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Non-autonomous dynamics in $\mathbb{P}^k$

HAN PETERS
University of Michigan, Ann Arbor, MI 48109, USA
(e-mail: hanpet@umich.edu)

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Abstract. We study the dynamics of compositions of a sequence of holomorphic mappings in $\mathbb{P}^k$. We define ergodicity and mixing for non-autonomous dynamical systems, and we construct totally invariant measures for which our sequence satisfies these properties.

1. *Introduction*

Non-autonomous dynamics differs from standard dynamics in that instead of iterating a single map, we consider compositions of a sequence of maps. The main goal of non-autonomous dynamics is to generalize theorems that hold in the autonomous setting or to find counterexamples. Here, we try to generalize theorems which state that for every complex mapping there exists a natural measure which is mixing and thus ergodic. This was first proved by Brolin for polynomials in the complex plane in [**Br**], by Bedford and Smillie for Hénon mappings in $\mathbb{C}^2$ in [**BS**], and for regular polynomial mappings of $\mathbb{C}^k$ [**FS3**] and holomorphic mappings of $\mathbb{P}^k$ [**FS2**] by Fornæss and Sibony. It has also been shown for endomorphisms of $\mathbb{P}^k$, see for instance the articles by Briend and Duval [**BD**] or Guedj and Sibony [**GS**].

Non-autonomous systems of polynomials in the complex plane have been studied in the past years by many authors, see for instance the survey article by Comerford [**Co**], which has an extensive bibliography. It has turned out that a good setting in which to work is that of bounded sequences of monic polynomials of some fixed degree which were first considered in [**FS1**]. We work with more general mappings, and our results imply the same results for such sequences.

Non-autonomous systems in higher complex dimensions have been studied only rarely. We look at a compact sequence of holomorphic mappings on $\mathbb{P}^k$, which we define more precisely in the next section. This setting has already been studied in [**FW**], but in a rather different way. There, the dynamics of all nearby mappings of a holomorphic mapping were studied at the same time, while we study the dynamics of one fixed sequence.

Let P_n be a compact sequence of holomorphic mappings, and let μ_n be the equilibrium measures for this sequence, which we define later. The main results of this paper are the following two theorems.

THEOREM 1. *The system* $(\{P_n\}, \{\mu_n\})$ *is randomly ergodic.*

THEOREM 2. *The system* $(\{P_n\}, \{\mu_n\})$ *is randomly mixing.*

In §2 we set our notation and give the precise definitions of randomly ergodic and randomly mixing, and in §3 we use pluripotential methods to introduce the equilibrium measures μ_n. In §4 we prove a series of lemmas considering the convergence of preimages, and we give the proofs of our two main theorems in §5. In the last section, we prove that an autonomous system which is randomly ergodic is in fact exact, which implies that it is mixing.

2. *Non-autonomous systems in* $\mathbb{P}^k$

We now introduce the setting for this paper. Let $\mathcal{P}$ be a compact family (in the coefficients topology) of holomorphic endomorphisms of $\mathbb{P}^k$ whose degrees are at least 2 and bounded from above, and let $P_0, P_1, \ldots$ be a sequence of polynomials in $\mathcal{P}$, where P_n has degree d_n.

We define

$$P(n) = P_n \circ \cdots \circ P_1; \quad d(n) = d_n \cdot \cdots \cdot d_1$$

and for n larger than m we write

$$P(m,n) = P_n \circ \cdots \circ P_{m+1}; \quad d(m,n) = d_n \cdot \cdots \cdot d_{m+1}.$$

For a point $z = z_0$ in $\mathbb{P}^k$ we also write z_n for $P(n)(z)$, which we say is a point at stage n. Thus, P_n is a mapping from stage $n-1$ to stage n.

