\}$ dense in $\mathbb{C}^2 \setminus K$. By Theorem 6.6, there exists a Fatou-Bieberbach domain $\Omega$ so that $\{p_j\} \subset \Omega \subset \mathbb{C}^2 \setminus K$, which implies that $\partial K \subset \partial\Omega$. □

Notice that this corollary only tells us that a part of the boundary of $\Omega$ is large; it does not give any information about the size of the boundary near other points. We now prove the main result of this chapter:

*Proof of Theorem 6.1*: Let $D = (\overline{D})^\circ \subset\subset \mathbb{C}$ be a simply connected set whose boundary has non-zero 2-dimensional Hausdorff measure. Then as before $K = \overline{D \times D}$ is a polynomially convex compact set whose boundary has non-zero 4-dimensional Hausdorff measure. For $p \in \mathbb{C}^2$ we let $K_\epsilon(p)$ denote an arbitrary such $K$ with $p \in K$ and $K \subset B(p, \epsilon)$.

Let $A$ be the linear map defined by

$$A\colon (z_1, z_2) \to (\frac{z_1}{2}, \frac{z_2}{2}).$$

It follows from Theorem 3.6 that we can choose a $\delta > 0$ so that given any sequence $\{f_j\} \subset \mathrm{Aut}_0(\mathbb{C}^2)$ with $\|f_j - A\|_{\overline{\mathbf{B}}} < \delta$ for all $j \in \mathbb{N}$, $\Omega_{\{f_j\}}$ is a Fatou-Bieberbach domain.

Choose a sequence of strictly positive numbers $\{\epsilon_j\}_{j\in\mathbb{N}}$ converging to zero. We will construct our sequence of automorphisms inductively with the following induction hypothesis $I_j$: We have automorphisms $\{f_1, ..., f_j\} \subset \mathrm{Aut}_0(\mathbb{C}^2)$, a polynomially convex compact set $K^j = K_1^j \cup \cdots \cup K_m^j \subset B(j+1)$, where each $K_i^j$ is equal to $K_\epsilon(p)$ for some point $p$ and an $\epsilon > 0$, a set of points $T_j = \{t_i\}_{i=1}^l \subset B(j+1)$, and a ball $B_j$

so that $\overline{B}_j \cap \overline{\mathbf{B}} = \emptyset$. In addition they satisfy the conditions:

(6.5) $\|f_i - A\|_{\overline{\mathbf{B}}} < \delta$ for $i = 1, \dots, j$,

(6.6) $B(j+1) \setminus f(j)^{-1}(\overline{\mathbf{B}}) \neq \emptyset$,

(6.7) $f(j)(t_i) \in \mathbf{B}$ for $t_i \in T_j$,

(6.8) For any $p \in B(j+1) \setminus (K^j)^\circ$ there is a $t_i \in T_j$ so that $\|t_i - p\| < \epsilon_j$,

(6.9) For any $p \in B(j+1) \setminus f(j)^{-1}(\mathbf{B})$ we have that $d(p, K^j) < \epsilon_j$,

(6.10) $f(j)(K^j) \subset B_j$.

We will now show how to construct $K^{j+1}, T_{j+1}, B_{j+1}$ and $f_{j+1}$ so as to satisfy $I_{j+1}$. It will be clear from the construction how to define $K^1, T_1, B_1$ and $f_1$.

Choose a set of points $\{p_i\}_{i=1}^n \subset B(j+2) \setminus (K^j \cup f(j)^{-1}(\overline{\mathbf{B}}))$, so that for any point $p \in B(j+2) \setminus ((K^j)^\circ \cup f(j)^{-1}(\mathbf{B}))$ there is a $p_i$ with $\|p - p_i\| < \epsilon_{j+1}$. Let $q_i = f(j)(p_i)$ for $i = 1, ..., n$. Then Remark 6.5 tells us that $\overline{\mathbf{B}} \cup f(j)(K^j) \cup \{q_i\}_{i=1}^n$ is a polynomially convex compact set and there exists a $\rho > 0$ so that:

(6.11) $\overline{\mathbf{B}} \cup f(j)(K^j) \cup (\cup_{i=1}^n \overline{B(q_i, \rho)})$ is polynomially convex,

(6.12) $\overline{B(q_i, \rho)} \cap (\overline{\mathbf{B}} \cup f(j)(K^j)) = \emptyset$ for $i = 1, ..., n$,

(6.13) $\overline{B(q_i, \rho)} \cap \overline{B(q_l, \rho)} = \emptyset$ for $i \neq l$.

For each $i$, let $\tilde{K}_i = K_\rho(q_i)$ and define $K^{j+1} = K^j \cup f(j)^{-1}(\cup_{i=1}^n \tilde{K}_i)$, which is also polynomially convex by Remark 6.5. This takes care of (6.9).

Let $T_{j+1} = \{t_i\}_{i=1}^l \subset B(j+2) \setminus K^{j+1}$ be such that for any $p \in B(j+2) \setminus (K^{j+1})^\circ$, there is an $i \leq l$ with $\|p - t_i\| < \epsilon_{j+1}$. This ensures (6.8). Let $\tilde{t}_i = F(j)(t_i)$ for $i = 1, ..., k$.

We will now construct $f_{j+1}$, referring to Theorem 6.4 for each statement made about the existence of an automorphism. For any $\mu > 0$ there exists a $\varphi \in \text{Aut}_0(\mathbb{C}^2)$ with $\|\varphi(z) - z\| < \mu$ for all $z \in \overline{\mathbf{B}} \cup f(j)(K^j)$, so that there exists a large ball $B_{j+1} \subset \mathbb{C}^2$ with $\overline{B}_{j+1} \cap \overline{\mathbf{B}} = \emptyset$ and $\tilde{K}^{j+1} = \varphi(\cup_{i=1}^n \tilde{K}_i \cup f(j)(K^j)) \subset B_{j+1}$. There also exists a $\phi \in \text{Aut}_0(\mathbb{C}^2)$ with $\|\phi(z) - z\| < \mu$ for all $z \in \overline{B}_{j+1}$ and $\|\phi - A\|_{\overline{\mathbf{B}}} < \mu$. Finally, there exists a $\psi \in \text{Aut}_0(\mathbb{C}^2)$ with $\|\psi(z) - z\| < \mu$ for all $z \in \overline{\mathbf{B}} \cup \tilde{K}^{j+1}$ and $\psi(\varphi(\tilde{t}_i)) \in \phi^{-1}(\mathbf{B})$ for $i = 1, \ldots, l$. If we choose $\mu$ small enough then $f_{j+1} := \phi\psi\varphi$ satisfies (6.5),(6.6),(6.7) and (6.10), completing the induction step.

