NON-AUTONOMOUS BASINS OF ATTRACTION AND THEIR BOUNDARIES

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Abstract. We study whether the basin of attraction of a sequence of automorphisms of $\mathbb{C}^k$ is biholomorphic to $\mathbb{C}^k$. In particular we show that given any sequence of automorphisms with the same attracting fixed point, the basin is biholomorphic to $\mathbb{C}^k$ if every map is iterated sufficiently many times. We also construct Fatou-Bieberbach domains in $\mathbb{C}^2$ whose boundaries are 4-dimensional.

1. Introduction

In the 1920's Fatou and Bieberbach proved the existence of proper subdomains of $\mathbb{C}^2$ that are biholomorphically equivalent to $\mathbb{C}^2$, later known as Fatou-Bieberbach domains. Their examples were the basins of attraction of some automorphisms of $\mathbb{C}^2$ which have more than one fixed point. In fact, the basin of attraction of an attracting fixed point of an automorphism of $\mathbb{C}^k$ is always biholomorphic to $\mathbb{C}^k$. This follows from the work of Sternberg in 1957 [Sg] and was proved independently by Rosay and Rudin in 1988 [RR].

More recently, Stensønes [St] showed that there exist Fatou-Bieberbach domains whose boundaries are smooth, so in particular they have Hausdorff dimension 3. Wolf [Wo] showed that for any $h \in (3, 4)$, there exists a Fatou-Bieberbach domain whose boundary has Hausdorff dimension h. The results of Stensønes and Wolf leave open two questions about the Hausdorff dimension of the boundary of a Fatou Bieberbach domain:

i) Is it possible for the dimension of the boundary to be less than 3?

ii) Is it possible that the dimension of the boundary is exactly 4?

The Fatou-Bieberbach domains constructed in [Wo] are basins of attraction of some particular polynomial automorphisms of $\mathbb{C}^2$. We will show that one can use a sequence of these same maps to get a Fatou-Bieberbach domain whose boundary has upper-box dimension 4. Since the Hausdorff dimension can in general be larger than the upper-box dimension, this does not answer question (ii) above.

However, we will use a more direct approach to obtain the following theorem:

Theorem 1. *There exists a Fatou-Bieberbach domain $\Omega \subset \mathbb{C}^2$ with $\partial\Omega = \partial\overline{\Omega}$ for which the 4-dimensional Hausdorff measure of $\partial\Omega$ near any of its points is non-zero.*

The Fatou-Bieberbach domain in the above theorem will also be a basin of attraction of a sequence of mappings. The boundary with upper-box dimension 4 is constructed as the limit of lower dimensional sets, but to prove Theorem 1

Date: October 3, 2004.

1991 *Mathematics Subject Classification.* 32H50, 32H02.

we will start with sets of Hausdorff dimension 4 and then construct a sequence of mappings in such a way that these sets lie in the boundary of the basin of attraction.

We will need two theorems that tell us that the basins we construct are biholomorphic to $\mathbb{C}^k$. For a given sequence of automorphisms of $\mathbb{C}^k$ having a single attracting fixed point, say 0, define the basin of attraction as

$$\Omega = \Omega_{\{f_n\}} = \{z \in \mathbb{C}^k \mid f_n \circ \ldots \circ f_1(z) \to 0\}.$$

One could ask whether such a basin is always biholomorphic to $\mathbb{C}^k$. In general the answer to this question is no. However, it is possible to prove that the basin is always biholomorphic to $\mathbb{C}^k$ if one puts some restrictions to the sequence of maps. Recall for instance the following theorem from [Wd]:

Theorem 2. *Let $f_1, f_2, \ldots$ be a sequence of automorphisms of $\mathbb{C}^k$. Suppose that there exist $0 < a < b < 1$ with $b^2 < a$ such that every f_n satisfies*

$$a\|z\| \leq \|f_n(z)\| \leq b\|z\|,$$

for every z in the unit ball. Then the basin of attraction of the sequence is biholomorphic to $\mathbb{C}^k$.

We will use this result in the proof of Theorem 1.

Theorem 2 is very useful for the construction of Fatou-Bieberbach domains, but its strong hypotheses significantly limit its scope from a dynamical point of view.

Let $f_1, f_2, \ldots$ be a sequence of automorphisms of $\mathbb{C}^k$ having a single attracting fixed point, say 0, and let Ω_{f_j} be the basin of attraction of the mapping f_j. For every $j \in \mathbb{N}$, let $r_j > 0$ such that $B(r_j) \subset\subset \Omega$. We will prove the following result:

Theorem 3. *There exist large enough integers $n_1, n_2, \ldots$ such that the $\{r_j\}$-calibrated basin of attraction of the sequence $f_1^{n_1}, f_2^{n_2}, \ldots$ is biholomorphic to $\mathbb{C}^k$.*

In Section 4 we show that Theorem 3 does not hold if we take the most obvious definition of basin of attraction (see Section 2 for the definition of the $\{r_j\}$-calibrated basin of attraction).

Later we will show how Theorem 3 can be used to obtain a Fatou-Bieberbach domain whose boundary has upper-box dimension 4. In fact, we employ the maps used in [Wo].

Attracting basins of sequences of mappings have an interesting connection to dynamics of a single mapping. The following conjecture was posed by Bedford [Be]:

Conjecture 1. Let F be an automorphism of a complex manifold, which is hyperbolic on a compact set K. Then for every $p \in K$ the stable manifold is biholomorphically equivalent to complex Euclidean space.

So far the best answer to this conjecture has been given by Jonsson and Varolin [JV], who proved that with respect to any invariant probability measure, almost every stable manifold is biholomorphic to $\mathbb{C}^k$. It follows from the work of Fornæss and Stensønes [FSt] that the above conjecture can be answered positively by proving the following conjecture about non-autonomous basins of attraction:

Conjecture 2. Let $f_1, f_2, \ldots$ be a sequence of automorphisms of $\mathbb{C}^k$ and assume that there exist $0 < a < b < 1$ such that for every n and every z in the unit ball the following holds:

$$a\|z\| \leq \|f_n(z)\| \leq b\|z\|.$$

Then the basin of attraction of 0 for this sequence is biholomorphically equivalent to $\mathbb{C}^k$.