Recall that a measure-preserving automorphism f of a space X with probability measure μ is called *ergodic* if all totally invariant measurable subsets A of X either have full or empty measure, and that f is called *mixing* if for all measurable sets A and B we have that

$$\mu(f^{-n}(A) \cap B) - \mu(A) \cdot \mu(B) \to 0.$$

We would like to study these two properties in the non-autonomous setting, but in this setting the above definitions do not make much sense. First of all, in general a sequence of maps $f_1, f_2, \ldots$ will not have a probability measure that is invariant for all f_n. We say that $\{f_n\}$ is *measure preserving* for a sequence of probability measures $\mu_0, \mu_1, \ldots,$ if $f_{n*}\mu_{n-1} = \mu_n$ holds for every n. Secondly, there will generally be no proper measurable subsets which are invariant for all f_n. We say that a sequence $A_0, A_1, \ldots$ is *totally invariant* if $f_n^{-1}(A_n) = A_{n-1}$ for every n. We make the following definitions.

Definition 3. A measure-preserving sequence $\{f_n\}$ is *randomly ergodic* if for all totally invariant sequences $A_0, A_1, \ldots,$ where A_n is a μ_n-measurable set, we have that $\mu_n(A_n)$ is 0 or 1.

Definition 4. A measure-preserving sequence $\{f_n\}$ is *randomly mixing* if for all continuous functions ϕ and ψ on X, we have that

$$\int (\phi \circ f(n)) \cdot \psi \, d\mu_0 - \int \phi \, d\mu_n \int \phi \, d\mu_0 \to 0.$$

Both definitions can also be studied in the autonomous setting, where a single map is iterated. Since continuous functions are dense in $\mathcal{L}^2(\mu)$, we have that randomly mixing and mixing are equivalent. However, randomly ergodic is a strictly stronger property than ergodic. It is easy to check that randomly ergodic implies ergodicity, but the only measure for which an automorphism is randomly ergodic is a point mass at a fixed point, which is certainly not the case for the classical definition. We note that a randomly mixing system is not necessarily randomly ergodic for the same reason.

It would be interesting to find a generalization of ergodicity that is useful for the study of the dynamics of a sequence of automorphisms.

3. *Equilibrium measures*

The following construction of the equilibrium measures is fairly standard in holomorphic dynamics and can be found in [**Si**], and can also be found for non-autonomous systems in [**FW**].

Since $\mathcal{P}$ is compact, we can extend all mappings P in $\mathcal{P}$ to homogeneous polynomial mappings $\tilde{P}$ of $\mathbb{C}^{k+1}$ in such a way that the coefficients of every $\tilde{P}$ are bounded by some uniform constant M, and such that the images of the unit sphere in $\mathbb{C}^{k+1}$ are bounded away from the origin. In other words, there exists some constant $t > 1$ such that

$$\frac{1}{t}\|z\|^{d_n} < \|\tilde{P}_n(z)\| < t\|z\|^{d_n}, \tag{1}$$

holds for any non-zero z in $\mathbb{C}^{k+1}$ and any n.

For every $i \in \mathbb{N}$ and $n \geq i$, we define the function

$$G_{n,i}(z) := \frac{1}{d(i, i+n)} \log \|\tilde{P}(i, i+n)(z)\|.$$

LEMMA 5. *As $n \to \infty$, the functions $G_{n,i}$ converge uniformly on $\mathbb{C}^{k+1}$ to a continuous and plurisubharmonic function G_i.*

Proof. Fix $\epsilon > 0$. It follows from (1) that for any z in $\mathbb{C}^{k+1}$ we have

$$|G_{n+1,i}(z) - G_{n,i}(z)| < \frac{\log(t)}{d(i, n+i+1)}.$$

Therefore, we have for any $m \geq n$ that

$$|G_{m,i}(z) - G_{n,i}(z)| < \frac{\log(t)}{d(i, i+n)(d_{n+1} - 1)}. \tag{2}$$

Since every d_n is at least 2, we can choose n large enough so that

$$|G_{m,i}(z) - G_{n,i}(z)| < \epsilon,$$

for any $m \geq n$. It follows that the sequence $G_{n,i}$ converges uniformly to a limit map G_i, and since all the functions $G_{n,i}$ are continuous and plurisubharmonic, the limit map is also continuous and plurisubharmonic. □