We have now constructed a sequence of automorphisms $\{f_j\}_{j \in \mathbb{N}}$, and by (6.5) and Theorem 3.6 the basin of attraction of this sequence is biholomorphic to $\mathbb{C}^2$. Since the basin of attraction is obviously not the whole of $\mathbb{C}^2$, it is a Fatou-Bieberbach domain that we will call $\Omega$. It is clear from (6.10) that none of the sets $K^j$ in the above construction are contained in $\Omega$, while from (6.7) it follows that all of the $t_i$'s chosen at each step are in $\Omega$. Let $K_i$ be one of the sets from step $j$, and let $p \in \partial K_i$. Because of (6.8) there is a sequence of $t_i$'s converging to $p$ all of which are in $\Omega$ so $p \in \partial\Omega$. This means the increasing union $K = \cup_{j=1}^{\infty} \partial K^j$ is a set whose 4-dimensional Hausdorff measure is non-zero at any point and $K \subset \partial\Omega$. Since (6.9) tells us that $K$ is dense in $\partial\Omega$ this completes the proof.

### 6.4 Hausdorff dimension 3

Let $\Omega$ be a Fatou-Bieberbach domain in $\mathbb{C}^k$. If the complement of $\Omega$ has non-empty interior, it is easy to see that the Hausdorff dimension of $\partial\Omega$ is at least $2k-1$: After a change of coordinates we may assume that $0 \in \Omega$ and $(0, \cdots, 0, i) \in (\mathbb{C}^k \setminus \Omega)^\circ$. Let $\epsilon > 0$ so that $B(\epsilon) \subset \Omega$ and $B(i, \epsilon) \subset (\mathbb{C}^k \setminus \Omega)$. Then for any $z = (z', x_k) \in B(\epsilon) \subset \mathbb{C}^{k-1} \times \mathbb{C}$, the curve $\{(z', x_k + it) \mid t \in [0, 1]\}$ intersects the boundary of $\Omega$. Therefore the image of $\partial\Omega$ under the projection map $\pi : \mathbb{C}^k \to \mathbb{C}^{k-1} \times \mathbb{R}$ defined by

$\pi(z) = (z', x_k)$ contains the ball of radius $\epsilon$ (with respect to the Euclidean norm on $\mathbb{C}^{k-1} \times \mathbb{R} \backsimeq \mathbb{R}^{2k-1}$). Therefore the Hausdorff dimension of $\partial\Omega$ is at least $2k-1$.

However, it is possible for $\Omega$ to be dense in $\mathbb{C}^k$ (see [RR88] or Theorem 6.6) so this does not guarantee that for an arbitrary $\Omega$ the dimension of $\partial\Omega$ is at least $2k-1$. We can, however, prove the following:

**Theorem 6.8.** *Let $\Omega$ be a Fatou-Bieberbach domain in $\mathbb{C}^k$ which is Runge. Then the Hausdorff dimension of $\partial\Omega$ is at least $2k-1$ near any point of the boundary.*

*Proof.* Assume that $0 \in \partial\Omega$. It is enough to show that the dimension of $\partial\Omega \cap B(\epsilon)$ is at least $2k-1$ for an arbitrary $\epsilon > 0$. If there are interior points of $\mathbb{C}^k \setminus \Omega$ in $B(\epsilon)$ then the result follows immediately, so we may assume that $\Omega \cap B(\epsilon)$ is dense in $B(\epsilon)$.

Let $U$ be an open subset of $B(\epsilon)$ consisting of annuli so that $U$ is uniformly bounded away from the hyperplane $\{z_k = 0\}$. Define $f : U \to \mathbb{C}^{k-1} \times \mathbb{R}$ by

$$f(z) = (\frac{z_1}{z_k}, \ldots, \frac{z_{k-1}}{z_k}, \|z\|).$$

This is a smooth open mapping where the preimages of points are circles centered at the origin. Since $\Omega$ is Runge, we have that every circle centered at the origin intersects $\partial\Omega$. Thus we have that $f(\partial\Omega \cap U) = f(U)$, which is an open subset of $\mathbb{R}^{2k-1}$. Since $\|f'\|_{\sup}$ is bounded from above on $\partial\Omega \cap U$, the dimension of $U \cap \partial\Omega$ is at least $2k-1$. □

Theorem 6.8 does not rule out the existence of a non-Runge Fatou-Bieberbach domain whose boundary has Hausdorff dimension strictly less than 3. Whether there exist Fatou-Bieberbach domains that are not Runge is a very interesting open question.

Another open question related to this chapter is whether there exists a basin of attraction for a single automorphism of $\mathbb{C}^2$ whose boundary has Hausdorff dimension exactly 4. In particular, it is unknown whether there exists a Hénon mapping with an attracting fixed point so that the boundary of the corresponding basin of attraction has Hausdorff dimension 4.

# CHAPTER VII

# Non-Autonomous Ergodic Theory

## 7.1 Ergodic Theory

Complex dynamical systems often provides an excellent test ground for properties from differentiable dynamics and ergodic theory. The complex structure and the availability of (pluri-) subharmonic functions often gives ways to construct invariant measures that have many interesting properties. We will look at two specific properties here, namely ergodicity and mixing measures.

Let $(X, f, \mu)$ be a dynamical system with invariant probability measure $\mu$, i.e. $\mu(f^{-1}(A)) = \mu(A)$ for all measurable sets $A$. Recall that a set $A$ is called invariant if $f(A) = A$, and that $A$ is called *totally invariant* if $f^{-1}(A) = A$. The system $(X, f, \mu)$ is called *ergodic* if all totally invariant measurable subsets $A$ of $X$ either have measure 0 or 1. The measure $\mu$ is called *mixing* for $f$ if for all measurable sets $A$ and $B$ we have that

$$\mu(f^{-n}(A) \cap B) - \mu(A) \cdot \mu(B) \to 0, \text{ as } n \to \infty.$$

Equivalently, $\mu$ is mixing if for all $\phi, \psi \in \mathcal{L}^2(\mu)$ one has that

$$\int \phi \circ f^n \cdot \psi d\mu \to \int \phi d\mu \cdot \int \psi d\mu, \text{ as } n \to \infty.$$

If we assume that $\mu$ is mixing for $f$ and we let $A$ be a completely invariant set, then

$$\mu(f^{-n}(A) \cap B) - \mu(A) \cdot \mu(B) \to 0,$$

so $\mu(A) = \mu(A)^2$. It follows that $\mu(A)$ must be either 0 or 1, and thus the system is ergodic. The mixing property is in general stronger than ergodicity. For example it is easy to check that the irrational rotation on the unit circle with the normalized Lebesgue measure is ergodic but not mixing.

