MAT 540, Homework 2, due Wednesday, Sept 20, in class

You can use everything we proved in class.

1. Let X be a CW space. Show that the following properties are equivalent:
(i) X is path connected;
(ii) X is connected;
(iii) The 1-skeleton X^1 is connected.

2. Let (X, A) be a CW pair. Prove that A has an open neighborhood U in X such that U deformation retracts onto A.

3. Given a CW complex X, give an induced CW structure on $X \times I$ (thinking of $I = [0,1]$ as a CW complex with two 0-cells $\{0\}$ and $\{1\}$ and one 1-cell $(0,1)$): describe the skeleta and the attaching maps for $X \times I$ in terms of the data for X.

Prove that the product topology on $X \times I$ is the same as the CW topology.

Warning: for arbitrary CW complexes X, Y one can similarly define a CW structure on $X \times Y$, but this CW-topology may be different from product topology! However, the product topology coincides with the CW topology on $X \times Y$ if X or Y is locally compact.

4. Prove that S^∞ is contractible. (Recall that S^∞ is a CW complex $S^0 \subset S^1 \subset S^n \subset \dots$, built inductively, with 2 cells in each dimension.)

5. Let X be any topological space, and let $\phi_0, \phi_1 : \partial D^n \to X$ be two homotopic continuous maps, where D^n is the n-disk. Show that

$$X \sqcup_{\phi_0} D^n \sim X \sqcup_{\phi_1} D^n,$$

that is, attaching a cell via two homotopic maps produces homotopy equivalent spaces.

6. Let X, X' be two homotopy equivalent topological spaces, $f : X \to X'$ a homotopy equivalence. Let $\phi : \partial D^n \to X$ be continuous. Show that

$$X \sqcup_\phi D^n \sim X' \sqcup_{f \circ \phi} D^n,$$

more precisely, the map $X \sqcup_\phi D^n \to X' \sqcup_{f \circ \phi} D^n$ determined by f and id_{D^n} is a homotopy equivalence.

Note for 5 and 6: these statements hold for general X, X' but are most useful in practice for cell complexes. Instead of attaching a single cell, similar results are true for gluing a CW pair (Y, A):

- if $\phi_0, \phi_1 : A \to X$ are two homotopic maps, then $X \sqcup_{\phi_0} Y \sim X \sqcup_{\phi_1} Y$;
- if $f : X \to X'$ is a homotopy equivalence, $\phi : A \to X$, then $X \sqcup_\phi Y \sim X' \sqcup_{f \circ \phi} Y$.

Proofs of these more general statements would require either a procedure similar to the one we used to prove the homotopy extension property, or a direct application of the homotopy extension property to some related spaces. (Even more generally, the above statements are true for any pair (Y, A) that has the homotopy extension property, with A closed.)