It follows from (2) that $G(z) = \log \|z\| + O(1)$. Also, since every $\tilde{P}_n$ is homogeneous, we have that $G_i(\lambda z) = \log(\lambda) + G_i(z)$. We get the equation

$$\tilde{P}_n^* G_n = d_n G_{n-1}. \tag{3}$$

Let π be the projection from $\mathbb{C}^{k+1}$ to $\mathbb{P}^k$. We can define $(1, 1)$ currents T_i on $\mathbb{P}^k$ which satisfy

$$\pi^* T_i := dd^c G_i.$$

T_i is a current of mass 1 on $\mathbb{P}^k$, that does not depend on our choices for $\tilde{P}_n$. It follows from (3) that

$$P_n^* T_n = d_n T_n.$$

Since G_n is continuous, it follows from [**BT**] that we can define $\mu_n = (T_n)^k$. Since T_i has unit mass, we get that μ_n is a probability measure and since G_n is locally bounded it follows from Proposition 4.6.4 in the book by Klimek [**Kl**] that μ_n does not assign any mass to locally pluripolar sets.

We call μ_n the *equilibrium measure* at stage n and we have that $P_n^* \mu_n = d_n^k \mu_{n-1}$, and that $P_{n*} \mu_{n-1} = \mu_n$.

4. *Uniform convergence of preimages*

Recall the following theorem, which was proved by Brolin [**Br**] for polynomials and by Lyubich [**Ly**] and independently by Freire *et al* [**FLM**] for rational functions.

THEOREM 6. *Let $R(z)$ be a rational function of degree $d \geq 2$, and let R^n be its nth iterate. Then for all $a \in \mathbb{P} \setminus \mathcal{E}_R$, $\operatorname{card}(\mathcal{E}_R) \leq 2$,*

$$\frac{1}{d^n} (R^n)^* \delta_a \to \mu.$$

Here δ_x is the Dirac mass at x. It follows from Theorem 1.2 of [**RS**] that this theorem can be generalized to our setting. However, to prove Theorems 1 and 2 we need the uniform versions of this theorem which we prove in this section. Our proofs are similar to the method used by Lyubich to prove the above theorem, and which was later used by Briend and Duval in [**BD**] to prove similar results for endomorphisms of $\mathbb{P}^k$.

Define $\eta_{x,n,i}$ to be the probability measure with mass $1/d(i, n+i)^k$ at all the preimages $P(i, n+i)^{-1}(x)$ counting multiplicity. In other words,

$$\eta_{x,n,i} = \frac{P(i, n+i)^* \delta_x}{d(i, n+i)^k}.$$

(For simplicity of notation, we write $\eta_{x,n}$ for $\eta_{x,n,0}$.)

For two probability measures μ_1, μ_2 on $\mathbb{P}^k$, we define the distance

$$d(\mu_1, \mu_2) = \sup_{\phi} \left| \int \phi \, d\mu_1 - \int \phi \, d\mu_2 \right|,$$

where the supremum is taken over all $\mathcal{C}^1(\mathbb{P}^k)$ functions ϕ for which $|\phi(z)|$ and $|\nabla \phi(z)|$ are bounded by 1. It is clear that the topology induced by this distance is weaker than the strong topology on probability measures. In fact, a sequence of probability measures ν_n

converges weakly to μ if and only if $d(\nu_n, \mu) \to 0$ since we are working in a compact space.

The following proposition shows that as n gets large, the measures $\eta_{x,n}$ depend less and less on the point x.

PROPOSITION 7. *Let $\epsilon > 0$. Then there exists a $N \in \mathbb{N}$ and subsets X_n of $\mathbb{P}^k$ such that for every n larger than N we have that $\mu_n(X_n) < \epsilon$, and also*

$$d(\eta_{n,x}, \eta_{n,y}) < \epsilon,$$

for every x, y outside of X_n.

