1. Let f and g be bounded functions in $\mathcal{L}^1(\mathbb{R})$. Show that $f \cdot g$ is in $\mathcal{L}^1(\mathbb{R})$. Does it follow that

$$\int_{\mathbb{R}} |fg| \mathrm{d}m \leq \int_{\mathbb{R}} |f| \mathrm{d}m \cdot \int_{\mathbb{R}} |g| \mathrm{d}m?$$

2. Recall that the measure δ_a is defined by $\delta_a(E) = 1$ if $a \in E$ and $\delta_a(E) = 0$ if $a \notin E$. Define probability measures μ_n on $[0,1]$ given by

$$\mu_n(E) = \frac{1}{n} \sum_{k=1}^{n} \delta_{\frac{k}{n}}(E).$$

Show that for any open interval $I = (a, b) \subset [0,1]$ we have that

$$\lim_{n\to\infty} \mu_n(I) = b - a.$$

3. Let $x \in [0,1]$. If $x^n \in \mathbb{Q}$ for some $n \in \mathbb{N}$ then we write $x^n = \frac{p}{q}$, where n is the smallest integer for which $x^n \in \mathbb{Q}$ and $p, q \in \mathbb{N}$ are relatively prime. In that case we define $f(x) = \frac{1}{q}$. If $x^n \notin \mathbb{Q}$ for any $n \in \mathbb{N}$ then we define $f(x) = 0$.
(a) Is the set $\{x \in [0,1] \mid f(x) \neq 0\}$ countable or uncountable?
(b) Is f Lebesque integrable on $[0,1]$? If so, what is $\int_0^1 f \mathrm{d}m$?
(c) Is f Riemann integrable on $[0,1]$? If so, what is $\int_0^1 f(x) \mathrm{d}x$?

4. Let (X, F, ϕ) and (X, G, ψ) be probability spaces. Define the set functions h and k by $h(E) = \phi(E) + \psi(E)$ and $k(E) = \phi(E)\psi(E)$.
(a) Show that the family of subsets $E \subset X$ for which $h(E)$ and $k(E)$ are defined is a σ-field.
(b) Prove or disprove: h is a probability measure.
(c) Prove or disprove: k is a probability measure.