By using biholomorphisms that map the local stable manifolds onto the local stable tangent bundle, one can translate the setting of Conjecture 1 into a sequence of biholomorphic mappings from the unit ball into the unit ball, which satisfy the conditions of Conjecture 2. In [FSt] it is shown how one can define the basin of attraction of such a sequence of biholomorphic mappings in terms of the *tail space*. It is easy to see that this basin is biholomorphic to the stable manifold. It is also shown that this basin of attraction is biholomorphic to the basin of attraction of a sequence of global automorphisms which satisfy the conditions in Conjecture 2. To extend the local biholomorphic mappings to global automorphisms, they used the following theorem, due to Forstneric [Fc] and Weickert [We]:

Theorem 4. *Let $P = (P_1, \ldots, P_k)$, $k \geq 2$ be a holomorphic polynomial mapping of $\mathbb{C}^k$ to itself, with $P'(0)$ invertible. Let $d \geq max_i(deg(P_i))$. Then there exists $\phi \in \mathrm{Aut}(\mathbb{C}^k)$ such that the d-jet of ϕ at 0 equals P.*

We will use Theorem 4 in the proof of Theorem 3.

In the second section we will introduce the notation used throughout the article. In the third section we will prove Theorem 3. In Section (4) we will look at two interesting examples of basins of attractions that are not biholomorphic to complex Euclidean space. These examples demonstrate that Theorem 3 does not hold when we loosen the conditions. In the fifth section we will construct Fatou-Bieberbach domains whose boundaries have upper box dimension 4. In the Section (6) we will prove Theorem 1. In the last section we will give an elementary proof of the fact that the Hausdorff dimension of the boundary of a Fatou-Bieberbach domain that is Runge is always at least 3.

`Acknowledgement` We would like to thank Christian Wolf for suggesting that a Fatou-Bieberbach domain with dimension-4 boundary could be constructed as the basin of attraction of a sequence of Hénon maps. It was his suggestion that got us started on the subject. We would also like to thank the referee for many helpful comments.

2. Preliminaries

From now on we fix an integer $k \geq 2$, and write $\mathrm{Aut}_0(\mathbb{C}^k)$ for the group of automorphisms of $\mathbb{C}^k$ that fix the origin. Throughout the paper we will write products and powers for the composition of maps, e.g. fg and f^2 for $f \circ g$ and $f \circ f$.

We will work with sequences of maps $f_1, f_2, \ldots \in \mathrm{Aut}_0(\mathbb{C}^k)$ and study orbits $z, f_1(z), f_2 f_1(z), \ldots$. For $f(n)(z) = f_n \cdots f_1(z)$ we will often write z_n.

When a sequence of integers $n_1, n_2, \ldots$ is given and the maps f_j are iterated n_j times then we write $F_j = f_j^{n_j}$. We also use the notation $N_j = n_j + \cdots + n_1$, and we write

$$F(j) = F_j \cdots F_1.$$

We think of $F(j)$ as the map that takes stage 0 to stage j. For $m \geq n$, we will write $F(n, m)$ for the map that takes stage n to stage m, i.e.

$$F(n, m) = F(m) F(n)^{-1}.$$

For $n = N_j + m$ (where $m < n_{j+1}$) we also let

$$f(n) = f_{j+1}^m F(j) \text{ and } f(p,q) = f(q)f(p)^{-1}.$$

Notice that $f(n)$ is the composition of n maps f_j, while $F(n)$ is the composition of n maps F_j.

We will write $\|\cdot\|$ for the Euclidean norm in $\mathbb{C}^k$, and for $z \in \mathbb{C}^k$ and $r > 0$ we will write $B(z,r)$ for the ball of radius r centered at z. For simplicity we will write $B(r)$ for $B(0,r)$.

Let $f_1, f_2, \ldots \in \mathrm{Aut}_0(\mathbb{C}^k)$ and assume that every f_n is attracting at 0. Suppose that there exist non-negative real numbers $r_1, r_2, \ldots$ such that $f_j(B(r_j)) \subset B(r_{j+1})$ for all $j \in \mathbb{N}$. Then the $\{r_j\}$*-calibrated basin of attraction* of the sequence $f_1, f_2, \ldots$ is defined as

$$\Omega(\{r_j\}) = \bigcup f(j)^{-1} B(r_{j+1}).$$

Remark 1. If $f(j)(z) \in B(r_{j+1})$ guarantees that the orbit of z converges to 0, then the $\{r_j\}$-calibrated basin of attraction is a subset of the basin of attraction. If we also have that the radii $\{r_j\}$ are uniformly bounded from below then $\Omega(\{r_j\}) = \Omega$. However, if the radii r_j converge to 0 then there may be orbits that converge to 0 but that do not reach the balls $B(r_j)$ at the appropriate stage. We will later give examples where this is indeed the case, and where $\Omega(\{r_j\})$ is biholomorphic to $\mathbb{C}^k$ while Ω is not.

3. Basins Biholomorphic to $\mathbb{C}^k$

In this section we will be working with invertible maps T_j, g_j and G_j, where $G_j = g_j^{n_j}$, and we will use the notation

$$G(j) = T_j^{-1} G_j T_j \cdots T_1^{-1} G_1 T_1, \text{ and } G(n,m) = G(m)G(n)^{-1},$$

and for $n = N_j + m$ we let

$$g(n) = T_{j+1}^{-1} g_j^m T_{j+1} G(j) \text{ and } g(p,q) = g(q)g(p)^{-1}.$$

Usually we will be working with inverse mappings for the G's, so then we can think of $G(n,m)^{-1}$ as the map that takes us back from stage m to stage n. The definition for $G(n)$ may not be entirely consistent with the definition for $F(n)$, but this will not cause any problems.

Remark 2. Before we give the proof of Theorem 3 we must first show that $\Omega(\{r_j\})$ is well defined. Recall that we have chosen the radii $\{r_j\}$ such that $B(r_j) \subset\subset \Omega_j$ for every j. Therefore we can choose n_j large enough such that $F_j(B(r_j) \subset B(r_j)$, so that the $\{r_j\}$-calibrated basin of attraction is well-defined.

By choosing n_j larger if necessary, we can also guarantee that $\|F_j(z)\| \leq \frac{1}{2}\|z\|$ for any $j \in \mathbb{N}$ and $z \in B(r_j)$. Hence we can get that $\Omega(\{r_j\}) \subset \Omega$. Note that if there exists a radius $r > 0$ such that $B(r) \subset \Omega_j$ for every $j \in \mathbb{N}$, then Theorem 3 implies that $\Omega \cong \mathbb{C}^2$.