The proof is given below.

Fix $\epsilon > 0$, and let $l = l(\epsilon)$ be some large enough number that we define later. For n greater or equal to l, let $V_{l,n}$ be the set of critical values of the holomorphic mapping $P(n-l, n)$.

LEMMA 8. *There exists a δ such that the μ_n mass of the δ-neighborhood of $V_{l,n}$ is less than ϵ for any n larger than l.*

Proof. We have seen that the measures μ_n do not assign any mass to pluripolar sets. Therefore, there exists for each $n \geq l$ a δ_n such that the δ_n-neighborhood of $V_{l,n}$ has μ_n mass less than ϵ. Let $\mathcal{S}$ be the set of sequences of polynomials of $\mathcal{P}$ with the product topology, so that $\mathcal{S}$ is a compact set.

The maps $G_{n,i}$ depend continuously on the sequence in $\mathcal{S}$, and since $G_{n,i}$ converges uniformly to the map G_i, we have that G_i also depends continuously on $\mathcal{S}$. Let $\{S^j\}$ be a sequence of sequences in $\mathcal{S}$ that converges uniformly to $S \in \mathcal{S}$. Write G_i^j, G_i, μ_i^j, μ_i for the Green's functions and equilibrium measures corresponding to the sequences S^j and S. Then we have that $G_i^j \to G_i$ uniformly on $\mathbb{C}^k$, and therefore it follows from [**CLN**] that μ_i^j converges weakly to μ_i.

Since the sets of critical values $V_{l,n}$ also vary continuously as a function in $\mathcal{S}$, we have that δ_n is also sufficient for an open neighborhood of our sequence S. Since $\mathcal{S}$ is compact, this means that we can choose one δ that suffices for all sequences, in particular for the sequences $P_j, P_{j+1}, \ldots$, which completes the proof. □

Fix δ as in the above lemma, and we now fix l such that $4\tau 2^{-l} < \epsilon$, where τ is the maximum possible algebraic degree of the sets V_n, the critical values of P_n. Let γ be the maximum possible degrees of the algebraic sets $V_{l,n}$. We can choose τ and γ since the degrees of the polynomials P_n are bounded from above, which follows from the compactness of $\mathcal{P}$.

We call a holomorphic disc Δ in a complex line L δ-extendable if the δ-neighborhood of Δ in L is simply connected.

LEMMA 9. *There exists a constant $c \in \mathbb{R}$ such that for every n large enough, every complex line L and every $\delta/4\gamma$-extendable holomorphic disc $\Delta \subset L$ that does not intersect a $\delta/2\gamma$-neighborhood of $L \cap V_{l,n}$, there exist at least $(1-\epsilon)d(n)^k$ inverse branches of $P(n)$ on Δ for which the preimages $\Delta_i = P(n)_i^{-1}(\Delta)$ satisfy*

$$\operatorname{diam}(\Delta_i) < c d(n)^{k/2}.$$

Proof. We can exactly follow the proof of the lemma in [BD] to get that for every such disc Δ, there exists a constant c such that there are at least $(1-\epsilon)d^n$ preimages Δ_i of diameter less than $cd^{n/2}$. To see that we can choose c independently of Δ, note that we can take the larger disc $\tilde{\Delta}$ in that proof as the $\delta/4\gamma$-neighborhood of Δ in L. It follows that $\mathrm{Mod}(\tilde{\Delta}-\Delta)$ is bounded from below by some strictly positive constant and this gives a bound on c, which completes the proof of the lemma. □

Note that for every line L that intersects $V(l,n)$ in a finite number of points and every x, y in the complement of the δ/γ-neighborhood of $V(l,n)$ in L, we can choose a $\delta/(4\gamma)$-extendable holomorphic disc outside of the $\delta/(2\gamma)$-neighborhood of $V(l,n)$. Indeed, we can take the shortest curve in L from x to y that avoids the $3\delta/(4\gamma)$-neighborhood of $V(l,n)$ and take the $\delta/(4\gamma)$-neighborhood of the curve as our extendable disc.