The construction of interesting ergodic and mixing measures for complex differentiable mappings is an important topic in complex dynamical systems. In the one dimensional case, such measures were first constructed and investigated by Brolin for polynomials in the complex plane in [Bro65], and later in more general settings by Lyubich [Lju83] and Freire, Lopez and Mañe [FLM83].

The case of Hénon mappings in $\mathbb{C}^2$ was studied by Fornæss and Sibony [FS92] and Bedford and Smillie [BS92], and the theory was later generalized for regular polynomial mappings of $\mathbb{C}^k$ by Fornæss and Sibony [FS95]. The construction of ergodic and mixing measures for holomorphic mappings in $\mathbb{P}^k$ was done by Fornæss and Sibony [FS94] and deeper properties were studied by Briend and Duval [BD01].

We would like to study ergodicity and mixing measures in the non-autonomous setting, but in this setting the above definitions do not make much sense. First of all, a sequence of maps $f_1, f_2, \ldots$ will generally not have a probability measure that is invariant for all $f_n$. Secondly, there will generally be no proper measurable subsets which are invariant for all $f_n$. We shall say that $\{f_n\}$ is *measure preserving* for a sequence of probability measures $\mu_0, \mu_1, \ldots$, if $f_{n*}\mu_{n-1} = \mu_n$ holds for every $n$ (in other words, $\mu_{n-1}(f_n^{-1}(A)) = \mu_n(A)$ for all $\mu_n$-measurable sets $A$), and we say that a sequence $A_0, A_1, \ldots$ is *totally invariant* if $f_n^{-1}(A_n) = A_{n-1}$ for every $n$. We can now make the following definitions:

**Definition 7.1.** A measure preserving sequence $\{f_n\}$ is *randomly ergodic* if for all totally invariant sequences $A_0, A_1, \ldots$, where $A_n$ is a $\mu_n$-measurable set, we have that $\mu_n(A_n)$ is 0 or 1.

**Definition 7.2.** A measure preserving sequence $\{f_n\}$ is *randomly mixing* if for all continuous functions $\phi$ and $\psi$ on $X$, we have that

$$\int (\phi \circ f(n)) \cdot \psi \, d\mu_0 - \int \phi \, d\mu_n \int \psi \, d\mu_0 \to 0.$$

Both definitions can also be studied in the autonomous setting, where only a single map and a single measure are considered. Since continuous functions are dense in $\mathcal{L}^2(\mu)$, we have that randomly mixing and mixing are equivalent in the autonomous setting. However, random ergodicity is a strictly stronger property than ergodicity in the autonomous setting. It is easy to check that randomly ergodic implies ergodicity, but the reverse does not hold. It is easy to see that the only measure for which an *automorphism* $f$ is randomly ergodic is a point mass at a fixed point. Otherwise, there would be a measurable set $A$ whose measure is not 0 or 1, and for which the sequence $A, f(A), f^2(A), \ldots$ is totally invariant, contradicting random ergodicity. However, the Lebesque measure on the unit circle is ergodic for an irrational rotation, which shows that ergodic measures do not need to be randomly ergodic. We note that a randomly mixing system is not necessarily randomly ergodic for similar reasons.

In this chapter we will construct invariant sequences of randomly mixing and randomly ergodic measures for sequences of holomorphic endomorphisms of complex projective space $\mathbb{P}^k$ that are *uniformly bounded*. We say that a sequence of endomorphisms of $\mathbb{P}^k$ is uniformly bounded if the degrees of the mappings are at least 2 and bounded from above, and if the sequence is compact in the locally uniform topology.

In the second and third section we will construct the invariant sequences of measures, and do the necessary preliminary work. In the fourth section we prove that the sequences constructed earlier are indeed randomly mixing and randomly ergodic. In the last section we take a closer look at random ergodicity in the autonomous setting, and prove that it is actually equivalent to a property called *exactness*.

Non-autonomous dynamical systems have been studied successfully to obtain results in autonomous complex dynamical systems. Examples are the study of skew-products (see for example the articles by Jonsson [Jon00] and Buzzi, Sester and Tsujii [BST03]) and the work of Dujardin, Dinh and Sibony [DDS04], [DS04] with respect to horizontal like mappings.

## 7.2 Equilibrium Measures

The following construction of equilibrium measures is fairly standard in holomorphic dynamics and can be found in [Sib99], and for non-autonomous systems in [FW00].

Let $P_1, P_2, \ldots$ be a uniformly bounded sequence of holomorphic mappings of $\mathbb{P}^k$ as defined in the previous section. As usual, we write $P(m,n)$ for $P_n \cdots P_{m+1}$, and we denote the degree of $P_n$ by $d_n$ and the degree of $P(m,n)$ by $d(m,n)$.

Since $\{P_n\}$ is uniformly bounded, we can extend all mappings $P_n$ to homogeneous polynomial endomorphisms $\tilde{P}_n$ of $\mathbb{C}^{k+1}$ in such a way that the coefficients of every $\tilde{P}_n$ are bounded by a uniform constant, and the $\tilde{P}_n$-images of the unit sphere in $\mathbb{C}^{k+1}$ are bounded away from the origin. In other words, there exists some constant $t > 1$ such that

$$\frac{1}{t}\|z\|^{d_n} < \|\tilde{P}_n(z)\| < t\|z\|^{d_n} \tag{7.1}$$

holds for any nonzero $z$ in $\mathbb{C}^{k+1}$ and any $n$.

