

NOTES ON SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS

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1. GENERALITIES

The general second order homogeneous linear differential equation with constant coefficients is

$$Ay'' + By' + Cy = 0,$$

where y is an unknown function of the variable x , and A , B , and C are constants. If $A = 0$ this becomes a first order linear equation, which in this case is separable, and so we already know how to solve. So we will consider the case $A \neq 0$. If we want, we can divide through by A and obtain the equivalent equation

$$y'' + by' + cy = 0,$$

where $b = B/A$ and $c = C/A$ (that is if we have nothing better to do, and like the first coefficient to be equal to 1).

Linear with constant coefficients means that each term in the Left Hand Side of the equation is a constant times y or a derivative of y . Homogeneous means that we exclude equations like $Ay'' + By' + Cy = f(x)$ which can be solved, in certain important cases, by an extension of the methods we will study here. Here we only will solve the case where the Right Hand Side $f(x)$ is identically 0. Homogeneous also means that the constant function $y = 0$ is always a solution to the equation.

By now we know to expect 2 degrees of freedom in the solution of this second order equation, i.e., the general family of solutions should have two arbitrary constants. We call the general **family** of solutions for short the **general solution**. That means that to find our general solution we have to find two **independent** functions $y = f_1(x)$ and $y = f_2(x)$ which are solutions, and then the general solution will be $y = C_1 \cdot f_1(x) + C_2 \cdot f_2(x)$. Now If $y = f_1(x)$ and $y = f_2(x)$ are indeed solutions, one can check by plugging in that $y = C_1 \cdot f_1(x) + C_2 \cdot f_2(x)$ will be a solution. The fact that all solutions are of this form (i.e., we haven't missed any solution) is harder to ascertain, but nevertheless true. Notice that of course we do need $f_1(x)$ and $f_2(x)$ to be **independent**. What does it mean for two functions to be independent? It means that one of them is not equal to a constant times the other. For example the functions $f_1(x) = e^x$

and $f_2(x) = 2e^x$ are not independent, because $f_2(x) = 2 \cdot f_1(x)$. On the other hand, for example the functions $g_1(x) = e^x$ and $g_2(x) = e^{2x}$ are independent (although we do not prove it, it seems intuitively clear that they are independent). To construct the general solution for a second order equation we do need two independent solutions. For example we cannot construct a general solution from say $f_1(x) = e^x$ and $f_2(x) = 2e^x$, because then the family $y = C_1 \cdot f_1(x) + C_2 \cdot f_2(x)$ will be $y = C_1 \cdot e^x + C_2 \cdot 2e^x = (C_1 + 2C_2)e^x = C_3e^x$ (here $C_3 = C_1 + 2C_2$). I.e., we only will get one degree of freedom if our functions are not independent, and we need two degrees of freedom. That is why it is important that we get two independent solutions of the equation somehow (by guessing, being smart or any way we can think of).

Let us look for the general solution of our second order homogeneous linear differential equation with constant coefficients. As often happens with differential equations, we try to make an educated guess as to the type of solutions. To do that one usually tries to see what happens with the equations with simplest possible coefficients. As we already agreed that we want $A \neq 0$, we can try checking what happens with the equation if we set the other coefficients to zero. And in fact so far we have already seen examples of 3 types of second order homogeneous linear differential equation with constant coefficients:

We have met in Calculus I with the equation (this is the simplest case when both b and c are equal to 0)

$$y'' = 0.$$

Here we don't need to guess, as we can solve by first getting $y' = C_2$, and then $y = C_1 + C_2 \cdot x$. Thus the general solution is $y = C_1 + C_2 \cdot x$.

We can next try to solve the differential equation we get when $b = -1$, $c = 0$:

$$y'' - y' = 0, \quad y'' = y'$$

We try guessing among our usual collection of functions—constant $y = 1$, exponentials $y = e^{kx}$, trigonometry $y = \sin kx$, $y = \cos kx$ —which one has the first derivative same as the second. We quickly come to the conclusion that of course the exponential $y = e^x$ (which has all its derivative equal among themselves), and the constant $y = 1$ both solve the equation. So the general solution is $y = C_1 \cdot e^x + C_2 \cdot 1$.

Next easiest in line to try to solve is the differential equation (with $b = 0$, $c = 1$):

$$y'' + y = 0, \quad y'' = -y$$

Again we try to guess among our usual collection of functions which of them have second derivative equal to minus the function: that is the trigonometry $y = \sin x$ and $y = \cos x$. These two are indeed solutions, and they are independent, so our general solution is $y = C_1 \sin x + C_2 \cos x$.