Proof of Proposition 7. Let X_n be the δ-neighborhood of $V_{l,n}$. We have that $\mu_n(X_n) < \epsilon$ for any $n \in \mathbb{N}$. Let x, y be points outside of X_n. We can choose z outside of X_n such that the lines L_1 and L_2 through, respectively, x, z and y, z intersect $V_{l,n}$ in at most γ points. This means that there exist $\delta/(4\gamma)$-extendable holomorphic discs $\Delta_1 \subset L_1$ and $\Delta_2 \subset L_2$ such that $x, z \in \Delta_1$ and $y, z \in \Delta_2$, and such that Δ_1 and Δ_2 avoid the $\delta/2\gamma$-neighborhood of $V_{l,n}$. Now it follows from the lemma that there are at least $(1-\epsilon)d^n$ preimages x_j^{-n}, y_j^{-n} and z_j^{-n} such that

$$\mathrm{dist}(x_j^{-n}, y_j^{-n}) \le \mathrm{dist}(x_j^{-n}, z_j^{-n}) + \mathrm{dist}(y_j^{-n}, z_j^{-n}) \le 2\frac{c}{d(n)^{k/2}}.$$

Hence, for any continuous function ϕ of norm 1 we have that

$$\left| \int \phi \, d\eta_{x,n} - \int \phi \, d\eta_{y,n} \right| \le 2\epsilon + \left| \frac{1}{d^n} \sum_j (\phi(y_j^{-n}) - \phi(y_j^{-n})) \right| \le 2\epsilon + 2\frac{c}{d(n)^{k/2}}.$$

For n large enough, this is smaller than 3ϵ, which completes the proof. □

Now, for some fixed small $\epsilon > 0$, let $\epsilon_1, \epsilon_2, \ldots$ be a monotone decreasing sequence such that the sum over all ϵ_j is smaller than ϵ. For every j, define a set $X_{n,j}$ as in Proposition 7 and N_j in $\mathbb{N}$ such that $\mu(X_{n,j}) < \epsilon_j$ and $d(\eta_{n,x}, \eta_{n,y}) < \epsilon_j$ for any n larger than N_j and x, y outside of $X_{n,j}$. Now set

$$U_n := \mathbb{P}^k - \bigcup_{N_j \le n} X_{n,j}.$$

We see, in particular, that $\mu_n(U_n)$ is larger than $1-\epsilon$ for every n. Fixing a sequence $x_1, x_2, \ldots$ such that x_n is an element of U_n, we get the following uniform version of Theorem 6.

LEMMA 10. *For every $\epsilon > 0$ there exists an N so that for every m and every $n \ge N$ we have that*

$$d(\eta_{x_n, n-m}, \mu_m) < \epsilon$$

Proof. We have that

$$\mu_m = \int \delta_y \, d\mu_m(y),$$

and, therefore, we have

$$\mu_m = \frac{P(m, n+m)^* \mu_{n+m}}{d(m, n+m)^k} = \int \eta_{y,n,m}\, d\mu_{n+m}(y).$$

It follows that

$$\mu_m - \eta_{x_{n+m},n,m} = \int (\eta_{y,n,m} - \eta_{x_n,n,m})\, d\mu_n(y).$$

We can choose a j such that $2\epsilon_j < \epsilon$, and by our construction of $X_{n,j}$ and U_n, it follows that for $n \geq N_j$ we have $d(\eta_{y,n,m}, \eta_{x_n,n,m}) < \epsilon_j$ for any y outside of $X_{n,j}$, while $\mu_{n+m}(X_{n+m,j}) < \epsilon_j$. Therefore, $d(\mu_m, \eta_{x_n,j}) < 2\epsilon_j$, which completes the proof. □

In the autonomous setting it is known that the equilibrium measure is the only totally invariant measure that does not charge the exceptional set [**BD**]. We cannot expect such a result to hold here. Consider, for instance, the map $z \mapsto z^2$ in $\mathbb{P}^1$. The equilibrium measures μ_n are all equal to the normalized Lebesgue measure on the unit circle. However, let ν_n be the normalized Lebesgue measure on the disc of radius $1/2^n$. Then $\{\nu_n\}$ is totally invariant and does not charge the exceptional set $\{0, \infty\}$.