For every $i \in \mathbb{N}$ and $n \geq 1$, we define the function

$$G_{n,i}(z) := \frac{1}{d(i, i+n)} \log \|\tilde{P}(i, i+n)(z)\|.$$

**Lemma 7.3.** *As $n \to \infty$, the functions $G_{n,i}$ converge uniformly on $\mathbb{C}^{k+1} \setminus \{0\}$ to a continuous and plurisubharmonic function $G_i$ on $\mathbb{C}^{k+1} \setminus \{0\}$.*

*Proof.* Fix $\epsilon > 0$. It follows from (7.1) and the definition of $G_{n,i}(z)$ that for any $z$ in $\mathbb{C}^{k+1} \setminus \{0\}$ we have

$$|G_{n+1,i}(z) - G_{n,i}(z)| < \frac{\log(t)}{d(i, n+i+1)}.$$

A standard geometric serious calculation gives that us for any $m \geq n$ we get

$$|G_{m,i}(z) - G_{n,i}(z)| < \frac{\log(t)}{d(i, i+n)(d_{n+1} - 1)}. \tag{7.2}$$

Since every $d_n$ is at least 2, we can choose $n$ large enough so that

$$|G_{m,i}(z) - G_{n,i}(z)| < \epsilon$$

for any $m \geq n$. It follows that the sequence $G_{n,i}$ converges uniformly on $\mathbb{C}^{k+1} \setminus \{0\}$ to a limit map $G_i$, and since all the functions $G_{n,i}$ are continuous and plurisubharmonic, the limit map is also continuous and plurisubharmonic. □

Equation (7.2) implies that

$$G(z) = \log \|z\| + O(1), \tag{7.3}$$

and since every $\tilde{P}_n$ is homogeneous we have that $G_i(\lambda z) = \log(\lambda) + G_i(z)$. It follows from the definition of the $G_{n,i}$'s that

$$\tilde{P}_n^* G_n = d_n G_{n-1}. \tag{7.4}$$

Let $\pi$ be the projection from $\mathbb{C}^{k+1}$ to $\mathbb{P}^k$. We can define $(1,1)$ currents $T_i$ on $\mathbb{P}^k$ which satisfy

$$\pi^* T_i := dd^c G_i.$$

Equation (7.3) gives that the $T_i$'s are currents of mass 1 on $\mathbb{P}^k$, and the currents $T_i$ do not depend on our choices for $\tilde{P}_n$. Equation (7.4) implies that

$$P_n^* T_n = d_n T_{n-1}.$$

Since $G_n$ is continuous, it follows from [BT77] that we can define $\mu_n = (T_n)^k$. Since $T_i$ has unit mass, we get that $\mu_n$ is a probability measure and since $G_n$ is locally bounded it follows from Proposition 4.6.4 in the book by Klimek [Kli91] that $\mu_n$ does not assign any mass to locally pluripolar sets.

We call $\mu_n$ the *equilibrium measure* at stage $n$ and we have that $P_n^* \mu_n = d_n^k \mu_{n-1}$ and $P_{n*}\mu_{n-1} = \mu_n$.

### 7.3 Uniform Convergence of Preimages

Recall the following theorem which was proved by H. Brolin [Bro65] for polynomials and by M. Lyubich [Lju83] and independently by A. Freire, A. Lopes and R. Mañé [FLM83] for rational functions of $\mathbb{P}^1$:

**Theorem 7.4.** *Let $R(z)$ be a rational function of degree $d \geq 2$ on $\mathbb{P}^1$, and let $R^n$ be its n-th iterate. Then there exists an exceptional set $\mathcal{E}_R$ with* $\operatorname{card}(\mathcal{E}_R) \leq 2$ *so that for all $a \in \mathbb{P}^1 \setminus \mathcal{E}_R$ we have*

$$\frac{1}{d^n}(R^n)^* \delta_a \to \mu.$$

(Here $\delta_x$ is the dirac mass at $x$, and $\mu$ is the equilibrium measure as defined in [Lju83].)

It follows from Theorem 1.2 of [RS95] that this theorem can be generalized to the non-autonomous setting in $\mathbb{P}^k$. However, to prove that the equilibrium measures defined in the rpevious section are randomly ergodic and randomly mixing, we will need the uniform versions of the above result which we will prove in this section. Our proofs will be similar to the method used by Lyubich to prove the above theorem, and that was later used by J. Briend and J. Duval in [BD01] to prove similar results for endomorphisms of $\mathbb{P}^k$.

Define $\eta_{x,n,i}$ to be the probability measure with mass $\frac{1}{d(i,n+i)^k}$ at all the preimages $P(i, n+i)^{-1}(x)$ counting multiplicity. In other words,

$$\eta_{x,n,i} = \frac{P(i, n+i)^* \delta_x}{d(i, n+i)^k}.$$

(For simplicity of notation, we shall write $\eta_{x,n}$ for $\eta_{x,n,0}$.)

For two probability measures $\mu_1, \mu_2$ on $\mathbb{P}^k$, we define the distance

$$d(\mu_1, \mu_2) = \sup_{\phi} \left| \int \phi \, d\mu_1 - \int \phi \, d\mu_2 \right|$$

where the supremum is taken over all $\mathcal{C}^1(\mathbb{P}^k)$ functions $\phi$ for which $|\phi(z)|$ and $|\nabla \phi(z)|$ are bounded by 1. It is clear that the topology induced by this distance is weaker than the strong topology on probability measures. Since we are working on a compact space, a sequence of probability measures $\nu_n$ in fact converges weakly to $\mu$ if and only if $d(\nu_n, \mu) \to 0$

The following proposition shows that as $n$ gets large, the measures $\eta_{x,n}$ depend less and less on the point $x$.

**Proposition 7.5.** *For any $\epsilon > 0$ there exists $N \in \mathbb{N}$ and subsets $X_n$ of $\mathbb{P}^k$ such that for every $n > N$ we have that $\mu_n(X_n) < \epsilon$, and*

$$d(\eta_{n,x}, \eta_{n,y}) < \epsilon,$$

*for every* $x, y \in \mathbb{P}^k \setminus X_n$.

We will give the proof of this proposition after we prove a few necessary Lemmas.

Fix $\epsilon > 0$ and choose $l$ large enough such that $4\tau 2^{-l} < \epsilon$, where $\tau$ is the maximum possible algebraic degree of the sets $V_n$, the critical values of $P_n$. Let $\gamma$ be the maximum possible degree of the algebraic sets $V_{l,n}$. We can choose $\tau$ and $\gamma$ since we required that the degrees of the polynomials $P_n$ are bounded from above.

For $n, \in \mathbb{N}$ with $n \geq l$ let $V_{l,n}$ be the set of critical values of the holomorphic mapping $P(n-l, n)$.

**Lemma 7.6.** *There exist a* $\delta > 0$ *such that the* $\mu_n$ *mass of the* $\delta$*-neighborhood of* $V_{l,n}$ *is less than* $\epsilon$ *for any* $n \in \mathbb{N}$ *with* $n \geq l$.

*Proof.* We have seen that the measures $\mu_n$ do not assign any mass to pluripolar sets, so for all $n \geq l \geq 0$ there exists a $\delta_{n,l}$ such that the $\delta_{n,l}$-neighborhood of $V_{l,n}$ has $\mu_n$ mass less than $\epsilon$.