In fact we saw another example of this situation:

$$4y'' + 25y = 0,$$

this was problem 7.1.4, and we saw that the solutions look like $y = C_1 \cdot \cos \frac{5}{2}x + C_2 \cdot \sin \frac{5}{2}x$.

We have also seen

$$2y'' + y' - y = 0,$$

this was problem 7.1.3, and we saw that the general solution was $y = C_1 \cdot e^{-x} + C_2 \cdot e^{\frac{1}{2}x}$.

Basically we see that the solutions in the different cases could be **polynomials**, they could be **trigonometry**, or they could be **exponentials**. Of course, by now we fully expect in more complicated cases **the mixtures** of the polynomial, trigonometry and exponential to show up: the solutions could possibly include functions of the kind of $x^n e^{rx} \cos kx$ and $x^n e^{rx} \sin kx$. Luckily, as we will see, in the case of **second order** the mixture of the three will **not appear**;

only the mixture of polynomial-exponential and exponential-trigonometry. But in the **higher order** linear differential equations, one indeed meets with solutions containing the mixture of all three: polynomial-exponential-trigonometry. In this class though, we will only solve homogeneous linear differential equation of second order.

Let us try to find solution in the general case now; i.e., we start to look for solutions of the general equation of second order

$$Ay'' + By' + Cy = 0,$$

Of course we always start with hoping for the best, so we try the easiest possible type of solution, which would be the exponential e^{rx} (easiest to differentiate). We plug in $y = e^{rx}$, trying to find what number r will make this a solution. That will give us:

$$Ar^2e^{rx} + Bre^{rx} + Ce^{rx} = 0, \quad (Ar^2 + Br + C)e^{rx} = 0.$$

But e^{rx} is never zero, so we can cancel it, and we get the equation

$$Ar^2 + Br + C = 0.$$

This is called the **characteristic equation**. Why?: because if e^{rx} is a solution to the differential equation, then r is a solution to this quadratic equation. Let us see what can possibly happen with this quadratic equation.

2. CHARACTERISTIC EQUATION WITH DIFFERENT ROOTS

If the Discriminant of this quadratic equation ($D = B^2 - 4AC$) is strictly positive this quadratic equation has two **different** solutions:

$$r_{1,2} = \frac{-B \pm \sqrt{D}}{2A},$$

thus we have at least two **independent** (different) choices for a solution of the differential equation: e^{r_1x} and e^{r_2x} . And as by now we know to expect 2 degrees of freedom, this is perfect, because the general family of solutions will be

$$y = C_1 \cdot e^{r_1x} + C_2 \cdot e^{r_2x}.$$

(It is easy to check by plugging in that this will be a solution of our differential equation for any C_1 and C_2 . The fact that all solutions are of this form (i.e., we haven't missed any solution) is harder to ascertain, but nevertheless true.)

Examples:

Example 1. Solve the equation

$$y'' - y = 0,$$

Solution: We solve the characteristic equation ("replacing orders with powers", and using that y is the zero order derivative):

$$r^2 - 1 = 0, \quad \text{which gives us } r_1 = 1; \quad r_2 = -1$$

That means

$$e^{1 \cdot x} = e^x \quad \text{and} \quad e^{-1 \cdot x} = e^{-x}$$

are two independent solutions. And thus the general solution is

$$y = C_1e^x + C_2e^{-x}.$$

One can check that by plugging in the differential equation.

Example 2. Let's go back to the equation

$$2y'' + y' - y = 0,$$

The characteristic equation is:

$$2r^2 + r - 1 = 0, \text{ which gives us } r_1 = -1; \quad r_2 = \frac{1}{2},$$

which of course gives the solutions

$$e^{-1 \cdot x} = e^{-x} \text{ and } e^{\frac{1}{2}x},$$

and thus the general solution is:

$$y = C_1 e^{-x} + C_2 e^{\frac{1}{2}x}.$$

One can check that by plugging in the differential equation.