We do have the following related uniqueness result.

COROLLARY 11. *Let* ν *be a probability measure on* $\mathbb{P}^k$ *that does not charge locally pluripolar sets. Then we have that*

$$\frac{P(m, n+m)^*}{d(m, n+m)^k}\nu \to \mu_m,$$

weakly.

The corollary follows from Proposition 7 as in the proof of Lemma 10.

5. *Proofs of Theorems 1 and 2*

Proof of Theorem 1. Let $A_0, A_1, \ldots$ be a sequence of measurable subsets of $\mathbb{P}^k$ such that $P_n^{-1}(A_n) = A_{n-1}$ for all n, and assume that $\mu_0(A_0)$ is not equal to 0. We need to show that A_0 has full measure. Define the measures ν_n by

$$\nu_n(X) = \frac{\mu_n(X \cap A_n)}{\mu_n(A_n)}.$$

Clearly, every ν_n is a probability measure. We see that

$$\begin{aligned} P_{n*}\nu_{n-1}(X) &= \nu_{n-1}(P_n^{-1}(X)) \\ &= \mu_{n-1}(P_n^{-1}(X) \cap A_{n-1})/\mu_{n-1}(A_{n-1}) \\ &= \mu_{n-1}(P_n^{-1}(X \cap A_n))/\mu_n(A_n) \\ &= P_{n*}\mu_{n-1}(X \cap A_n)/\mu_n(A_n) \\ &= \mu_n(X \cap A_n)/\mu_n(A_n) = \nu_n(X). \end{aligned}$$

Similarly, it follows from the total invariance of the sets A_n and the measures μ_n that

$$\frac{P_n^* \nu_n}{d(n)^k} = \nu_{n-1}.$$

As we have seen before in the proof of Lemma 10, we have the equation

$$\mu_0 = \int \eta_{x,n}\, d\mu_n(x),$$

and, similarly,

$$\nu_0 = \int \eta_{y,n}\, d\nu_n(y).$$

Therefore, we see that

$$\mu_0 - \nu_0 = \int\!\!\int (\eta_{x,n} - \eta_{y,n})\, d\mu_n(x) \otimes d\nu_n(y).$$

It now follows from Proposition 1 that for any $\epsilon > 0$, we have

$$\|\mu_0 - \nu_0\| < 3\epsilon.$$

Thus, $\nu_0 = \mu_0$ and $\mu_0(A_0)$ must equal 1, which completes the theorem. □

The argument of the proof of Theorem 2 is similar to that of Theorem 17.1 in [Br].

Proof of Theorem 2. Let ϕ, ψ be test functions of norm at most 1, and let $\epsilon > 0$. Construct sets U_n as we did for Lemma 10 such that $\mu_n(U_n) > 1 - \epsilon$ for each n. It follows from Lemma 10 that we can fix n so large that $\|\eta_{\zeta,n} - \mu_0\| < \epsilon$ for any $\zeta \in U_n$.

Let m be large enough so that

$$\int (\phi \circ P(n)) \cdot \psi\, d\mu_0 = \int (\phi \circ P(n)) \circ \psi\, d\eta_{x_{m+n},-(m+n)} + \epsilon_1,$$

where $|\epsilon_1| < \epsilon$. It follows from the definition of $\eta_{x_{m+n},-(m+n)}$ that the right-hand side is equal to

$$\sum_{\nu} \phi(P(n)(\zeta^{\nu}_{m+n,-(m+n)}))\psi(\zeta^{\nu}_{m+n,-(m+n)})\, d(m+n)^{-k} + \epsilon_1$$
$$= \sum_{\sigma} \phi(\zeta^{\sigma}_{m+n,-m})\, d(n, n+m)^{-k} \sum_{\zeta^{\sigma}_{m+n,-m}\ \text{fixed}} \psi(\zeta^{\nu}_{m+n,-(m+n)})\, d(n)^{-k} + \epsilon_1.$$