We may decrease the values of the r_j's in the proof, but we can always increase the integers $\{n_j\}$ later such that the definition of the basin of attraction with the smaller r_j's is equivalent to the definition of the basin of attraction with the original r_j's.

Proof of Theorem 3. For simplicity we assume that 0 is the attracting fixed point.

It follows from Lemma 3 in the appendix of [RR] that for every $j \in \mathbb{N}$ we can find:

(i) A polynomial automorphism g_j of $\mathbb{C}^k$ which is linearly conjugate to a lower triangular automorphism, with $g_j(0) = 0$ and $g_j'(0) = f_j'(0)$, and

(ii) For m_j as large as we desire a polynomial map $\phi_j : \mathbb{C}^k \to \mathbb{C}^k$, with $\phi_j(0) = 0, \phi_j'(0) = I$, and for which the following equation holds

$$g_j^{-1}\phi_j f_j - \phi_j = O(\|z\|^{m_j}).$$

It follows from Theorem 4 that for every $j \in \mathbb{N}$ we can find an automorphism T_j with $T_j - \phi_j = O(\|z\|^{m_j})$. Therefore we get the equation:

$$T_j^{-1}g_j^{-1}T_j f_j - I = O(\|z\|^{m_j}).$$

Recall from [RR] that there exist constants γ_j such that

$$\|g_j^{-n}(w) - g_j^{-n}(w')\| \leq \gamma_j^n \|w - w'\|,$$

for all w, w' in $B(1)$ and all natural numbers n. Also recall that the g_j's are attracting at 0, and that the basins of the maps g_j are all of $\mathbb{C}^k$.

By replacing f_j by a high iterate of f_j if necessary, we may assume that $\|f_j(z)\| \leq \frac{1}{2}\|z\|$ for all $j \in \mathbb{N}$ and all $z \in B(r_j)$. We now choose the m_j's so large that

$$\frac{\gamma_j}{2^{m_j}} < 1,$$

and $m_{j+1} > m_j$ for every $j \in \mathbb{N}$. Then it follows from equation (6) of the proof of the theorem in the appendix of [RR] that there exist constants C_j such that for every j and every $z \in B(r_j)$ the following holds

$$\|T_j^{-1}g_j^{-n-1}T_j f_j^{n+1}(z) - T_j^{-1}g_j^{-n}T_j f_j^n(z)\| \leq C_j(\frac{\gamma_j}{2^{m_j}})^n \|z\|^{m_j}.$$

Since $(\frac{\gamma_j}{2^{m_j}})^n$ is summable and the other terms do not depend on n, we can decrease the radii r_j if necessary such that for $z \in B(r_j)$ we have

$$(1) \qquad \sum_{n \geq 0} \|T_j^{-1}g_j^{-n-1}T_j f_j^{n+1}(z) - T_j^{-1}g_j^{-n}T_j f_j^n(z)\| \leq \frac{1}{2}\|z\|^{m_j - 1}.$$

It follows that for $z \in B(r_j)$ and any m larger than n we have

$$(2) \qquad \|T_j^{-1}g_j^{-m}T_j f_j^m(z) - T_j^{-1}g_j^{-n}T_j f_j^n(z)\| \leq \frac{1}{2}\|z\|^{m_j - 1}.$$

We now inductively construct large integers n_j. Suppose that we have already constructed n_1 through n_{j-1}. At this stage we have fixed the automorphism $G(j-1)$, and thus the set $T_j G(j-1)(B(2^j))$ is a fixed bounded set. The basin of attraction of the lower triangular mapping g_j is equal to $\mathbb{C}^k$, therefore we can choose n_j large enough such that

$$(3) \qquad T_j(G(j-1)(B(2^j))) \subset g_j^{-n_j} T_j(B(\frac{r_{j+1}}{2})).$$

We enlarge n_j if necessary so that

$$(4) \qquad f_j^{n_j}(B(r_j)) \subset B(r_{j+1}),$$

It follows from (1), (2), (4) and the fact that we chose m_{j+1} strictly larger than m_j that we can increase n_j if necessary and choose all subsequent $n_{j+1}, n_{j+2}, \ldots$ large enough so that the following inequality will hold throughout the construction for every for every m and every $z \in B(r_j)$

$$\|g(N_{j-1}, m)^{-1} f(N_{j-1}, m)(z) - z\| \leq \|z\|^{m_j - 1}. \tag{5}$$

We will now show that with these choices the basin $\Omega(\{r_j\})$ is biholomorphic to $\mathbb{C}^k$.

Let $K_j = F(j)^{-1}(B(r_{j+1}))$. Clearly $F(j)(K_j) = B(r_{j+1})$, and thus we have that

$$\|g(N_j, m)^{-1} f(m)(z) - F(j)(z)\| \leq \|F(j)(z)\|^{m_{j+1} - 1},$$

for every $z \in K_j$. It follows that $H_m = g(m)^{-1} f(m)$ is bounded on every K_j, so we can find a subsequence of $\{H_m\}$ that converges uniformly on K_1, say to a map H. We have that $K_1 \subset K_2 \subset \ldots$ and thus we can apply the same argument to every K_j, and by a diagonal argument, we see that H extends to $\bigcup K_j = \Omega(\{r_j\})$. We claim that H maps $\Omega(\{r_j\})$ biholomorphically onto $\mathbb{C}^k$.

It is a well known fact that the limit map of a sequence of biholomorphic maps is either degenerate everywhere or it is one to one. H is one to one since $H_m'(0) = I$ for all m.

We still need to show that H is surjective. Notice that it follows from (2) and (4) that

$$g(N_j, m)^{-1} f(m)(K_j) \supset B(\frac{r_{j+1}}{2}),$$

for all m. Therefore, it follows from (3) that $H_n(K_j)$ contains $B(2^j)$ for all $n \geq N_j$, and thus $B(2^j) \subset H(K_j)$, so H maps $\Omega(\{r_j\})$ biholomorphically onto $\mathbb{C}^k$. This completes the proof.

4. Counterexamples

In this section we present some examples showing that Theorem 3 is optimal. We first give an example by Fornæss that shows that Theorem 3 does not hold if we do not allow our maps to repeat. A second example shows that the theorem does not hold if we use the obvious definition for the basin of attraction.