Let $\mathcal{S} = \{P_n\}^{\mathbb{N}}$ be the set containing all possible sequences $P_{n_1}, P_{n_2}, \ldots$, and equip $\mathcal{S}$ with the product topology so that it is a compact set (here we use that $\{P_n\}$ is uniformly bounded).

For each sequence in $\mathcal{S}$ we can define the Green's functions as we did in Section 7.2. The maps $G_{n,i}$ depend continuously on the sequence in $\mathcal{S}$, so since $G_{n,i}$ converges uniformly to the map $G_i$, we have that $G_i$ also depends continuously on $\mathcal{S}$.

Choose an arbitrary $S \in \mathcal{S}$, let $\{S^j\}$ be a sequence in $\mathcal{S}$ that converges uniformly to $S$, and write $G_i^j, G_i$, $\mu_i^j, \mu_i$ for the Green's functions and equilibrium measures corresponding to the sequences $S^j$ and $S$. Then we have that $G_i^j \to G_i$ uniformly on $\mathbb{C}^{k+1}$, and therefore it follows from [CLN69] that $\mu_i^j$ converges weakly to $\mu_i$.

Since the sets of critical values $V_{l,n}$ also vary continuously in $\mathcal{S}$, we have the $\delta_n$-

neighborhoods will also have mass bounded by $\epsilon$ for an open neighborhood of $S$. Since $\mathcal{S}$ is compact, this means that we can choose one $\delta$ that suffices for sequences in $\mathcal{S}$, in particular for the sequences $P_j, P_{j+1}, \ldots, P_n$, which completes the proof. □

Fix $\delta$ as in the above lemma. Recall that a holomorphic disc is the image of a conformal mapping defined on the unit disc in $\mathbb{C}$. A holomorphic disc $\Delta$ in a complex line $L$ will be called $\delta$-extendable if the $\delta$-neighborhood of $\Delta$ in $L$ is simply connected.

**Lemma 7.7.** *There exists a constant $c \in \mathbb{R}$ such that for every $n$ large enough, every complex line $L$ and every $\frac{\delta}{4\gamma}$-extendable holomorphic disc $\Delta \subset L$ that does not intersect a $\frac{\delta}{2\gamma}$-neighborhood of $L \cap V_{l,n}$, there exist at least $(1-\epsilon)d(n)^k$ inverse branches of $P(n)$ on $\Delta$ for which the preimages $\Delta_i = P(n)_i^{-1}(\Delta)$ satisfy*

$$\mathrm{diam}\,(\Delta_i) < cd(n)^{k/2}.$$

*Proof.* We can exactly follow the proof of the lemma in [BD] to get that for every such disc $\Delta$, there exists a constant $c$ such that there are at least $(1-\epsilon)d^n$ preimages $\Delta_i$ of diameter less than $cd^{n/2}$. To see that we can choose $c$ independently of $\Delta$, note that we can take the larger disc $\tilde{\Delta}$ in that proof as the $\frac{\delta}{4\gamma}$-neighborhood of $\Delta$ in $L$. It follows that the modulus $\mathrm{Mod}(\tilde{\Delta} - \Delta)$ is bounded from below by some strictly positive constant, and this gives a bound on $c$ which completes the proof of the lemma. □

Note that for every line $L$ that intersects $V(l,n)$ in a finite number of points and every $x, y$ in the complement of the $\delta/\gamma$-neighborhood of $V(l,n)$ in $L$, we can choose a $\delta/(4\gamma)$-extendable holomorphic disc outside of the $\delta/(2\gamma)$-neighborhood of $V(l,n)$. Indeed, we can take the shortest curve in $L$ from $x$ to $y$ that avoids the $3\delta/(4\gamma)$-neighborhood of $V(l,n)$ and take the $\delta/(4\gamma)$-neighborhood of the curve as our extendable disc. We are now ready to prove our main proposition.

**Proof of Proposition 7.5:** Let $X_n$ be the $\delta$-neighborhood of $V_{l,n}$ and let $x, y \in \mathbb{P}^k \setminus X_n$. Lemma 7.6 gives that $\mu_n(X_n) < \epsilon$ for any $n \in \mathbb{N}$. It follows from Bezout's Theorem that we can choose $z$ outside of $X_n$ such that the lines $L_1$ and $L_2$ through respectively $x, z$ and $y, z$ intersect $V_{l,n}$ in at most $\gamma$ points. This means that there exist $\delta/(4\gamma)$-extendable holomorphic discs $\Delta_1 \subset L_1$ and $\Delta_2 \subset L_2$ such that $x, z \in \Delta_1$, $y, z \in \Delta_2$, and $\Delta_1$ and $\Delta_2$ avoid the $\frac{\delta}{2\gamma}$ neighborhood of $V_{l,n}$. Now it follows from Lemma 7.7 that there are at least $(1-\epsilon)d^n$ preimages $x_j^{-n}$, $y_j^{-n}$ and $z_j^{-n}$ such that

$$\mathrm{dist}(x_j^{-n}, y_j^{-n}) \leq \mathrm{dist}(x_j^{-n}, z_j^{-n}) + \mathrm{dist}(y_j^{-n}, z_j^{-n}) \leq 2\frac{c}{d(n)^{k/2}}.$$

This means that for any continuous function $\phi$ of norm 1 we have

$$|\int \phi d\eta_{x,n} - \int \phi d\eta_{y,n}| \leq 2\epsilon + |\frac{1}{d^n}\sum_j \left(\phi(x_j^{-n}) - \phi(y_j^{-n})\right)| \leq 2\epsilon + 2\frac{c}{d(n)^{k/2}}.$$

The right hand side is smaller than $3\epsilon$ for large enough $n \in \mathbb{N}$ which completes the proof. □

Now let $\epsilon_1, \epsilon_2, \ldots$ be a monotone decreasing sequence with $\sum \epsilon_j < \epsilon$. For every $j$ define a set $X_{n,j}$ as in Proposition 7.5 and $N_j$ in $\mathbb{N}$ such that $\mu(X_{n,j}) < \epsilon_j$ and $d(\eta_{n,x}, \eta_{n,y}) < \epsilon_j$ for any $n$ larger than $N_j$ and $x, y$ outside of $X_{n,j}$. Set

$$U_n := \mathbb{P}^k - \bigcup_{N_j \leq n} X_{n,j},$$

so that $\mu_n(U_n) > 1 - \epsilon$ for every $n$. Fixing a sequence $x_1, x_2, \ldots$ such that $x_n$ is an element of $U_n$, we get the following uniform version of Theorem 7.4:

**Lemma 7.8.** *For every $\epsilon > 0$ there exists an $N$ so that for all $m, n \in \mathbb{N}$ with $n \geq N$ we have*

$$d(\eta_{x_n, n-m}, \mu_m) < \epsilon.$$

*Proof.* We know that

$$\mu_m = \int \delta_y \, d\mu_m(y),$$

and therefore we get

$$\mu_m = \frac{P(m, n+m)^* \mu_{n+m}}{d(m, n+m)^k} = \int \eta_{y,n,m} \, d\mu_{n+m}(y).$$

It follows that

$$\mu_m - \eta_{x_{n+m},n,m} = \int (\eta_{y,n,m} - \eta_{x_n,n,m}) \, d\mu_n(y).$$

We can choose a $j$ such that $2\epsilon_j < \epsilon$, and by our construction of $X_{n,j}$ and $U_n$, it follows that for $n \geq N_j$ we have $d(\eta_{y,n,m}, \eta_{x_n,n,m}) < \epsilon_j$ for any $y$ outside of $X_{n,j}$. We also have that $\mu_{n+m}(X_{n+m,j}) < \epsilon_j$, so $d(\mu_m, \eta_{x_n,j}) < 2\epsilon_j$, which completes the proof. □

In the autonomous setting it is known that the equilibrium measure is the only totally invariant measure that does not charge the exceptional set [BD01]. We cannot expect such a result to hold here. Consider for example the map $z \mapsto z^2$ in $\mathbb{P}^1$. The equilibrium measures $\mu_n$ are all equal to the normalized Lebesgue measure on the unit circle. However, if we let $\nu_n$ be the normalized Lebesgue measure on the disc of radius $1/2^n$, then $\{\nu_n\}$ is totally invariant and does not charge the exceptional set $\{0, \infty\}$.

We do have the following related uniqueness result:

**Corollary 7.9.** *Let $\nu$ be a probability measure on $\mathbb{P}^k$ that does not charge locally pluripolar sets. Then we have that*

$$\frac{P(m, n+m)^*}{d(m, n+m)^k} \nu \to \mu_m,$$

*weakly.*

The corollary follows from Proposition 7.5 as in the proof of lemma 7.8

## 7.4 Random Ergodicity and Random Mixing

Now that we have done the necessary preliminary work we can prove the following theorem:

**Theorem 7.10.** *Let $P_1, P_2, \ldots$ be a uniformly bounded sequence of holomorphic endomorphisms of $\mathbb{P}^k$ and let $\mu_0, \mu_1, \ldots$ be the corresponding sequence of equilibrium measures. Then the non-autonomous dynamical system $(\mathbb{P}^k, \{P_n\}, \{\mu_n\})$ is randomly ergodic and randomly mixing.*

*Proof.* First we will prove random ergodicity. Let $A_0, A_1, \ldots$ be a sequence of measurable subsets of $\mathbb{P}^k$ such that $P_n^{-1}(A_n) = A_{n-1}$ for all $n$, and assume that $\mu_0(A_0)$ is not equal to 0. We need to show that $A_0$ has full measure. Define the conditional measures $\nu_n$ by

$$\nu_n(X) = \frac{\mu_n(X \cap A_n)}{\mu_n(A_n)}.$$

Clearly, every $\nu_n$ is a probability measure, and also

$$\begin{aligned} P_{n*}\nu_{n-1}(X) &= \nu_{n-1}(P_n^{-1}(X)) \\ &= \mu_{n-1}(P_n^{-1}(X) \cap A_{n-1})/\mu_{n-1}(A_{n-1}) \\ &= \mu_{n-1}(P_n^{-1}(X \cap A_n))/\mu_n(A_n) \\ &= P_{n*}\mu_{n-1}(X \cap A_n)/\mu_n(A_n) \\ &= \mu_n(X \cap A_n)/\mu_n(A_n) = \nu_n(X). \end{aligned}$$

Similarly, it follows from the total invariance of the sets $A_n$ and the measures $\mu_n$ that

$$\frac{P_n^*\nu_n}{d(n)^k} = \nu_{n-1}.$$

As we have seen before in the proof of Lemma 7.8, we have

$$\mu_0 = \int \eta_{x,n} d\mu_n(x),$$

and similarly,

$$\nu_0 = \int \eta_{y,n} d\nu_n(y).$$

Therefore we see that

$$\mu_0 - \nu_0 = \int \int (\eta_{x,n} - \eta_{y,n}) d\mu_n(x) \otimes d\nu_n(y).$$

It now follows from Proposition 1 that for any $\epsilon > 0$, the following holds

$$\|\mu_0 - \nu_0\| < 3\epsilon,$$

so $\nu_0 = \mu_0$ and $\mu_0(A_0)$ must equal 1, which proves that the system is randomly ergodic.

To prove that the system is randomly mixing, let $\phi, \psi$ be test functions of norm at most 1, and let $\epsilon > 0$. Construct sets $U_n$ as we did for Lemma 7.8 such that $\mu_n(U_n) > 1 - \epsilon$ for each $n$. It follows from Lemma 7.8 that we can fix $n$ so large that $\|\eta_{\zeta,n} - \mu_0\| < \epsilon$ for any $\zeta \in U_n$.

Let $m$ be large enough so that

$$\int (\phi \circ P(n)) \cdot \psi \, d\mu_0 = \int (\phi \circ P(n)) \circ \psi \, d\eta_{x_{m+n,-(m+n)}} + \epsilon_1,$$

where $|\epsilon_1| < \epsilon$. It follows from the definition of $\eta_{x_{m+n,-(m+n)}}$ that the right hand side is equal to

$$\sum_{\nu} \phi(P(n)(\zeta^{\nu}_{m+n,-(m+n)}))\psi(\zeta^{\nu}_{m+n,-(m+n)})d(m+n)^{-k} + \epsilon_1 \tag{7.5}$$

$$= \sum_{\sigma} \phi(\zeta^{\sigma}_{m+n,-m}) d(n, n+m)^{-k} \sum_{\zeta^{\sigma}_{m+n,-m} \text{fixed}} \psi(\zeta^{\nu}_{m+n,-(m+n)}) d(n)^{-k} + \epsilon_1. \tag{7.6}$$