Counting multiplicity, there are $d(n, n+m)^k$ preimages $\zeta^{\sigma}_{m+n,-m}$, and since $\mu_n(U_n) > 1 - \epsilon$, we can increase m if necessary so that at least $(1-\epsilon)\, d(n, n+m)^k$ of the $\zeta_{m+n,-m}$ are in U_n. It follows that the above right-hand side is equal to

$$\sum \phi(\zeta_{m+n,-m})\, d(m, n+m)^{-k} \left(\int \psi\, d\mu_0 + \epsilon_3 \right) + \epsilon_1 + \epsilon_2,$$

where ϵ_3, which depends on ν, and ϵ_2 all have absolute value less than ϵ. We can rewrite this as

$$\left(\int \psi\, d\mu_0 + \epsilon_3 \right) \sum \phi(\zeta_{m+n,-m})\, d(n, n+m)^{-k} + \epsilon_1 + \epsilon_2,$$

where ϵ_3 no longer depends on m. By increasing m if necessary we get

$$\left(\int \psi\, d\mu_0 + \epsilon_3 \right) \left(\int \phi\, d\mu_n + \epsilon_4 \right) + \epsilon_1 + \epsilon_2,$$

and so

$$\left| \int (\phi \circ P(n)) \cdot \psi \, d\mu_0 - \int \phi \, d\mu_n \int \psi \, d\mu_0 \right| < 4\epsilon.$$

This proves that

$$\int (\phi \circ P(n)) \cdot \psi \, d\mu_0 - \int \phi \, d\mu_n \int \psi \, d\mu_0 \to 0$$

for all test functions ϕ and ψ. The theorem follows since we can uniformly approximate any continuous function by test functions. □

Remark 12. It is not clear whether the theorem holds if we allow ϕ in the definition of randomly mixing to be in the intersection of all $\mathcal{L}^2(\mu_n)$, since, in general, we will not be able to approximate these functions by continuous functions that are close in every $\mathcal{L}^2(\mu_n)$ norm at the same time. The theorem does however hold for ψ in $\mathcal{L}^2(\mu_0)$.

6. *Random ergodicity in the autonomous setting*

We have already seen that random ergodicity is not equivalent to ergodicity in the classical case. Indeed, an automorphism can never have interesting measures that are randomly ergodic. We now show that random ergodicity is equivalent to a condition that is far stronger than ergodicity, namely *exactness*. Let $(X, \mathcal{B}, \mu)$ be a measurable space X, μ with sigma-algebra $\mathcal{B}$. Recall that a measurable transformation $T : X \to X$ is called exact (see, for instance, [**Wa**]) if

$$\bigcap_{n \geq 0} T^{-n}\mathcal{B} \doteq \mathcal{N},$$

where $\mathcal{N} = \{\emptyset, X\}$, and $\doteq$ means that the two sides are equal up to sets of measure zero.

PROPOSITION 13. *A sequence $T_1, T_2, \ldots$ of transformations of $(X, \mathcal{B}, \mu)$ is randomly ergodic if and only if*

$$\bigcap_{n \geq 0} T(n)^{-1}\mathcal{B} \doteq \mathcal{N}.$$

The proof follows directly from the definition of random ergodicity. We immediately get the following corollary for the autonomous setting.

COROLLARY 14. *An automorphism T of $(X, \mathcal{B}, \mu)$ is randomly ergodic if and only if T is exact.*

Exact automorphisms are strong mixing [**Wa**]. Hence, a random ergodic transformation is in particular mixing.

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Annotations from 28607e.pdf

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Annotation 1
Author: line 6/7.
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Annotation 1
Author: line 11/12.
Please clarify `Proposition 1' as there does not seem to be one in the text?