MAT 311, Homework 7 due 10/25

1. Compute the Legendre symbol to determine whether the following congruences have solutions. Note that 227 and 1009 are prime.

(a) $x^2 \equiv -7 \mod 227$; (b) $x^2 \equiv 11 \mod 41$; (c) $x^2 \equiv 54 \mod 1009$.

2. Find all primes p such that the congruence $x^2 \equiv 10 \mod p$ has solutions.

3. (a) Show that there exist infinitely many positive integers n which *cannot* be written as a sum of three squares,

$$n = x^2 + y^2 + z^2.$$

Note: In class, we proved Fermat's theorem (2.15, section 2.1) describing which integers can be represented as a sum of two squares. The complete answer for the sum of three squares is also known. A theorem of Lagrange says that *every* integer can be represented as a sum of four squares.

(b) Determine (with proof) what integers can be written as a difference of two squares, $n = x^2 - y^2$. This is much easier than the sum of squares!

4. Find all integer solutions of the following equations. Use basic divisibility and modular arithmetics.

(a) $3x^2 + 2 = y^2$; (b) $2x^2 + y^2 = 8t + 13$.

5. Using Pythagorean triples, prove that the equation $x^2 + y^2 = z^4$ has infinitely many integer solutions where x, y, and z share no common divisors $d > 1$.

6. Find all the integer solutions of the equation

$$x^2 + y^2 + z^2 = 2xyz.$$

Hint: Determine whether x, y, z can be odd or even. For positive integers, use the method of descent.

7. Show that the equation $x^3 = 3y^3 + 9z^3$ has no solutions where x, y, and z are positive integers.

(Optional challenge (do not submit): Mimic some of the steps in the proof of Fermat's theorem (Lemmas 2.13 and 2.14, Theorem 2.15) in Section 2.1 to solve questions 56-60 in section 2.1 and determine which integers can be represented in the form $n = a^2 + 2b^2$.