3. CHARACTERISTIC EQUATION WITH REPEATED ROOTS

Let us now try and solve the equation

$$y'' - 4y' + 4y = 0,$$

We write the characteristic equation

$$r^2 - 4r + 4 = 0,$$

and because the discriminant $D = 16 - 16 = 0$, we have $r_1 = r_2 = 2$. This only gives us the one solution e^{2x} , and no second solution for the second degree of freedom. How then to get the second solution? To get inspiration consider again the example

$$y'' = 0,$$

Here we know the solution, $y = C_1 \cdot 1 + C_2 \cdot x$. But let us solve it with the characteristic equation:

$$r^2 = 0 \text{ which gives us both } r_1 = 0 \text{ and } r_2 = 0.$$

Now as we know the general solution: $y = C_1 \cdot 1 + C_2 \cdot x$, we can see that the first independent solution is indeed $e^{0 \cdot x} = 1$, but the second one appears to be $x \cdot e^{0 \cdot x} = x$. (Indeed x is a solution to the $y'' = 0$ —we can check by plugging in).

We can now go back to the equation $y'' - 4y' + 4y = 0$ and try something similar: the first independent solution is e^{2x} , try with a second one: $x e^{2x}$. We check that it is a solution by plugging in:

We have

$$(x e^{2x})' = e^{2x} + 2x e^{2x}; \quad (x e^{2x})'' = 2e^{2x} + (2e^{2x} + 4x e^{2x}) = 4e^{2x} + 4x e^{2x}$$

Plugging in:

$$y'' - 4y' + 4y = (x e^{2x})'' - 4(x e^{2x})' + 4(x e^{2x}) = 4e^{2x} + 4x e^{2x} - 4(e^{2x} + 2x e^{2x}) + 4x e^{2x} = 0,$$

so $x e^{2x}$ is indeed a solution of the differential equation $y'' - 4y' + 4y = 0$. Then our general solution is $y = C_1 e^{2x} + C_2 x e^{2x}$.

We can do that every time we have repeated roots of the characteristic equation (i.e., when the discriminant $D = B^2 - 4AC = 0$):

If $r_1 = r_2 = r$ are the roots of the characteristic equation, then the first independent solution is $y = e^{rx}$, and the second one is $y = xe^{rx}$. The general solution is $y = C_1e^{rx} + C_2xe^{rx}$.

Example: Solve the equation

$$y'' - 6y' + 9 = 0,$$

Solution: the characteristic equation is

$$r^2 - 6r + 9 = 0, \quad r_1 = r_2 = 3;$$

The two independent solutions are

$$y_1 = e^{3x}, \quad y_2 = xe^{3x};$$

the general solution is:

$$y = C_1e^{3x} + C_2xe^{3x}.$$

One can check that by plugging in the differential equation.

4. CHARACTERISTIC EQUATION WITH NO REAL ROOTS

Let us now try and solve again the following equation:

$$4y'' + 25y = 0,$$

but let's try through the characteristic equation. The problem is that the characteristic equation is

$$4r^2 + 25 = 0, \quad \text{thus } r_{1,2} = \frac{-B \pm \sqrt{D}}{2A} = \frac{0 \pm \sqrt{-400}}{8} = \frac{\pm 5}{2}\sqrt{-1},$$

and it doesn't have real roots ($\sqrt{-1}$ is a slight problem :). But we know that the solution of the differential equation is $y = C_1 \cdot \cos \frac{5}{2}x + C_2 \cdot \sin \frac{5}{2}x$. So we notice that the two independent solutions are the trigonometry $y = \cos kx$, and $y = \sin kx$, where $k = \frac{5}{2}$. We can get these if ignore the $\sqrt{-1}$ in $r_{1,2} = \frac{\pm 5}{2}\sqrt{-1}$, **but** instead of exponentials (which we would have if the $\sqrt{-1}$ was not there) we have **trigonometry** now.

Thus the conclusion seems to be that **whenever we have real roots we take exponentials, whenever we have $\sqrt{-1}$ (imaginary roots) we take trigonometry.**

Then the question is what happens with the **mixtures**? For example, let us consider the equation

$$y'' + 4y' + 13y = 0,$$

The characteristic equation is

$$r^2 - 4r + 13 = 0, \quad r_{1,2} = \frac{-B \pm \sqrt{D}}{2A} = \frac{4 \pm \sqrt{-36}}{2} = 2 \pm 3\sqrt{-1}$$

Now we have a mixture: 2 is real (without $\sqrt{-1}$), and $3\sqrt{-1}$ is not. That suggests that we should have 2 appear in an exponential, and the 3 appear in a trigonometry (i.e., either $\cos 3x$ or $\sin 3x$). Thus we concoct the mixtures $y = e^{2x} \cos 3x$ and $y = e^{2x} \sin 3x$ (indeed these are the only two independent mixtures possible in this case). We have to check if they are solutions: we can do that by plugging in. We calculate for $y = e^{2x} \cos 3x$

$$y' = 2e^{2x} \cos 3x - 3e^{2x} \sin 3x$$

$$y'' = 4e^{2x} \cos 3x - 6e^{2x} \sin 3x - 6e^{2x} \sin 3x - 9e^{2x} \cos 3x = -5e^{2x} \cos 3x - 12e^{2x} \sin 3x$$

