MAT 540, Homework 9, due Wednesday, Nov 29

The first two questions are a review of the homotopy exact sequence. We'll need the homotopy exact sequence of a triple soon.

1. Suppose that (X, A, B) is a CW triple, with $B \subset A \subset X$ (each space is a CW subcomplex of the next), $x_0 \in B$ is the basepoint. Then there is an exact sequence of the triple

$$\cdots \to \pi_n(A, B, x_0) \to \pi_n(X, B, x_0) \to \pi_n(X, A, x_0) \to \pi_{n-1}(A, B, x_0) \to \cdots \to \pi_1(A, B, x_0),$$

generalizing the homotopy exact sequence of a pair.

Describe the connecting homomorphism $\partial : \pi_n(X, A, x_0) \to \pi_{n-1}(A, B, x_0)$ explicitly (the other two maps are induced by the obvious inclusions), and prove exactness at the term $\pi_n(X, A, x_0)$.

Check exactness at other terms, too, but do not submit these other proofs to reduce writing and grading.

2. Show that the homotopy exact sequence of a triple is *natural* in the following sense: a continuous map of triples $f : (X, A, B, x_0) \to (Y, C, D, y_0)$ induces a map between the corresponding exact sequences (via maps $f_* : \pi_n(X, A, x_0) \to \pi_n(Y, C, y_0)$, etc), such that all squares in the resulting diagram commute.
The exact sequence of a pair is also natural (this follows by the same argument, or by setting $B = D = x_0$).

3. Prove that every continuous map $f : X \to Y$ is homotopy equivalent to a Serre fibration, in the following sense. Consider the space

$$\tilde{X} = \{(x, \gamma)|\ x \in X, \ \ \gamma \text{ a path in } Y \text{ starting at } f(x)\}.$$

Then, we have maps $\phi : \tilde{X} \to X$ given by $\phi(x, \gamma) = x$ and $p : \tilde{X} \to Y$ given by $p(x, \gamma) = \gamma(1)$. Check that

(1) ϕ is a homotopy equivalence;
(2) p is a strong Serre fibration;
(3) $f \circ \phi$ is homotopic to p.

You can find proofs in Hatcher or Fomenko–Fuchs (sections 9.4, 9.7), but most details are still left to the reader. It will be easier to make the complete proofs directly (but feel free to treat this as a combination of reading and filling the gaps).

4. This is a question that Dennis Sullivan proposed for the comps a couple of years ago (the question didn't make it into the comps, but it's a good one to think about now). It's about the homotopy lifting property (with respect to disks or CW spaces).

(a) Show that the map $p : \mathbb{C} \to \mathbb{C}$ given by $p(z) = z^2$ does **not** have the homotopy lifting property.

(b) How can this map be altered to have the homotopy lifting property?

5. Compute $\pi_2(S^2 \vee S^1)$.

6. Let M_g be the closed orientable surface of genus $g \geq 0$. For what g, h does there exist a covering $p : M_g \to M_h$?