Counting multiplicity, there are $d(n, n+m)^k$ preimages $\zeta^{\sigma}_{m+n,-m}$, and since $\mu_n(U_n) > 1 - \epsilon$, we can increase $m$ if necessary so that at least $(1-\epsilon)d(n, n+m)^k$ of the $\zeta_{m+n,-m}$ are in $U_n$. It follows that 7.6 is equal to

$$\sum \phi(\zeta_{m+n,-m}) d(m, n+m)^{-k} \left( \int \psi \, d\mu_0 + \epsilon_3 \right) + \epsilon_1 + \epsilon_2,$$

where $\epsilon_3$, which depends on $\nu$, satisfies $|\epsilon_3| < \epsilon$ and also $|\epsilon_2| < \epsilon$. We can rewrite the above equation as

$$\left(\int \psi \, d\mu_0 + \epsilon_3\right) \sum \phi(\zeta_{m+n,-m}) d(n, n+m)^{-k} + \epsilon_1 + \epsilon_2,$$

where $\epsilon_3$ no longer depends on $m$. By increasing $m$ if necessary we get

$$\left(\int \psi \, d\mu_0 + \epsilon_3\right) \left(\int \phi \, d\mu_n + \epsilon_4\right) + \epsilon_1 + \epsilon_2,$$

with $|\epsilon_4| < \epsilon$, and so

$$|\int (\phi \circ P(n)) \cdot \psi \, d\mu_0 - \int \phi \, d\mu_n \int \psi \, d\mu_0| < 4\epsilon.$$

This proves that

$$\int (\phi \circ P(n)) \cdot \psi \, d\mu_0 - \int \phi \, d\mu_n \int \psi \, d\mu_0 \to 0$$

for all test functions $\phi$ and $\psi$ with norm bounded by 1, and thus for all test functions. The theorem follows since we can uniformly approximate any continuous function by test functions. □

*Remark* 7.11. It is not clear if the theorem holds when we allow $\phi$ in the definition of randomly mixing to be in the intersection of all $\mathcal{L}^2(\mu_n)$, since in general we will not be able to approximate these functions by continuous functions that are close in every $\mathcal{L}^2(\mu_n)$ norm at the same time. The theorem does however hold for $\psi$ in $\mathcal{L}^2(\mu_0)$.

## 7.5 Random Ergodicity in the Autonomous Setting

We have already seen that random ergodicity is not equivalent to ergodicity in the classical case. Indeed, an automorphism can never have interesting measures that are randomly ergodic. We will now show that random ergodicity is equivalent to a condition that is far stronger than ergodicity, namely *exactness*. Let $(X, \mathcal{B}, \mu)$ be a measurable space $(X, \mu)$ with $\sigma$-algebra $\mathcal{B}$. Recall that a measurable transformation $T : X \to X$ is called exact (see for instance [Wal82]) if

$$\bigcap_{n \geq 0} T^{-n}\mathcal{B} \doteq \mathcal{N},$$

where $\mathcal{N} = \{\emptyset, X\}$, $\doteq$ means that the two sides are equal up to sets of measure zero, and the left hand side is the $\sigma$-algebra consisting of all sets $A \in \mathcal{B}$ that have for every $n \in \mathbb{N}$ a set $A_n \in \mathcal{B}$ with $A = T^{-n}(A_n)$.

**Proposition 7.12.** *A sequence $T_1, T_2, \ldots$ of transformations of $(X, \mathcal{B}, \mu)$ is randomly ergodic if and only if*

$$\bigcap_{n \geq 0} T(n)^{-1}\mathcal{B} \doteq \mathcal{N},$$

The proof follows directly from the definition of random ergodicity. We immediately get the following corollary for the autonomous setting:

**Corollary 7.13.** *An automorphism $T$ of $(X, \mathcal{B}, \mu)$ is randomly ergodic if and only if $T$ is exact.*

Exact automorphisms are strong mixing [Wal82]. Hence a random ergodic transformation is in particular mixing.

# BIBLIOGRAPHY

# BIBLIOGRAPHY


[Aba02] Marco Abate, *Basins of attraction in quadratic dynamical systems with a Jordan fixed point*, Nonlinear Anal. **51** (2002), no. 2, Ser. A: Theory Methods, 271–282.

[AL92] Erik Andersén and László Lempert, *On the group of holomorphic automorphisms of* $\mathbf{C}^n$, Invent. Math. **110** (1992), no. 2, 371–388.

[And90] Erik Andersén, *Volume-preserving automorphisms of* $\mathbf{C}^n$, Complex Variables Theory Appl. **14** (1990), no. 1-4, 223–235.

[BD01] Jean-Yves Briend and Julien Duval, *Deux caractérisations de la mesure d'équilibre d'un endomorphisme de* $\mathrm{P}^k(\mathbf{C})$, Publ. Math. Inst. Hautes Études Sci. (2001), no. 93, 145–159.

[Bed] E. Bedford, *Open problem session of the* biholomorphic mappings *meeting at the American Institute of Mathematics Palo Alto, ca, july 2000.*

[Bro65] Hans Brolin, *Invariant sets under iteration of rational functions*, Ark. Mat. **6** (1965), 103–144 (1965).

[BS91] Eric Bedford and John Smillie, *Polynomial diffeomorphisms of* $\mathbf{C}^2$*: currents, equilibrium measure and hyperbolicity*, Invent. Math. **103** (1991), no. 1, 69–99.

[BS92] ———, *Polynomial diffeomorphisms of* $\mathbf{C}^2$*. III. Ergodicity, exponents and entropy of the equilibrium measure*, Math. Ann. **294** (1992), no. 3, 395–420.

[BST03] Jérôme Buzzi, Olivier Sester, and Masato Tsujii, *Weakly expanding skew-products of quadratic maps*, Ergodic Theory Dynam. Systems **23** (2003), no. 5, 1401–1414.

[BT77] Eric Bedford and B. A. Taylor, *The Dirichlet problem for a complex Monge-Ampère equation*, Several complex variables (Proc. Sympos. Pure Math., Vol. XXX, Williams Coll., Williamstown, Mass., 1975), Part 1, Amer. Math. Soc., Providence, R.I., 1977, pp. 109–113.

[CLN69] S. S. Chern, Harold I. Levine, and Louis Nirenberg, *Intrinsic norms on a complex manifold*, Global Analysis (Papers in Honor of K. Kodaira), Univ. Tokyo Press, Tokyo, 1969, pp. 119–139.

[DDS04] Tien-Cuong Dinh, Romain Dujardin, and Nessim Sibony, *On the dynamics near infinity of some polynomial mappings in* $\mathbb{C}^2$, 2004.

[DS04] Tien-Cuong Dinh and Nessim Sibony, *Dynamics of horizontal-like maps in higher dimension (I)*, 2004.

[FLM83] Alexandre Freire, Artur Lopes, and Ricardo Mañé, *An invariant measure for rational maps*, Bol. Soc. Brasil. Mat. **14** (1983), no. 1, 45–62.