Plugging in:

$$\begin{aligned} y'' + 4y' + 13y &= -5e^{2x} \cos 3x - 12e^{2x} \sin 3x - 4(2e^{2x} \cos 3x - 3e^{2x} \sin 3x) + 13e^{2x} \cos 3x = \\ &= -5e^{2x} \cos 3x - 12e^{2x} \sin 3x - 8e^{2x} \cos 3x + 12e^{2x} \sin 3x + 13e^{2x} \cos 3x = 0 \end{aligned}$$

So indeed $y = e^{2x} \cos 3x$ is a solution, and we can also check that $y = e^{2x} \sin 3x$ is a solution. Thus the general solution is

$$y = C_1 \cdot e^{2x} \cos 3x + C_2 \cdot e^{2x} \sin 3x = e^{2x}(C_1 \cdot \cos 3x + C_2 \cdot \sin 3x).$$

So the rule then if the quadratic characteristic equation doesn't have real roots (the Discriminant $D = B^2 - 4AC < 0$) is:

Write the solutions of the characteristic equation carefully:

$$r_{1,2} = \frac{-B \pm \sqrt{D}}{2A} = \frac{-B}{2A} \pm \frac{\sqrt{D}}{2A} = \frac{-B}{2A} \pm \left(\frac{\sqrt{-D}}{2A}\right)\sqrt{-1}$$

Then the two independent solutions are

$$y_1 = e^{\frac{-B}{2A}x} \cos\left(\frac{\sqrt{-D}}{2A}x\right), \quad y_2 = e^{\frac{-B}{2A}x} \sin\left(\frac{\sqrt{-D}}{2A}x\right);$$

and the general solution is

$$y = C_1 y_1 + C_2 y_2 = C_1 e^{\frac{-B}{2A}x} \cos\left(\frac{\sqrt{-D}}{2A}x\right) + C_2 e^{\frac{-B}{2A}x} \sin\left(\frac{\sqrt{-D}}{2A}x\right) = e^{\frac{-B}{2A}x} \left(C_1 \cos\left(\frac{\sqrt{-D}}{2A}x\right) + C_2 \sin\left(\frac{\sqrt{-D}}{2A}x\right)\right).$$

Example: Solve the equation

$$y'' - 6y' + 10y = 0,$$

Solution: the characteristic equation is

$$r^2 - 6r + 10 = 0, \quad r_{1,2} = \frac{6 \pm \sqrt{-4}}{2} = 3 \pm 1\sqrt{-1};$$

The two independent solutions are

$$y_1 = e^{3x} \cos 1x = e^{3x} \cos x, \quad y_2 = e^{3x} \sin x;$$

the general solution is:

$$y = C_1 e^{3x} \cos x + C_2 e^{3x} \sin x = e^{3x}(C_1 \cos x + C_2 \sin x).$$

5. SUMMARY ON SOLVING THE LINEAR SECOND ORDER HOMOGENEOUS DIFFERENTIAL EQUATION

To solve the equation:

$$Ay'' + By' + Cy = 0,$$

first solve the characteristic equation:

$$Ar^2 + Br + C = 0.$$

We have 3 cases:

Case 1: Two different real roots ($D > 0$)

$$r_{1,2} = \frac{-B \pm \sqrt{D}}{2A}.$$

The two independent solutions are

$$y_1 = e^{r_1 x}, \quad y_2 = e^{r_2 x};$$

the general solution is

$$y = C_1 y_1 + C_2 y_2 = C_1 e^{r_1 x} + C_2 e^{r_2 x}$$

Case 2: Repeated roots ($D = 0$)

$$r_{1,2} = \frac{-B \pm \sqrt{D}}{2A} = \frac{-B}{2A}, \quad r_1 = r_2 = r = \frac{-B}{2A}.$$

The two independent solutions are

$$y_1 = e^{rx}, \quad y_2 = x e^{rx};$$

the general solution is

$$y = C_1 y_1 + C_2 y_2 = C_1 e^{rx} + C_2 x e^{rx} = e^{rx}(C_1 + C_2 x)$$

Case 3: No real roots ($D < 0$)

$$r_{1,2} = \frac{-B \pm \sqrt{D}}{2A} = \frac{-B}{2A} \pm \frac{\sqrt{D}}{2A} = \frac{-B}{2A} \pm \left(\frac{\sqrt{-D}}{2A}\right)\sqrt{-1}.$$