Recall the following result from [Fo]

Theorem 5. *Let $a_1, a_2, \ldots$ be complex numbers such that $0 < |a_n| < 1$ and such that $|a_{n+1}| < |a_n|^t$ for some fixed $t > 2$. For $j \in \mathbb{N}$ let f_j be a polynomial automorphism of $\mathbb{C}^2$ defined by $f_j(z, w) = (z^2 + a_j w, a_j z)$. Then the basin of attraction of the sequence $f_1, f_2, \ldots$ is not biholomorphic to $\mathbb{C}^2$.*

It is shown in [Fo] that the basin of attraction in Theorem 5 is an example of a so-called *short* $\mathbb{C}^2$, i.e. an increasing union of holomorphic balls whose Kobayashi metric vanishes identically but that is not biholomorphic to $\mathbb{C}^2$.

Let us recall some facts about Hénon mappings in $\mathbb{C}^2$ (see for instance [BS]). Let $F \in \mathrm{Aut}(\mathbb{C}^2)$ be of the form

$$F(z, w) = (aw + P(z), az),$$

with non-zero $a \in \mathbb{C}$ and P a polynomial of degree at least 2. For $R > 0$ one defines:

$$V^+ = \{(z, w) \in \mathbb{C}^2 \mid |z| \geq |w|, |z| \geq R\},$$
$$V^- = \{(z, w) \in \mathbb{C}^2 \mid |w| \geq |z|, |w| \geq R\},$$

and

$$D = \{(z,w) \in \mathbb{C}^2 \mid |z|, |w| \leq R\}.$$

Then one can choose R large enough such that the following properties hold:

$$F(V^+) \subset V^+, \tag{6}$$

$$F^{-1}(V^-) \subset V^-. \tag{7}$$

The orbit of $z \in \mathbb{C}^2$ converges to infinity in positive time (or negative time) if and only if for some $n \in \mathbb{N}$ one has that $F^n(z) \in V^+$ (resp. $F^{-n}(z) \in V^-$). The sets D, V^+ and V^- together are called a *filtration* for F.

For the following theorem, let $f_j \in \mathrm{Aut}_0(\mathbb{C}^k)$ be defined by $f_j(z,w) = (z^2 + a_j w, a_j z)$, where $a_j \in (0,1)$ and $a_j \to 1$ as $j \to \infty$. Recall that Ω is defined as $\{z \in \mathbb{C}^2 \mid f(n)(z) \to 0\}$.

Theorem 6. *Ω is not open, and is therefore not biholomorphic to $\mathbb{C}^2$.*

Proof. Let Ω_∞ be the set of points in $\mathbb{C}^2$ whose orbits converge to infinity. Since the constants a_n converge to 1 as $n \to \infty$, there exists a uniform filtration for all the mappings f_j. In fact $R = 3$ will suffice. Therefore we have that if $z \in \Omega_\infty$, then there is some $z_n = (x_n, y_n)$ such that $|x_n| > 3$ and $|x_n| > |y_n|$. Also, if $z = (x,y)$ is such that $|x| > 3$ and $|x| > |y|$, then we have that $\|f_n(z)\| > \|z\| + 1$. Therefore

$$\Omega_\infty = \bigcup_{n \geq 0} f(n)^{-1}\left(\{z = (x,y) \in \mathbb{C}^2 \mid |x| > 3, |x| > |y|\}\right),$$

and in particular, Ω_∞ is open.

Let $z = (x,y) \in \mathbb{R}^2_+ := \{((x,y) \in \mathbb{R}^2 \mid x, y \geq 0\}$. We write $\|z\|_1$ for $x + y$. Assume that z_n does not converge to 0. Then there exists an $\epsilon > 0$ such that $\|z_n\|_1 > \epsilon$ for arbitrary large n. Take j so large that for all $n \geq j$ we have

$$(a_n)^2(1 + \frac{1}{4}\epsilon^2) > 1 + \frac{1}{5}\epsilon^2,$$

and choose $n \geq j$ such that $\|z_n\|_1 > \epsilon$. Notice that

$$f_{n+2}f_{n+1}(z_n) = ((x_n^2 + a_{n+1}y_n)^2 + a_{n+2}a_{n+1}x_n, a_{n+2}x_n^2 + a_{n+2}a_{n+1}y_n),$$

and therefore

$$\begin{aligned}\|f_{n+2}f_{n+1}(z_n)\|_1 &\geq a_{n+1}^2 y_n^2 + a_{n+2}x_n^2 + a_{n+2}a_{n+1}(x_n + y_n)\\ &> \min(a_{n+1}, a_{n+2})^2(x_n + y_n + x_n^2 + y_n^2)\\ &\geq \min(a_{n+1}, a_{n+2})^2(1 + \frac{1}{4}\epsilon^2)\|z\|_1 > (1 + \frac{1}{5}\epsilon^2)\|z\|_1\end{aligned}$$

Therefore $\|z_n\|_1$ converges to infinity and $z \in \Omega_\infty$. Hence

$$\mathbb{R}^2_+ = (\Omega \cap \mathbb{R}^2_+) \cup (\Omega_\infty \cap \mathbb{R}^2_+).$$

We have that $\Omega_\infty \cap \mathbb{R}^2_+$ is a nonempty relatively open subset of $\mathbb{R}^2_+$, and therefore $(\Omega \cap \mathbb{R}^2_+)$ is a proper relatively closed subset of $\mathbb{R}^2_+$. Since $0 \in \Omega$, it follows that Ω is not open. ☐

Theorem 6 shows that Theorem 3 does not hold if we use the more standard definition of basin of attraction instead of the $\{r_j\}$-calibrated basin of attraction even if we allow the maps to repeat.

Remark 3. Let $f_n \in \mathrm{Aut}_0(\mathbb{C}^k)$ be defined by $f_n(z,w) = (a_n w + z^2, a_n z)$, and assume that there exist $a, b \in \mathbb{R}$ such that $0 < a \leq a_n \leq b < 1$ for all $n \in \mathbb{N}$. Then the conditions in Conjecture 2 are satisfied. It is due to Fornæss [Fo] that in this case the basin of attraction of the sequence $f_1, f_2, \ldots$ is biholomorphic to $\mathbb{C}^k$.

5. Upper Box Dimension 4

First we recall the definition of upper box dimension. Let K be a compact subset of some metric space. For $\epsilon > 0$ we write $\mathcal{B}_\epsilon$ for the set of all coverings $\{B_i\}$ of K with balls of radius ϵ. For $h \geq 0$ we define

$$\gamma_h^\epsilon(K) = \epsilon^h \inf_{\mathcal{B}_\epsilon} \#\{B_i\}, \text{ and}$$

$$\mu_h(K) = \limsup_{\epsilon \to 0} \gamma_h^\epsilon(K).$$

$\mu_h(K)$ is called the h-upper box content (or upper Minkowski content) of K.