[FM89] Shmuel Friedland and John Milnor, *Dynamical properties of plane polynomial automorphisms*, Ergodic Theory Dynam. Systems **9** (1989), no. 1, 67–99.

[For99] Franc Forstnerič, *Interpolation by holomorphic automorphisms and embeddings in* $\mathbf{C}^n$, J. Geom. Anal. **9** (1999), no. 1, 93–117.

[For04] John Erik Fornæss, *Short* $\mathbb{C}^k$, Complex analysis in several variables—Memorial Conference of Kiyoshi Oka's Centennial Birthday, Adv. Stud. Pure Math., vol. 42, Math. Soc. Japan, Tokyo, 2004, pp. 95–108.

[FR93] Franc Forstnerič and Jean-Pierre Rosay, *Approximation of biholomorphic mappings by automorphisms of* $\mathbf{C}^n$, Invent. Math. **112** (1993), no. 2, 323–349.

[FS92] John Erik Fornæss and Nessim Sibony, *Complex Hénon mappings in* $\mathbf{C}^2$ *and Fatou-Bieberbach domains*, Duke Math. J. **65** (1992), no. 2, 345–380.

[FS94] ______, *Complex dynamics in higher dimensions*, Complex potential theory (Montreal, PQ, 1993), NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., vol. 439, Kluwer Acad. Publ., Dordrecht, 1994, Notes partially written by Estela A. Gavosto, pp. 131–186.

[FS95] John Erik Fornaess and Nessim Sibony, *Complex dynamics in higher dimension. II*, Modern methods in complex analysis (Princeton, NJ, 1992), Ann. of Math. Stud., vol. 137, Princeton Univ. Press, Princeton, NJ, 1995, pp. 135–182.

[FS04] John Erik Fornæss and Berit Stensønes, *Stable manifolds of holomorphic hyperbolic maps*, Internat. J. Math. **15** (2004), no. 8, 749–758.

[FW00] John Erik Fornæss and Brendan Weickert, *Random iteration in* $\mathbf{P}^k$, Ergodic Theory Dynam. Systems **20** (2000), no. 4, 1091–1109.

[Hak98] Monique Hakim, *Analytic transformations of* $(\mathbf{C}^p, 0)$ *tangent to the identity*, Duke Math. J. **92** (1998), no. 2, 403–428.

[Jon00] Mattias Jonsson, *Ergodic properties of fibered rational maps*, Ark. Mat. **38** (2000), no. 2, 281–317.

[JV02] Mattias Jonsson and Dror Varolin, *Stable manifolds of holomorphic diffeomorphisms*, Invent. Math. **149** (2002), no. 2, 409–430.

[Kli91] Maciej Klimek, *Pluripotential theory*, London Mathematical Society Monographs. New Series, vol. 6, The Clarendon Press Oxford University Press, New York, 1991, Oxford Science Publications.

[Lat11] S. Lattès, *Sur les formes réduites des transformations ponctuelles à deux variables*, Comptes Rendus **152** (1911).

[Lju83] M. Ju. Ljubich, *Entropy properties of rational endomorphisms of the Riemann sphere*, Ergodic Theory Dynam. Systems **3** (1983), no. 3, 351–385.

[Mat95] Pertti Mattila, *Geometry of sets and measures in Euclidean spaces*, Cambridge Studies in Advanced Mathematics, vol. 44, Cambridge University Press, Cambridge, 1995, Fractals and rectifiability.

[MNTU00] S. Morosawa, Y. Nishimura, M. Taniguchi, and T. Ueda, *Holomorphic dynamics*, Cambridge Studies in Advanced Mathematics, vol. 66, Cambridge University Press, Cambridge, 2000, Translated from the 1995 Japanese original and revised by the authors.

[Ran86] R. Michael Range, *Holomorphic functions and integral representations in several complex variables*, Graduate Texts in Mathematics, vol. 108, Springer-Verlag, New York, 1986.

[RR88] Jean-Pierre Rosay and Walter Rudin, *Holomorphic maps from* $\mathbf{C}^n$ *to* $\mathbf{C}^n$, Trans. Amer. Math. Soc. **310** (1988), no. 1, 47–86.

[RS95] Alexander Russakovskii and Mikhail Sodin, *Equidistribution for sequences of polynomial mappings*, Indiana Univ. Math. J. **44** (1995), no. 3, 851–882.

[Shi98] Mitsuhiro Shishikura, *The Hausdorff dimension of the boundary of the Mandelbrot set and Julia sets*, Ann. of Math. (2) **147** (1998), no. 2, 225–267.

[Sib99] Nessim Sibony, *Dynamique des applications rationnelles de* $\mathbf{P}^k$, Dynamique et géométrie complexes (Lyon, 1997), Panor. Synthèses, vol. 8, Soc. Math. France, Paris, 1999, pp. ix–x, xi–xii, 97–185.

[Ste57] Shlomo Sternberg, *Local contractions and a theorem of Poincaré*, Amer. J. Math. **79** (1957), 809–824.

[Ste97] Berit Stensönes, *Fatou-Bieberbach domains with* $C^\infty$*-smooth boundary*, Ann. of Math. (2) **145** (1997), no. 2, 365–377.

[Ued86] Tetsuo Ueda, *Local structure of analytic transformations of two complex variables. I*, J. Math. Kyoto Univ. **26** (1986), no. 2, 233–261.

[Wal82] Peter Walters, *An introduction to ergodic theory*, Graduate Texts in Mathematics, vol. 79, Springer-Verlag, New York, 1982.

[Wei97] Brendan Weickert, *Automorphisms of* $\mathbb{C}^n$., Ph.D. thesis, 1997.

[Wei98] Brendan J. Weickert, *Attracting basins for automorphisms of* $\mathbf{C}^2$, Invent. Math. **132** (1998), no. 3, 581–605.

[Wol00] Christian Wolf, *Dimension of Julia sets of polynomial automorphisms of* $\mathbf{C}^2$, Michigan Math. J. **47** (2000), no. 3, 585–600.

[Wol04] Erlend Fornæss Wold, *Fatou-Bieberbach domains.*, 2004.

**ABSTRACT**

Non-Autonomous Complex Dynamical Systems

by

Han Peters

Chair: John Erik Fornæss

We study the orbits of sequences of complex differentiable endomorphisms of complex manifolds. We are in particular interested in the following question: is a stable manifold of a hyperbolic automorphism always biholomorphic to complex Euclidean space? We show that the answer to this question is yes if a related conjecture in non-autonomous complex dynamics holds. We investigate this second conjecture, and prove several intermediate results.