The two independent solutions are

$$y_1 = e^{\frac{-B}{2A}x} \cos\left(\frac{\sqrt{-D}}{2A}x\right), \quad y_2 = e^{\frac{-B}{2A}x} \sin\left(\frac{\sqrt{-D}}{2A}x\right);$$

and the general solution is

$$y = C_1 y_1 + C_2 y_2 = C_1 e^{\frac{-B}{2A}x} \cos\left(\frac{\sqrt{-D}}{2A}x\right) + C_2 e^{\frac{-B}{2A}x} \sin\left(\frac{\sqrt{-D}}{2A}x\right) = e^{\frac{-B}{2A}x} \left(C_1 \cos\left(\frac{\sqrt{-D}}{2A}x\right) + C_2 \sin\left(\frac{\sqrt{-D}}{2A}x\right)\right).$$

6. SOLVING INITIAL VALUE PROBLEMS

Initial Value problem is a system of

- Differential Equation
- Initial Conditions

Since a second order linear equation has two degrees of freedom, for an initial value problem we need two initial conditions: for y and for y' .

To solve the initial value problem:

- first: solve the differential equation. The general solution will have two constants C_1 and C_2 (because we have two degrees of freedom).
- second: plug in the initial conditions and solve for the constants C_1 and C_2 .

Examples:**Example 1.** Solve the initial value problem:

$$2y'' + y' - y = 0, \quad y(0) = 0, \quad y'(0) = 1$$

Solution: 1. General solution: we already solved the equation, and we got the general solution

$$y = C_1 e^{-x} + C_2 e^{\frac{1}{2}x}.$$

2. Initial conditions: now we plug in to find the constants C_1 and C_2 :

$$y(0) = C_1 e^{-0} + C_2 e^{\frac{1}{2} \cdot 0} = C_1 + C_2 = \text{should equal to } 0$$

$$y'(x) = -C_1 e^{-x} + C_2 \cdot \frac{1}{2} e^{\frac{1}{2}x}$$

$$y'(0) = -C_1 e^{-0} + C_2 \cdot \frac{1}{2} e^{\frac{1}{2} \cdot 0} = -C_1 + \frac{1}{2} \cdot C_2 = \text{should equal to } 1$$

Thus we get the system for C_1 and C_2 :

$$\begin{aligned} C_1 + C_2 &= 0 \\ -C_1 + \frac{1}{2} \cdot C_2 &= 1 \end{aligned}$$

We solve this:

$$\begin{aligned} C_2 &= -C_1 \\ -C_1 - \frac{1}{2} \cdot C_1 &= 1, \quad -\frac{3}{2} \cdot C_1 = 1, \quad C_1 = -\frac{2}{3}, \quad C_2 = \frac{2}{3}. \end{aligned}$$

Thus the solution to the Initial Value problem is

$$y = -\frac{2}{3} e^{-x} + \frac{2}{3} e^{\frac{1}{2}x}.$$

One can check that by plugging in both the differential equation and the initial conditions.

Example 2. Solve the initial value problem:

$$y'' - 4y' + 4y = 0, \quad y(0) = 1, \quad y'(0) = 0$$

Solution: 1. General solution: we already solved the equation, and we got the general solution

$$y = C_1 e^{2x} + C_2 x e^{2x}.$$

2. Initial conditions: now we plug in to find the constants C_1 and C_2 :

$$y(0) = C_1 e^0 + C_2 \cdot 0 \cdot e^0 = C_1 + 0 = \text{should equal to } 1$$

$$y'(x) = C_1 \cdot 2e^{2x} + C_2 \cdot (e^{2x} + 2x \cdot e^{2x})$$

$$y'(0) = 2C_1 e^0 + C_2 \cdot (e^0 + 0 \cdot e^0) = 2C_1 + C_2 = \text{should equal to } 0$$

Thus we get the system for C_1 and C_2 :

$$\begin{aligned} C_1 &= 1 \\ 2C_1 + C_2 &= 0 \end{aligned}$$

We solve this:

$$\begin{aligned} C_1 &= 1 \\ C_2 &= -2C_1 = -2, \quad C_2 = -2. \end{aligned}$$

Thus the solution to the Initial Value problem is

$$y = e^{2x} - 2xe^{2x}.$$

One can check that by plugging in both the differential equation and the initial conditions.

