MAT 540, Homework 8, due Wednesday, Nov 15

1. In class, we derived the homotopy exact sequence of a Serre fibration $p : E \to B$ with fiber F from the isomorphism $p_* : \pi_n(E, F, x_0) \to \pi_n(B, b_0)$ and the homotopy exact sequence of a pair, and then used it to compute $\pi_2(S^2) = \pi_1(S^1) = Z$.

(a) Describe the map $\pi_n(B, b_0) \to \pi_{n-1}(F, x_0)$ explicitly, $n \geq 0$.

(b) Use (a) to find the generator of $\pi_2(S^2)$.

2. A Stiefel manifold $V_n(\mathbb{R}^k)$ is the space of all n-frames (n-tuples of orthogonal unit vectors) in $\mathbb{R}^k$. Prove that there is a fiber bundle $V_n(\mathbb{R}^k) \to V_m(\mathbb{R}^k)$ for $m < n$; the projection is given by taking an n-frame to the m-frame given by its first m vectors.

Using this fiber bundle and its homotopy exact sequence, show that $\pi_r(V_n(\mathbb{R}^k)) = 0$ for $r \leq k - n - 1$. What is $\pi_1(V_{k-1}(\mathbb{R}^k))$?

This is discussed in Hatcher Example 4.54, although most details are omitted (leaving a fair amount of work to do), so feel free to check Hatcher if you need a hint.

3. Let X be an arbitrary topological space with a base point x_0. Consider the space of paths in X starting at x_0:

$$E = \{\gamma : I \to X \text{ path},\ \gamma(0) = x_0\}.$$

The topology on E is the usual compact-open topology (think space of continuous maps $I \to X$). Define the map $p : E \to X$ by $p(\gamma) = \gamma(1)$.

Prove that $p : E \to X$ is a "strong" Serre fibration (ie homotopy lifting holds for all spaces, not only CW complexes).

You can find a proof in Fomenko–Fuchs (section 9.4), with most details left to the reader. It will be easier to make the complete proofs directly (but feel free to treat this as a combination of reading and filling the gaps). If you need your space to be Hausdorff, locally compact, etc, feel free to assume it.

4. Show that S^2 and $S^3 \times \mathbb{C}P^\infty$ have the same homotopy groups in every dimension but are not homotopy equivalent. (Again, you really need to have that map in the Whitehead theorem!)

5. Please also do question 24 from Hatcher section 1.3, p. 81.