The upper box dimension of K, written as $\overline{\dim}_B(K)$, is defined as the unique value $h \geq 0$ such that $\mu_{h'}(K) = 0$ for all $h' > h$ and $\mu_{h'}(K) = \infty$ for all $h' < h$. The upper box dimension of K may in general be larger than the Hausdorff dimension of K, but for many sets the two concepts of dimension agree. Upper box dimension is often used instead of Hausdorff dimension because it is better suited for computer approximations.

For two compact subsets $A, B \subset \mathbb{C}^k$, we use the usual definition for the Hausdorff distance between A and B, namely

$$d_H(A,B) = \max\{d(A,B), d(B,A)\},$$

where

$$d(A,B) = \sup_{x \in A} \inf_{y \in B} d(x,y).$$

Next we recall the definitions of the Julia sets of a Hénon mapping f (see for instance [BS]). We define the sets K^+ and K^- as

$$K^{\pm} = \{z \in \mathbb{C}^2 \mid \{f^{\pm n}(z)\} \text{ is bounded}\}.$$

It follows from the filtration properties discussed in Section 4 that $z \in K^+$ if and only if $f^n(z) \in D$ for all n large enough.

We define the forward and backward Julia sets J^+ and J^- as $J^{\pm} = \partial K^{\pm}$, and the Julia set as $J = J^+ \cap J^-$.

The dynamics of conjugated Hénon maps of the form $f(z,w) = (z^2 + c + aw, az)$, where c is in the main cardioid of the Mandelbrot set and the constant $a > 0$ is very small, was studied by Fornæss and Sibony in [FSy]. The complement of the forward Julia set consists of exactly two connected components, called Fatou components, namely the basin of attraction of the unique attracting fixed point and the basin of a point at infinity.

It was proved in [Wo] that for every $h \in (3,4)$ there exists a conjugated Hénon map f as above such that the Hausdorff dimension of the forward Julia set J^+ of f is exactly equal to h. In fact, it follows from the proof of Theorem 4.1 in [Wo] that the dimension of J^+ is equal to h in any neighborhood of a point $p \in J^+$.

Now let $h_1, h_2, \ldots$ be an increasing sequence of real numbers that converge to 4. Let $f_1, f_2, \ldots$ be a sequence of Hénon maps as above such that the Hausdorff dimension near any point of the Julia set of f_j is equal to $\frac{h_j+4}{2}$.

Conjugate the maps f_j with translations that move the attracting fixed point to zero to get that f_j has an attracting fixed point at zero. Of course, the Hausdorff-dimension properties of the maps are unchanged by this conjugation. For every map there is a small ball centered at 0, say $B(r_j)$, such that f_j attracts to 0 on $B(r_j)$.

We will denote by J_j^+ the forward Julia set of f_j. Choose an increasing sequence $R_1, R_2, \dots$ with $\lim R_j = \infty$ such that the constant R_j defines a filtration D_j, V_j^+, V_j^- for the map f_j.

In the theorem below we will construct integers $n_1, n_2, \dots$ and study the forward Julia set of the sequence $f_1^{n_1}, f_1^{n_2}, \dots$, which we will denote by J^+. As before, we will write F_j for $f_j^{n_j}$

Theorem 7. *We can choose integers $n_1, n_2, \dots$ large enough such that the upper box dimension of J^+ is equal to 4 near any point in J^+.*

Proof. First of all, make sure that all n_j's are so large that $F_j(B(r_j)) \subset B(r_{j+1})$ and $F_j(V_j^+) \subset V_{j+1}^+$ and such that if $z_j \in B(r_{j+1})$ (or in V_{j+1}^+) then the orbit $\{z_n\}$ converges to the origin (or to the attracting point at infinity respectively).

We will define the sequence $n_1, n_2, \dots$ inductively. Suppose that we have fixed $n_1, \dots, n_{j-1}$.

Let I_j be the forward Julia set of the sequence $F_1, F_2, \dots F_{j-1}, f_j, f_j, \dots$. We have that $z \in I_j$ if and only if $F(j-1)(x) \in J_j^+$, and thus $I_j = F(j-1)^{-1} J_j^+$. Therefore we have that the Hausdorff dimension of I_j near any point of I_j is also equal to $\frac{h_j+4}{2}$.

Let $\mathcal{B}_j$ be a finite collection of open balls of radius $(\frac{1}{2})^j$ that covers $I_j \cap D_j$ so that every element of $\mathcal{B}_j$ intersects I_j. Since the Hausdorff dimension of I_j is strictly larger than h_j near any point in I_j, we can choose some small $\hat{\epsilon}_j > 0$ such that $\gamma_{h_j}^{\hat{\epsilon}_j}(B \cap I_j) > 2^{j+1}$ for any $B \in \mathcal{B}_j$.

Now let $\epsilon_j < \hat{\epsilon}_j$ such that

$$\left(\frac{\hat{\epsilon}_j}{\epsilon_j}\right)^{h_j} < 2,$$

and let $\delta_j < \hat{\epsilon}_j - \epsilon_j$.

Define the set K_j by

$$K_j = \{z \in D_j \mid d(z, I_j) \geq \delta_j\}.$$

For every $z \in K_j$, we have that $f_j^n F(j-1)(z)$ converges either to 0 or to the attracting point at infinity, and K_j is compact. Therefore we can choose $n_j \in \mathbb{N}$ such that $F(j)(z) = f_j^{n_j} F(j-1)(z)$ lies in $B(r_{j+1})$ or V_{j+1}^+ for any $z \in K_j$. It follows that we can further increase n_j if necessary so that for any z in the compact set $I_j \cap D_j$ there exist $x, y \in B(z, \delta_j)$ such that $F(n)(x) \in B(r_{j+1})$ and $F(n)(y) \in V_{j+1}^+$. It follows that $d_H(I_j \cap D_j, J^+ \cap D_j) < \delta_j$.

Now let $z \in J^+$, let $h \in (0, 4)$, and let $\epsilon > 0$. It suffices to show that the upper-box dimension of $J^+ \cap B(z, \epsilon)$ is larger than or equal to h.