Example 3. Solve the initial value problem:

$$y'' - 6y' + 10y = 0, \quad y(0) = 1, \quad y'(0) = 0$$

Solution: 1. General solution: we already solved the equation, and we got the general solution

$$y = C_1 e^{3x} \cos x + C_2 e^{3x} \sin x = e^{3x}(C_1 \cos x + C_2 \sin x).$$

2. Initial conditions: now we plug in to find the constants C_1 and C_2 :

$$y(0) = C_1 e^0 \cos 0 + C_2 e^0 \sin 0 = C_1 + 0 = \text{should equal to } 1$$

$$y'(x) = (C_1 \cdot 3e^{3x} \cos x - C_1 \cdot e^{3x} \sin x) + (C_2 \cdot 3e^{3x} \sin x + C_2 e^{3x} \cos x)$$

$$y'(0) = (C_1 \cdot 3 \cdot 1 - C_1 \cdot 0) + (C_2 \cdot 3 \cdot 0 + C_2 \cdot 1 \cdot 1) = 3C_1 + C_2 = \text{should equal to } 0$$

Thus we get the system for C_1 and C_2 :

$$\begin{aligned} C_1 &= 1 \\ 3C_1 + C_2 &= 0 \end{aligned}$$

We solve this:

$$\begin{aligned} C_1 &= 1 \\ C_2 &= -3C_1 = -3, \quad C_2 = -3. \end{aligned}$$

Thus the solution to the Initial Value problem is

$$y = e^{3x} \cos x - 3e^{3x} \sin x = e^{3x}(\cos x - 3 \sin x).$$

One can check that by plugging in both the differential equation and the initial conditions.

Problems cribbed from Salas-Hille-Etgen, page 1133

In exercises 1-10, find the general solution. Give the real form.

1. $y'' - 13y' + 42y = 0.$

2. $y'' + 7y' + 3y = 0.$

3. $y'' - 3y' + 8y = 0.$

4. $y'' - 12y = 0.$

5. $y'' + 12y = 0.$

6. $y'' - 3y' + \frac{9}{4}y = 0.$

7. $2y'' + 3y' = 0.$

8. $y'' - y' - 30y = 0.$

9. $y'' - 4y' + 4y = 0.$

10. $5y'' - 2y' + y = 0.$

In exercises 11-16, solve the given initial-value problem.

11. $y'' - 5y' + 6y = 0, \quad y(0) = 1, \quad y'(0) = 1$

12. $y'' + 2y' + y = 0, \quad y(2) = 1, \quad y'(2) = 2$

13. $y'' + \frac{1}{4}y = 0, \quad y(\pi) = 1, \quad y'(\pi) = -1$

14. $y'' - 2y' + 2y = 0, \quad y(0) = -1, \quad y'(0) = -1$

15. $y'' + 4y' + 4y = 0, \quad y(-1) = 2, \quad y'(-1) = 1$

16. $y'' - 2y' + 5y = 0, \quad y(\pi/2) = 0, \quad y'(\pi/2) = 2$

8. THE CONNECTION BETWEEN EXPONENTIALS AND TRIGONOMETRY: EULER'S FORMULA

One may ask how come in some of the cases the solutions to the differential equation are trigonometric functions, and in other cases the solutions are exponential functions; but in both cases the solutions are connected to the characteristic equation. The answer to that question is that there is a connection between the exponential $e^{(\lambda\sqrt{-1})x}$ and the trigonometric functions $\sin \lambda x$ and $\cos \lambda x$.

Of course to find that connection, one first need to make sense of what is $e^{(\lambda\sqrt{-1})x}$; i.e., one needs to work with numbers like $\lambda\sqrt{-1}$, which are not real numbers. That is achieved by giving the number $\sqrt{-1}$ formally a name \mathbf{i} , so we write $\mathbf{i} = \sqrt{-1}$; and we work with it as if nothing happened. We call this number \mathbf{i} the **imaginary unit** (that's where the name i for the $\sqrt{-1}$ comes from). So we now work with numbers of the form $z = a + b \cdot \mathbf{i}$, where the numbers a and b are any real numbers, and $\mathbf{i} = \sqrt{-1}$. We can add and subtract such numbers as usual (as if the \mathbf{i} is a variable, with of course the property $\mathbf{i}^2 = (\sqrt{-1})^2 = -1$):

$$\begin{aligned} z_1 + z_2 &= (a_1 + \mathbf{i} \cdot b_1) + (a_2 + \mathbf{i} \cdot b_2) = (a_1 + a_2) + \mathbf{i} \cdot (b_1 + b_2), \\ z_1 - z_2 &= (a_1 + \mathbf{i} \cdot b_1) - (a_2 + \mathbf{i} \cdot b_2) = (a_1 - a_2) + \mathbf{i} \cdot (b_1 - b_2). \end{aligned}$$

