MAT 511

Homework 3

due Thursday, Sept 25

Please explain everything and give careful proofs.

In class, we considered sequences (x_n) with positive terms, $x_n > 0$, and worked with definitions of increasing, decreasing, and bounded sequences. Let us state some more definitions:

(S-INC) A sequence (x_n) is *strictly increasing* if $\forall n\ x_n < x_{n+1}$.
(This is slightly different from increasing sequences we considered in class, as we no longer allow $x_n = x_{n+1}$.)

(BDD) A sequence (x_n) is bounded if $\exists M\ \forall n\ x_n < M$.
(This is the definition we had in class.)

(∞) A sequence (x_n) goes to ∞ if $\forall M\ \exists N\ \forall n > N\ x_n > M$.

1. Give a definition of a sequence that does not go to infinity by constructing a (useful) denial of the definition **(∞)** above. Give examples (with proofs) of a sequence that goes to ∞ and a sequence that does not go to ∞.

2. Are the followong statements true or false? Prove or give counterexample.
(a) A strictly increasing sequence can be bounded.
(b) A bounded sequence cannot go to infinity.
(c) Every sequence that goes to ∞ is strictly increasing.

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In class, we stated the following Pigeonhole principle:
If there are more than nk rabbits in k cages, then there is a cage that contains more than n rabbits.
(We gave a simple proof by contradiction: if every cage contains no more than n rabbits, then there would be no more than nk rabbits total.)

Use a similar argument to solve the following geometric problem.

3. There are 5 points in a 1in$\times$1in square. Prove that there exist two points such that the distance between them is no more than $\sqrt{2}/2$in. Is it true that there two points with distance less than $\sqrt{2}/2$in?
(Try to cut the square into appropriate "cages". You can use the fact that the largest distance between two points in a square is given by its diagonal. See also question 11 of §1.5 – you don't have to use the Pigeonhole principle for that one, but if you do, you'll use a very similar idea.)

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Please also do question 11 of §1.5, and question 8(efgh) of §1.6.