Let $j \in \mathbb{N}$ be large enough so that we have $h_j > h$, $z \in D_j$ and $\epsilon > 3(\frac{1}{2})^j$. Then there exists a $B \in \mathcal{B}_j$ such that $B \subset B(z, \epsilon)$. Let $\{B_i\}$ be an ϵ_j covering of $J^+ \cap B(z, \epsilon)$. We write $\tilde{B}_i$ for the ball with the same center as B_i but with radius $\hat{\epsilon}_j$. Since $\delta_j < \hat{\epsilon}_j - \epsilon_j$ and $d_H(I_j \cap D_j, J^+ \cap D_j) < \delta_j$ we have that $\{\tilde{B}_i\}$ is an

$\hat{\epsilon}_j$-covering of $B \cap I_j$, and thus that $\hat{\epsilon}_j^{h_j} \# \tilde{B}_i > 2^{j+1}$. Since $\hat{\epsilon}_j^{h_j}/\epsilon_j^{h_j} < 2$ we have that

$$\gamma_h^{\epsilon_j}(J^+ \cap B(z,\epsilon)) > \gamma_{h_j}^{\epsilon_j}(J^+ \cap B(z,\epsilon)) > 2^j. \tag{8}$$

Since Equation (8) holds for j large enough, we have that $\mu_h(J^+ \cap B(z,\epsilon)) = \infty$, and this completes the proof. □

We can now prove:

Corollary 1. *There exists a Fatou-Bieberbach domain whose boundary has upper box dimension* 4 *near any boundary point.*

Proof. It follows from Theorem 3 that we can enlarge the integers $n_1, n_2, \dots$ obtained in Theorem 7 if necessary to get that the $\{r_j\}$-calibrated basin of attraction of the sequence $f_1^{n_1}, f_2^{n_2}, \dots$ is biholomorphic to $\mathbb{C}^k$. Since there are orbits converging to infinity, we have that $\Omega(\{r_j\})$ is a proper subset of $\mathbb{C}^2$ and thus a Fatou-Bieberbach domain.

In the proof of Theorem 7 we made sure that every open ball in $\mathbb{C}^k$ contains a point whose orbit either converges to the origin or to the attracting point at infinity. Therefore there are only two Fatou-components, and one of them is $\Omega(\{r_j\})$. Hence we have that $\partial\Omega(\{r_j\})$ is exactly equal to J^+ and we are done. □

It would be interesting to know whether the Hausdorff dimension of the boundary of Ω is also equal to 4.

6. Hausdorff Dimension 4

In our construction we will make use of the following version of Theorem 2.3 in [FR]:

Theorem 8. *Let $K_1, K_2, ..., K_m$ be pairwise disjoint polynomially convex compact sets in $\mathbb{C}^k$ whose union is polynomially convex, and assume that $K_1, K_2, .., K_l$ are star-shaped ($l \leq m$). Let $\phi_i \in Aut(\mathbb{C}^k)$ be automorphisms for $i = 1, ..., l$ so that the sets $K_i' = \phi_i(K_i)$ and the sets $K_{l+1}, ..., K_m$ are pairwise disjoint, and their union is polynomially convex. Let $\epsilon > 0$. Then there exists an automorphism $\phi \in Aut(\mathbb{C}^k)$, so that $\|\phi(z) - \phi_i(z)\| < \epsilon$ for all $z \in K_i$ and $i = 1, .., l$, and $\|\phi(z) - z\| < \epsilon$ for all $z \in K_i$ and $i = l+1, .., m$.*

The proof of this theorem is a small modification of the proof of Theorem 2.3 in [FR], and we give a short outline of how one makes this modification:

Define $K_0 = \cup_{j=l+1}^m K_j$. Choose $R > 0$ so that $K_0 \subset B(R)$, and let p_i denote the contracting center of K_i for $i = 1, ..., l$. By choosing appropriate $\mathcal{C}^2$ paths from the p_i's to separate points outside of $\overline{B(R)}$, and using these to define an isotopy of biholomorphisms, it is clear that by the arguments from the proof of Theorem 2.3 in [FR] one can find an automorphism that stays close to the identity on K_0 and maps the other K_i's outside of $B(R)$. Now Theorem 2.3 applies and we can find an automorphism that moves K_0 far away while staying close to the identity on the images of the other K_i's. If we have moved K_0 far enough, we can use the inverses of our paths to map the images of the other K_i's approximately back to their original positions. Now we can apply Theorem 2.3 to get a single automorphism that approximates each ϕ_i well on each K_i for $i = 1, ...m$ and that stays close to the identity on the image of K_0 (notice that the K_i's may no longer be star-shaped, so we might have to pass to slightly bigger compacts). Now repeat this procedure to map the image of K_0 approximately back to K_0, i.e. define paths to move the images of the K_i's so far away for $i = 1, ...m$ that we can move the image of K_0 approximately back to K_0, and then use the inverse paths to finish the construction.

Remark 4. Recall these basic facts concerning polynomially convex compact sets:
(i) The union of a polynomially convex compact set and a finite set of points is polynomially convex,
(ii) If $K_1 \cup K_2$ is polynomially convex and compact, $K_1 \cap K_2 = \emptyset$, and $K_1' \subset K_1$ is polynomially convex and compact, then $K_1' \cup K_2$ is polynomially convex,
(iii) A polynomially convex compact set has a neighborhood basis consisting of polynomially convex compact sets.
One can prove these facts as follows:
To show (i), let $q \in \mathbb{C}^k \setminus (K \cup \{p\})$ and $f \in \mathcal{O}(\mathbb{C}^k)$ so that $f(p) = 0$ and $f(q) \neq 0$. Since K is polynomially convex, there exists a $g \in \mathcal{O}(\mathbb{C}^k)$ so that $\|g\|_K < 1$ and so that $g(q) = 1$. If m is large enough, $g^m \cdot f$ separates q from $\{p\} \cup K$, and the result follows by induction. To prove (ii), let $p \in \mathbb{C}^k \setminus (K_1' \cup K_2)$. We may assume that $p \in K_1 \setminus K_1'$. There exists an $f \in \mathcal{O}(\mathbb{C}^k)$ separating p from K_1'. We get (ii) by defining a constant function on K_2 and applying the Oka-Weil Theorem [Ra, p.220]. To prove (iii), observe that a polynomially convex compact set has a neighborhood basis $U_1 \supset\supset U_2 \supset\supset ...$ consisting of analytic polyhedra defined by entire functions, i.e. a Runge and Stein neighborhood basis [Ra, p.71], which means that the sets $\{\widehat{\overline{U}_i}\}$ satisfy (iii).