We can also multiply such numbers:

$$\begin{aligned} z_1 \cdot z_2 &= (a_1 + \mathbf{i} \cdot b_1) \cdot (a_2 + \mathbf{i} \cdot b_2) = a_1 \cdot a_2 + \mathbf{i} \cdot a_1 \cdot b_2 + \mathbf{i} \cdot b_1 \cdot a_2 + \mathbf{i}^2 \cdot b_1 \cdot b_2 = \\ &= a_1 \cdot a_2 + \mathbf{i} \cdot a_1 \cdot b_2 + \mathbf{i} \cdot b_1 \cdot a_2 - b_1 \cdot b_2 = (a_1 \cdot a_2 - b_1 \cdot b_2) + \mathbf{i} \cdot (a_1 \cdot b_2 + b_1 \cdot a_2). \end{aligned}$$

Basically each time we just multiply out or add, but we keep the parts with \mathbf{i} separate, and the part without \mathbf{i} separate as well (of course we always keep in mind that $\mathbf{i}^2 = (\sqrt{-1})^2 = -1$).

The part without \mathbf{i} is called real part, the part which goes with the \mathbf{i} is called imaginary part. So for example the number $2 + 3\mathbf{i}$ is a complex number, with real part equal to 2, and imaginary part equal to 3.

The collection of numbers $z = a + \mathbf{i} \cdot b$ (or if one prefers to write it $a + b \cdot \mathbf{i}$), where the numbers a and b are any real numbers, is called **complex numbers**.

So with this newly obtained knowledge we can write the solutions of the characteristic equation when the discriminant is negative, i.e., $D = B^2 - 4AC < 0$, as

$$r_{1,2} = \frac{-B \pm \sqrt{D}}{2A} = \frac{-B}{2A} \pm \frac{\sqrt{D}}{2A} = \frac{-B}{2A} \pm \left(\frac{\sqrt{-D}}{2A}\right)\sqrt{-1} = \frac{-B}{2A} \pm \left(\frac{\sqrt{-D}}{2A}\right)\mathbf{i}.$$

So we see that when the discriminant is negative, the solutions are complex numbers, with the same real part $\frac{-B}{2A}$ and imaginary parts $\pm \frac{\sqrt{-D}}{2A}$. Call for short the real part $\mu = \frac{-B}{2A}$, and $\lambda = \frac{\sqrt{-D}}{2A}$. We want to understand what is $e^{\mu+\lambda\mathbf{i}}$. Of course we know what is e^μ , when μ is real, but we don't know yet what is e^z , when z is complex. As we know how to add and multiply complex numbers we can use the Taylor expansion of the exponential to define what is e^z :

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots + \frac{z^n}{n!} + \cdots$$

One can check that this exponential function is defined for all complex numbers, and it has all the usual properties the real exponential had, for example:

$$e^{z_1+z_2} = e^{z_1} \cdot e^{z_2}.$$

Now if we put in the exponential $z = \mathbf{i}y$, where y is real, we get

$$\begin{aligned} e^{\mathbf{i}y} &= \sum_{n=0}^{\infty} \frac{(\mathbf{i}y)^n}{n!} = 1 + \mathbf{i}y + \frac{(\mathbf{i}y)^2}{2!} + \frac{(\mathbf{i}y)^3}{3!} + \frac{(\mathbf{i}y)^4}{4!} + \frac{(\mathbf{i}y)^5}{5!} + \cdots + \frac{(\mathbf{i}y)^n}{n!} + \cdots = \\ &= 1 + \mathbf{i}y + \frac{\mathbf{i}^2 y^2}{2!} + \frac{\mathbf{i}^3 y^3}{3!} + \frac{\mathbf{i}^4 y^4}{4!} + \frac{\mathbf{i}^5 y^5}{5!} + \cdots + \frac{\mathbf{i}^n y^n}{n!} + \cdots \end{aligned}$$

Now we can use that

$$\mathbf{i}^2 = -1, \quad \mathbf{i}^3 = \mathbf{i}^2 \cdot \mathbf{i} = -\mathbf{i}, \quad \mathbf{i}^4 = \mathbf{i}^2 \cdot \mathbf{i}^2 = 1, \quad \mathbf{i}^5 = \mathbf{i}, \quad \dots$$