Theorem 8.5 in [RR] states that for any sequence of points $\{p_j\}_{j \in \mathbb{N}} \subset \mathbb{C}^2$ and any strictly convex compact set K such that $\{p_j\}_{j \in \mathbb{N}} \cap K = \emptyset$ one can find a Fatou-Bieberbach Domain $\Omega \subset \mathbb{C}^k \setminus K$ so that $\{p_j\}_{j \in \mathbb{N}} \subset \Omega$. We can use Theorem 8 and Theorem 2 to prove a slight generalization of this theorem:

Theorem 9. *Let K be a polynomially convex compact subset of $\mathbb{C}^k$, and let $\{p_j\}_{j \in \mathbb{N}} \subset \mathbb{C}^k \setminus K$. Then there exists a Fatou-Bieberbach domain Ω so that $\{p_j\} \subset \Omega \subset \mathbb{C}^k \setminus K$.*

Proof. We may assume that K does not intersect the unit ball in $\mathbb{C}^k$. Let A be the linear map defined by

$$A : (z_1, \cdots, z_k) \to (\frac{z_1}{2}, \cdots, \frac{z_k}{2})$$

Theorem 2 implies that there exists a $\delta > 0$ so that if $\{f_j\}_{j \in \mathbb{N}} \subset Aut_0(\mathbb{C}^k)$ is a sequence of automorphisms with $\|f_i - A\|_{\overline{B}} < \delta$ for all $i \in \mathbb{N}$, then $\Omega_{\{f_j\}}$ is biholomorphic to $\mathbb{C}^k$.

We will construct a sequence of automorphisms by induction, with induction hypothesis I_j as follows: We have constructed automorphisms $\{f_1, ..., f_j\}$ so that the following are satisfied:

(9) $$\|f_i - A\|_{\overline{B}} < \delta \text{ for } i = 1, ...j$$

(10) $$f(j)(p_i) \subset B \text{ for } i = 1, .., j$$

(11) $$f(j)(K) \subset \mathbb{C}^k \setminus \overline{B}$$

I_1 is satisfied by assuming K to be far enough away from the unit ball, letting p_1 be the origin, and defining $f_1 = A$. Now, assume that I_j is satisfied. Theorem 8 gives that for any $\mu > 0$ there exists a $\psi \in Aut_0(\mathbb{C}^k)$ so that $\|\psi - A\|_{\overline{B}} < \mu$, and $\|\psi - id\|_{f(j)(K)} < \mu$. There also exists a $\phi \in Aut_0(\mathbb{C}^k)$ so that $\|\phi - id\|_{f(j)(K) \cup \overline{B}} < \mu$

and $\phi(f(j)(p_{j+1})) \in \psi^{-1}(B)$. Then we get I_{j+1} by letting μ be sufficiently small and $f_{j+1} = \psi\phi$.

Now by (9) and the choice of δ, $\Omega = \Omega_{\{f_j\}}$ is biholomorphic to $\mathbb{C}^k$, and (10) and (11) give that Ω satisfies the claims of the theorem. $\square$

Corollary 2. *There exists a Fatou-Bieberbach domain $\Omega \subset \mathbb{C}^2$ so that the 4-dimensional Hausdorff measure of $\partial\Omega$ is non-zero.*

Proof. Let $D = (\overline{D})^\circ \subset\subset \mathbb{C}$ be a simply connected open set so that ∂D has non-zero 2-dimensional Hausdorff measure. Then $K = \overline{D \times D} \subset \mathbb{C}^2$ is a polynomially convex compact set whose boundary has non-zero 4-dimensional Hausdorff measure. Let $P = \{p_j\}$ be dense in $\mathbb{C}^2 \setminus K$. By Theorem 9, there exists a Fatou-Bieberbach domain Ω so that $P \subset \Omega \subset \mathbb{C}^2 \setminus K$, which implies that $\partial K \subset \partial\Omega$. $\square$

Notice that this corollary only tells us that a part of the boundary of Ω is large; it does not give any information about the size of the boundary near other points. We now prove Theorem 1:

Proof of Theorem 1: Let $D = (\overline{D})^\circ \subset\subset \mathbb{C}$ be a simply connected set whose boundary has non-zero 2-dimensional Hausdorff measure. Then $K = \overline{D \times D}$ is a polynomially convex compact set whose boundary has non-zero 4-dimensional Hausdorff measure. Let $p \in \mathbb{C}^2$. We will let $K_\epsilon(p)$ denote an arbitrary such K with $p \in K$ and $K \subset B(p, \epsilon)$.

Let A be the linear map defined by

$$A\colon (z_1, z_2) \to (\frac{z_1}{2}, \frac{z_2}{2})$$

It follows from Theorem 2 that we can choose a $\delta > 0$ so that if we construct a sequence of automorphisms $\{f_j\} \subset Aut_0(\mathbb{C}^2)$ with $\|f_i - A\|_{\overline{B}} < \delta$ for all $i \in \mathbb{N}$, then $\Omega_{\{f_j\}}$ is a Fatou-Bieberbach domain.

Choose a sequence of strictly positive numbers $\{\epsilon_j\}_{j\in\mathbb{N}}$ converging to zero. We will construct a sequence of automorphisms inductively with the following induction hypothesis I_j: We have defined automorphisms $\{f_1, ..., f_j\} \subset Aut_0(\mathbb{C}^2)$, and a polynomially convex compact set $K^j = K_1^j \cup \cdots \cup K_m^j \subset B(j+1)$, where each K_i^j is equal to $K_\epsilon(p)$ for some point p and an $\epsilon > 0$. We also have a set of points $\{t_j\}_{j=1}^l \subset B(j+1)$, and a ball B_{R_j} so that $\overline{B}_{R_j} \cap \overline{B} = \emptyset$. In addition, the following are satisfied:

(12) $\|f_i - A\|_{\overline{B}} < \delta$ for $i = 1, \ldots, j$,

(13) $B(j+1) \setminus f(j)^{-1}(\overline{B}) \neq \emptyset$,

(14) $f(j)(t_i) \in B$ for $i = 1, \ldots, l$,

(15) For any $p \in B(j+1) \setminus (K^j)^\circ$ there is a t_i so that $\|t_i - p\| < \epsilon_j$,

(16) For any $p \in B(j+1) \setminus f(j)^{-1}(B)$ we have that $d(p, K^j) < \epsilon_j$,

(17) $f(j)(K^j) \subset B_{R_j}$.

We will now show how to construct f_{j+1} so as to satisfy I_{j+1}. It will be clear from the construction how to define f_1.

Choose a set of points $\{p_i\}_{i=1}^n \subset B(j+2) \setminus (K^j \cup f(j)^{-1}(\overline{B}))$, so that for any point $p \in B(j+2) \setminus ((K^j)^\circ \cup f(j)^{-1}(B))$ there is a p_i with $\|p - p_i\| < \epsilon_{j+1}$. Let $q_i = f(j)(p_i)$ for $i = 1, ..., n$. Then Remark 4 tells us that $\overline{B} \cup f(j)(K^j) \cup \{q_i\}_{i=1}^n$

is a polynomially convex compact set, and that we can choose a $\rho > 0$ so that the following are satisfied:

$$\overline{B} \cup f(j)(K^j) \cup (\cup_{i=1}^n \overline{B(q_i, \rho)}) \text{ is polynomially convex,} \tag{18}$$

$$\overline{B(q_i, \rho)} \cap (\overline{B} \cup f(j)(K^j)) = \emptyset \text{ for } i = 1, ..., n, \tag{19}$$

$$\overline{B(q_i, \rho)} \cap \overline{B(q_l, \rho)} = \emptyset \text{ for } i \neq l. \tag{20}$$

For each i, let $\tilde{K}_i = K_\rho(q_i)$, and define $K^{j+1} = K^j \cup f(j)^{-1}(\cup_{i=1}^n \tilde{K}_i)$, which is also is polynomially convex by Remark 4. This takes care of (16).

Let $\{t_j\}_{j=1}^k \subset B(j+2) \setminus K^{j+1}$ such that for any $p \in B(j+2) \setminus (K^{j+1})^\circ$, there is some i so that $\|p - t_i\| < \epsilon_{j+1}$. This ensures (15). Let $\tilde{t}_i = F(j)(t_i)$ for $i = 1, ..., k$.

We will now construct f_{j+1}, referring to Theorem 8 for each statement made about the existence of an automorphism. For any $\mu > 0$ there exists a $\varphi \in Aut_0(\mathbb{C}^2)$ with $\|\varphi(z) - z\| < \mu$ for all $z \in \overline{B} \cup F(j)(K^j)$, so that there exists an R_{j+1} with $\overline{B}_{R_{j+1}} \cap \overline{B} = \emptyset$, and $\tilde{K}^{j+1} = \varphi(\cup_{j=1}^n \tilde{K}_i \cup F(j)(K^j)) \subset B_{R_{j+1}}$. There exists some $\phi \in Aut_0(\mathbb{C}^2)$ so that $\|\phi(z) - z\| < \mu$ for all $z \in \overline{B}_{R_{j+1}}$ and so $\|\phi - A\|_{\overline{B}} < \mu$. Lastly, there exists a $\psi \in Aut_0(\mathbb{C}^2)$ with $\|\psi(z) - z\| < \mu$ for all $z \in \overline{B} \cup \tilde{K}^{j+1}$ and $\psi(\varphi(\tilde{t}_i)) \in \phi^{-1}(B)$ for $i = 1, \ldots, k$. If we choose μ small enough and define $f_{j+1} = \phi\psi\varphi$, we have now satisfied (12),(13),(14) and (17), completing the induction step.

We have now constructed a sequence of automorphisms $\{f_j\}_{j\in\mathbb{N}}$, and by (12) and Theorem 2 the basin of attraction of zero is biholomorphic to $\mathbb{C}^2$. Since it is obviously not the whole of $\mathbb{C}^2$, it is a Fatou-Bieberbach domain which we will denote by Ω. It is clear from (17) that none of the sets K^j in the above construction are contained in Ω, and it is clear from (14) that all of the t_i's chosen at each step are. Let K_i be one of the sets from step j, and let $p \in \partial K_i$. Because of (15), there is a sequence of t_i's converging to p, and since all of these are in Ω we have that $p \in \partial\Omega$. Thus the increasing union $K = \cup_{j=1}^\infty \partial K^j$ is a set whose 4-dimensional Hausdorff measure is non-zero at any point and $K \subset \partial\Omega$. It follows from (16) that K is dense in $\partial\Omega$, and this completes the proof.

7. Hausdorff dimension 3

Let Ω be a Fatou-Bieberbach domain in $\mathbb{C}^k$. If the complement of Ω has non-empty interior it is easy to see that the Hausdorff dimension of $\partial\Omega$ is at least $2k-1$. However it is possible for Ω to be dense in $\mathbb{C}^k$ (see [RR] or Theorem 9) so this does not guarantee that for an arbitrary Ω, the dimension of $\partial\Omega$ is at least $2k-1$. We can, however, prove the following:

Theorem 10. *Let Ω be a Fatou-Bieberbach domain in $\mathbb{C}^k$ which is Runge. Then the Hausdorff dimension of $\partial\Omega$ is at least $2k-1$ near any point of the boundary.*

Proof. Assume that $0 \in \partial\Omega$. It is enough to show that the dimension of $\partial\Omega \cap B(\epsilon)$ is at least $2k-1$ in for an arbitrary $\epsilon > 0$. If there are interior points of $\mathbb{C}^k \setminus \Omega$ in $B(\epsilon)$ then the result follows immediately, so we may assume that $\Omega \cap B(\epsilon)$ is dense in $B(\epsilon)$.

Let U be an open subset of $B(\epsilon)$ consisting of annuli that are uniformly bounded away from the hyperplane $\{z_k = 0\}$. Define $f : U \to \mathbb{C}^{k-1} \times \mathbb{R}$ by

$$f(z) = (\frac{z_1}{z_k}, \ldots, \frac{z_{k-1}}{z_k}, \|z\|).$$

This is a smooth open mapping, and the preimages of points are circles centered at the origin. Since Ω is Runge we have that every circle centered at the origin intersects $\partial\Omega$. Thus we have that $f(\partial\Omega \cap U) = f(U)$ which is an open subset of $\mathbb{R}^{2k-1}$, and therefore the dimension of $U \cap \partial\Omega$ is at least $2k-1$. □

Remark 5. Theorem 10 does not rule out the existence of a non-Runge Fatou-Bieberbach domain whose boundary has Hausdorff dimension strictly less than 3. Whether there exist Fatou-Bieberbach domains which are not Runge is an open question.

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