That will mean that in the series formula for $e^{\mathbf{i}y}$ all the even powers will not have \mathbf{i} , and all the odd powers will have \mathbf{i} in them. I.e., the even powers will go into the **real part** of the $e^{\mathbf{i}y}$, and the odd powers will go into the **imaginary part** of the $e^{\mathbf{i}y}$:

$$\begin{aligned} e^{\mathbf{i}y} &= 1 + \mathbf{i}y - \frac{y^2}{2!} - \frac{\mathbf{i}y^3}{3!} + \frac{y^4}{4!} + \frac{\mathbf{i}y^5}{5!} + \cdots = \\ &= \left(1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \frac{y^6}{6!} + \cdots\right) + \mathbf{i}\left(y - \frac{y^3}{3!} + \frac{y^5}{5!} - \cdots\right) \end{aligned}$$

Here we recognize that the first part, the real part, is the Taylor series of $\cos y$, and the second part, the imaginary part, is the Taylor series of $\sin y$. So we get the following famous result:

$$e^{\mathbf{i}y} = \cos y + \mathbf{i} \sin y.$$

This is called **Euler's formula**.

The Euler's formula explains what happens when the characteristic equation has complex roots: Suppose the roots of the characteristic equation are

$$r_{1,2} = \frac{-B}{2A} \pm \left(\frac{\sqrt{-D}}{2A}\right)\mathbf{i}.$$

As before, call for short the real part $\mu = \frac{-B}{2A}$, and $\lambda = \frac{\sqrt{-D}}{2A}$, thus

$$r_1 = \mu + \mathbf{i} \cdot \lambda, \quad r_2 = \mu - \mathbf{i} \cdot \lambda.$$

Then the general solution to the differential equation can be written using the **complex exponential** as

$$\begin{aligned} y &= C_1 e^{r_1 x} + C_2 e^{r_2 x} = C_1 e^{(\mu + \mathbf{i} \cdot \lambda)x} + C_2 e^{(\mu - \mathbf{i} \cdot \lambda)x} = C_1 e^{\mu x} e^{+\mathbf{i} \cdot \lambda x} + C_2 e^{\mu x} e^{-\mathbf{i} \cdot \lambda x} = \\ &= C_1 e^{\mu x} (\cos \lambda x + \mathbf{i} \cdot \sin \lambda x) + C_2 e^{\mu x} (\cos(-\lambda x) + \mathbf{i} \cdot \sin(-\lambda x)) \end{aligned}$$

Here we used the Euler's formula. Now we use the fact that $\cos(-X) = \cos X$ and $\sin(-X) = -\sin X$ and we rearrange:

$$y = C_1 e^{\mu x} (\cos \lambda x + \mathbf{i} \cdot \sin \lambda x) + C_2 e^{\mu x} (\cos \lambda x - \mathbf{i} \cdot \sin \lambda x) = (C_1 + C_2) e^{\mu x} \cos \lambda x + \mathbf{i} \cdot (C_1 - C_2) e^{\mu x} \sin \lambda x$$

Recall that we named $\mu = \frac{-B}{2A}$, and $\lambda = \frac{\sqrt{-D}}{2A}$; so we see that

$$y = C_1 e^{r_1 x} + C_2 e^{r_2 x} = (C_1 + C_2) e^{\frac{-B}{2A} x} \cos\left(\frac{\sqrt{-D}}{2A} x\right) + \mathbf{i} \cdot (C_1 - C_2) e^{\frac{-B}{2A} x} \sin\left(\frac{\sqrt{-D}}{2A} x\right).$$

Now if we call $\tilde{C}_1 = (C_1 + C_2)$ and $\tilde{C}_2 = \mathbf{i}(C_1 - C_2)$, we get:

$$y = C_1 e^{r_1 x} + C_2 e^{r_2 x} = \tilde{C}_1 e^{\frac{-B}{2A} x} \cos\left(\frac{\sqrt{-D}}{2A} x\right) + \tilde{C}_2 e^{\frac{-B}{2A} x} \sin\left(\frac{\sqrt{-D}}{2A} x\right).$$

That shows that the general solution of the second order differential equation can be written in two equally correct ways: either we use the two complex exponentials, or the two exponential/trigonometry mixtures. The complex exponentials have the benefit of having the same form as the solutions for real characteristic roots. The mixtures though have the benefit of giving **real solutions** . Usually when solving the differential equation, or an initial value problem, we need a real solution, so the exponential/trigonometry mixtures are to be preferred.